Optimizing the fundamental limits for quantum and private communication

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The quantum capacity of a noisy quantum channel determines the maximal rate at which we can code reliably over asymptotically many uses of the channel, and it characterizes the channel's ultimate ability to transmit quantum information coherently. In this paper, we derive single-letter computable upper bounds on the quantum and private capacities of quantum channels. The quantum capacity of a quantum channel is always no larger than the quantum capacity of its extended channels, such as its flagged channel, since the flags (or extensions) of the channel can be considered as assistance from the environment. By optimizing the degrading channel of the flagged quantum channel as well as the parametrized flag states, we obtain new upper bounds on the quantum capacity of the main channel. Furthermore, we generalize our approach to private communication and derive upper bounds on the private classical capacity of a quantum channel. As notable applications, we establish improved upper bounds to the quantum and private capacities for fundamental quantum channels of great interest in quantum information, some of which are also the sources of noise in superconducting quantum computing. In particular, our upper bounds on the quantum capacities of the depolarizing channel and the generalized amplitude damping channel are strictly better than previously best-known bounds.

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I. INTRODUCTION

Quantum communication is an integral part of quantum information theory. The quantum capacity of a noisy quantum channel is the maximum rate at which it can convey quantum information reliably over asymptotically many uses of the channel. The theorem by Lloyd, Shor, and Devetak (LSD) [1–3] and the work in Refs. [4–6] show that the quantum capacity is equal to the regularized coherent information, i.e., the quantum capacity of a noisy quantum channel $\mathcal{N}$ is given by

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{Q^{(1)}(\mathcal{N}^{\otimes n})}{n}, \tag{1}$$

where $Q^{(1)}(\mathcal{N}) := \max_{\rho_A} \mathrm{H}(\mathcal{N}(\rho_A)) - H(\mathcal{N}^c(\rho_A))$ denotes the coherent information of the channel, $\mathcal{N}^c$ is the complementary channel of $\mathcal{N}$, and H is the von Neumann entropy.

Quantum capacity is notoriously complicated and hard to evaluate since it is characterized by the above multi-letter, regularized expression. Note that such regularization is necessary in general due to the superadditivity of channel's coherent information [7, 8] and it is worse that an unbounded number of channel uses may be required to detect a channels capacity [9]. It has been an open problem for more than 20 years to determine the capacities of some of these low-noise channels such as the depolarizing channels.

To evaluate the quantum capacity, substantial efforts have been made in the past two decades. During the development of estimating the quantum capacity, there are two threads of studying the upper bounds on quantum capacity. One is to develop single-letter upper bounds that are efficiently computable for general quantum channels (see, e.g., [10–16]), which aims to provide efficiently computable benchmarks for arbitrary quantum noise. Another is to develop single-letter upper bounds that are relatively tight or computable for specific classes of quantum channels (see, e.g., [17–25]), which help us better understand quantum communication via these quantum channels.

The depolarizing channel $\mathcal{D}_p^d(\rho) := (1-p)\rho + p\,\mathrm{Tr}(\rho)I_d/d$ is one of the most important and fundamental quantum channels [26, 27], which is useful in modelling noise for quantum hardware such as the superconducting quantum processor [28]. However, even for the qubit depolarizing channel, the quantum capacity remains unsolved despite substantial efforts [8, 18–23]. Recently, Fanizza et al. [23] considered the degradable extensions with non-orthogonal quantum flags and obtained improved upper bound on the quantum capacity of qubit depolarizing channel in an intermediate regime of noise.

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A. Summary of results

In this paper, we introduce single-letter computable converse bounds on the quantum and private capacities of a noisy quantum channel. As notable applications, we establish improved upper bounds to the quantum and private capacities for fundamental channels of great interest in quantum communication.

In Section II, we introduce an approach to upper bound the quantum capacity by optimizing the flagged (or extended) quantum channel, whose capacity is no smaller than the main channel since the flags (or extensions) of the channel can be considered as the assistance from the environment. Our method allows approximate degradability of the flagged channel and focuses on optimizing the quantum or private capacities of the approximate degradable flagged quantum channels with parameterized flag states. Built on previously works [19, 23], our method, on one hand, improves the approximate degradable bound by allowing more freedom or space in optimizing or searching the approximate degradable channels. In particular, for the low-noise depolarizing channels, our bounds outperform previously best known upper bounds via approximate degradable channels [19] (cf. Figure. 1). On the other hand, our method improves the previous quantum flag upper bound by involving the optimization over the flagged channels. For the depolarizing channels with intermediate noise, our bound outperforms previously best known upper bounds obtained via unoptimized flag states [23] as well as the bound obtained via degradable and anti-degradable decompositions [22] (cf. Figure. 2).

We further apply our bounds to other quantum channels of interest. We show that our bounds lead to better upper bounds on the quantum capacity of the shift depolarizing channel in Section II D. As another notable application, Section II E establishes improved upper bounds on the quantum capacity of the generalized amplitude damping channel, which are tighter than previously known upper bounds [29] via data-processing approach [29] and Rains information [25].

In Section III, we extend our upper bounds for quantum communication to private communication. Similarly, we obtain efficiently computable upper bounds on the private classical capacity by optimizing the flagged quantum channel. As applications, we derive improved upper bounds to the private classical capacity for the low-noise depolarizing channels and the BB84 channels.

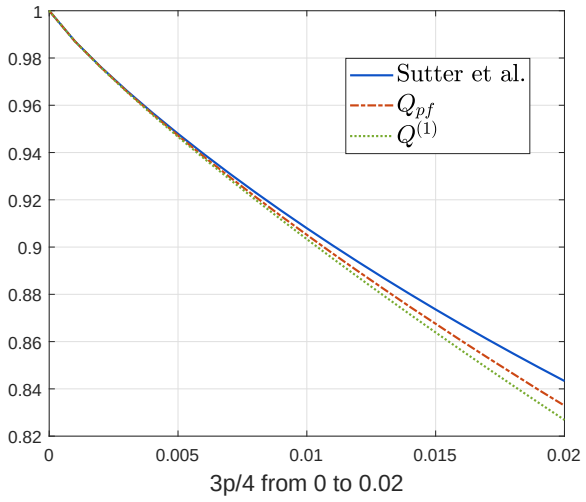


FIG. 1: Upper and lower bounds on the quantum capacity $Q(\mathcal{D}_p)$ of the low-noise qubit depolarizing channel for the interval $3p/4 \in [0, 0.02]$. The channel's coherent information $Q^{(1)}$ (green, dotted) provides a lower bound on the quantum capacity $Q(\mathcal{D}_p)$. The red dashed line depicts our bound Q_{pf} in Proposition 3, which improves the bound via approximate degradable channels obtained in [19].

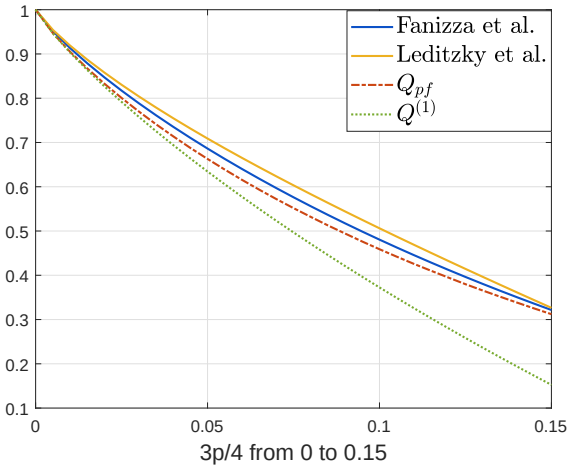


FIG. 2: Upper and lower bounds on the quantum capacity $Q(\mathcal{D}_p)$ of the intermediate-noise qubit depolarizing channel for the interval $3p/4 \in [0, 0.15]$. The channel's coherent information $Q^{(1)}$ (green, dotted) provides a lower bound on the quantum capacity $Q(\mathcal{D}_p)$. The red dashed line depicts our bound Q_{pf} in Proposition 3, which improves the bound obtained via mixed flag states [23] (blue, solid) as well as the bound obtained in [22] (yellow, solid).

II. OPTIMIZED QUANTUM FLAG UPPER BOUND ON QUANTUM CAPACITY

We begin with the preliminaries on quantum information and degradable quantum channels in Section II A. Then, in Section II B, we give a brief review of the recent results on the quantum flag upper bounds for depolarizing channels. In Section II C, we introduce the optimized flag upper bound on quantum capacity, which is the main contribution of this work. In Section II D we show that our bound on the quantum capacity of the depolarizing channel and the shift depolarizing channel outperforms previously best known bound. In Section II E, we establish improved upper bounds on the quantum capacity of the generalized amplitude damping channel.

A. Backgrounds

In the following, we will frequently use symbols such as A (or A') and B (or B') to denote (finite-dimensional) Hilbert spaces associated with Alice and Bob, respectively. We use d_A to denote the dimension of system A. The set of linear operators acting on A is denoted by $\mathcal{L}(A)$. The set of positive operators acting on A is denoted by $\mathcal{P}(A)$. The set of quantum states on A is denoted by $\mathcal{S}(A)$. We usually write an operator with a subscript indicating the system that the operator acts on, such as M_{AB}, and write $M_A := \mathrm{Tr}_B M_{AB}$. Note that for a linear operator $X \in \mathcal{L}(A)$, we define $|X| = \sqrt{X^\dagger X}$, where $X^\dagger$ is the adjoint operator of X, and the trace norm of X is given by $\|X\|_1 = \mathrm{Tr}\,|X|$. A quantum channel $\mathcal{N}_{A'\to B}$ is simply a completely positive (CP) and trace-preserving (TP) linear map from $\mathcal{L}(A')$ to $\mathcal{L}(B)$. The Choi-Jamiołkowski matrix of $\mathcal{N}$ is given by $J_{\mathcal{N}} = \sum_{ij} |i_A\rangle\langle j_A| \otimes \mathcal{N}(|i_{A'}\rangle\langle j_{A'}|)$, where $\{|i_A\rangle\}$ and $\{|i_{A'}\rangle\}$ are orthonormal bases on isomorphic Hilbert spaces A and A', respectively.

Degradable quantum channels are among the only channels whose quantum capacity is known [30]. They are characterized by the property that the complementary channel can be written as a composition of the original channel with a degrading channel. For degradable quantum channels, the channels coherent information is known to be additive. Thus, the quantum capacity of a degradable channel is given by the channel's coherent information [30].

Definition 1 (Degradable and anti-degradable channels) *A channel $\mathcal{N}_{A\to B}$ is called degradable if there exists another quantum channel $\mathcal{D}_{B\to E}$ such that*

$$\mathcal{N}^c_{A\to E} = \mathcal{D}_{B\to E} \circ \mathcal{N}_{A\to B}, \tag{2}$$

where the complementary channel $\mathcal{N}^c_{A\to E}(\rho) := \mathrm{Tr}_B\, V\rho V^\dagger$ *and* V *is an isometry that ensures* $\mathcal{N}_{A\to B}(\rho) := \mathrm{Tr}_E\, V\rho V^\dagger$. *Here, the channel* $\mathcal{D}_{B\to E}$ *is called degrading channel.*

On the other hand, the channel $\mathcal{N}_{A\to B}$ *is called anti-degradable if there exists a quantum channel* $D_{E\to B}$ *such that*

$$\mathcal{N}_{A\to B} = D_{E\to B} \circ \mathcal{N}^c_{A\to E}. \tag{3}$$

The class of degradable channels provides us a chance to explore the quantum capacity of other channels since its quantum capacity is given by a single letter formula. In a previous work [19], Sutter et al. made use of the continuity bound of channel capacities [31] to estimate the quantum capacity of approximate degradable quantum channels and gave previously tightest upper bound for low noise depolarizing channels. Moreover, degradable extensions with orthogonal flags have also been used to provide estimations on the quantum capacity of depolarizing channels [21, 22, 32]. By decomposing the depolarizing channel to the convex combination of degradable and anti-degradable channels, the work by Leditzky et al. [22] established the state-of-the-art upper bound in the high-noise regime. Recently, Fanizza et al. [23] showed that the degradable extensions with non-orthogonal quantum flags could be used to obtain better upper bound on the quantum capacity of the intermediate-noise qubit depolarizing channel.

B. Quantum flag upper bound for the depolarizing channel

The intuition behind the so-called *quantum flag model* [23] can be traced back to environment-assisted quantum information processing [33–35], where an environment (or channel) is assumed to "friendly" provide partial helpful information to the receiver.

To be specific, for a given noisy quantum channel $\mathcal{N}_{A\to B} = \sum_j p_j \mathcal{N}_j$ with probability distribution $\{p_j\}$ and individual CPTP maps $\mathcal{N}_j$, the quantum flag model assumes that the state of the partial environment is σ_j for each individual map $\mathcal{N}_j$ and the receiver has access to this partial environment. In this case, the complete channel can be written as

$$\widehat{\mathcal{N}} = \sum_j p_j \mathcal{N}_j \otimes \sigma_j. \tag{4}$$

With such assistance from the environment, the receiver Bob receives a quantum flag σ_j which encodes the information about which channel is acting. With this extra information, the flagged channel can certainly convey more quantum information then the original channel $\mathcal{N}$, i.e.,

$$Q(\mathcal{N}) \le Q(\widehat{\mathcal{N}}). \tag{5}$$

Note that a special kind of flagged channel was considered in [21, 32] where the flag states are assumed to be orthogonal pure states. However, the flag states are not necessarily orthogonal.

For the depolarizing channel, one of its flagged versions can be represented in the following form

$$\widehat{\mathcal{D}}^d_p(\rho) = (1-p)\rho \otimes \sigma_0 + p\,\mathrm{Tr}(\rho)\frac{I_d}{d} \otimes \sigma_1. \tag{6}$$

The main idea of [23] is to study the coherent information of the degradable flagged depolarizing channel $\widehat{\mathcal{D}}^d_p$. The reason is twofold. On one hand, the quantum capacity of the flagged depolarizing channel is larger than or equal to the capacity of the original channel, i.e., $Q(\mathcal{D}^d_p) \le Q(\widehat{\mathcal{D}}^d_p)$. On the other hand, the quantum capacity gets simplified to be single-letter coherent information when the channel is degradable. By choosing $\sigma_0 = |e_1\rangle\langle e_1|$, $\sigma_1 = c|e_1\rangle\langle e_1| + (1-c)|e_1^\perp\rangle\langle e_1^\perp|$ and constructing a special kind of degrading channel $\mathcal{W}$ such that

$$(\widehat{\mathcal{D}}^d_p)^c = \mathcal{W} \circ \widehat{\mathcal{D}}^d_p, \tag{7}$$

the authors of [23] derived sufficient conditions for the degradability of $\widehat{\mathcal{D}}^d_p$ and determine a feasible region of c, which leads to an outer region of $Q(\widehat{\mathcal{D}}^d_p)$. The minimum value in this region is then used to upper bound the quantum capacity of depolarizing channel [23], i.e.,

$$Q(\mathcal{D}^d_p) \le \log d + \zeta((1-p)c+p) - \zeta((1-p)c + \frac{p}{d^2}) - (d^2-1)\zeta(\frac{p}{d^2}), \tag{8}$$

where $\zeta(z) = -z\log z$. They also discussed the case of pure flag states and reported a less handy expression.

C. General quantum flag upper bound

The above approach gives a pretty good example of how to utilize the non-orthogonal quantum flags. However, the step of manually creating a degrading channel leaves the space for finding the optimal flagged degradable channel with minimal coherent information. Moreover, as the step of constructing the degrading channel is artificial, this method can not be easily applied to other quantum channels.

To better exploit the power of the extended or flagged quantum channel, we are going to introduce a new approach that optimizes the flagged states and utilizes approximate degradable quantum channels. The central idea of our approach is to consider the optimization of all degrading channels with some error tolerance rather than manually constructing the degrading channel. We also introduce a more general quantum flag model involving the decomposition of a quantum channel to the sum of a set of CP maps.

Given a quantum channel $\mathcal{N}_{A\to B}$, its degradability parameter [19] is defined to be

$$\eta(\mathcal{N}_{A\to B}) := \min_{\mathcal{D}_{B\to E}} \frac{1}{2}\|\mathcal{N}^c_{A\to E} - \mathcal{D}_{B\to E}\circ\mathcal{N}\|_\diamond, \tag{9}$$

where $\|\mathcal{F}\|_\diamond := \sup_{k\in\mathbb{N}}\sup_{\|X\|_1\le 1}\|(\mathcal{F}\otimes \mathrm{id}_k)(X)\|_1$ denotes the diamond norm. Moreover, a quantum channel $\mathcal{N}$ is called ε-degradable if $\eta(\mathcal{N}) = \varepsilon$. Note that for any two quantum channels $\mathcal{N}_1, \mathcal{N}_2$ from A to B, the diamond norm of their difference can be expressed as a semidefinite program (SDP) of the following form [36]:

$$\begin{aligned}\frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond = \inf\big\{\ \mu\ \big|\ &\mathrm{Tr}_B\, Z \le \mu \mathbb{1}_A,\\ &Z \ge J_{\mathcal{N}_1} - J_{\mathcal{N}_2},\ Z \ge 0\ \big\},\end{aligned} \tag{10}$$

where $J_{\mathcal{N}_1}$ and $J_{\mathcal{N}_2}$ are the corresponding Choi-Jamiołkowski matrices. The degradability parameter of $\mathcal{N}$ thus can be computed by the following SDP [19]:

$$\begin{aligned}
\inf\quad & \mu && \text{(11a)}\\
\text{s.t.}\quad & \mathrm{Tr}_E\, Z_{AE} \le \mu I_A,\ Z_{AE} \ge 0, && \text{(11b)}\\
& \mathrm{Tr}_E\, J^{BE}_{\mathcal{D}} = I_B,\ J^{BE}_{\mathcal{D}} \ge 0, && \text{(11c)}\\
& Z_{AE} \ge J^{AE}_{\mathcal{N}^c} - J_{\mathcal{D}\circ\mathcal{N}} && \text{(11d)}
\end{aligned}$$

where $\Phi_{EE'}$ is the unnormalized maximally entangled state with dimension d_E. Note that the last condition Eq. (11d) has semidefinite presentations (e.g., $\mathrm{Tr}_{BE}(J^{AB}_{\mathcal{N}}\otimes\Phi_{EE'})((J^{BE}_{\mathcal{D}})^T\otimes I_{AE})$, see alternative presentations via transfer matrix in [19]).

For approximate degradable channels, a useful property related to quantum capacity was established in [19]: let $\mathcal{N}_{A\to B}$ be a ε-degradable quantum channel whose environment dimension is d_E, then

$$Q(\mathcal{N}) \le Q^{(1)}(\mathcal{N}) + \varepsilon\log(d_E - 1) + h(\varepsilon) + 2\varepsilon\log d_E + g(\varepsilon), \tag{12}$$

where

$$h(p) := -p\log p - (1-p)\log(1-p) \tag{13}$$

is the binary entropy and

$$g(p) := (1+p)\log(1+p) - p\log p = (1+p)h(p/(1+p)) \tag{14}$$

is the bosonic entropy. This slightly simplified representation of the continuity bound is from [29, 37].

Using the above property, we further introduce the approximate degradable flag channels and obtain the following upper bound on quantum capacity. Note that we use CP map decomposition of the channel instead of the convex decomposition of CPTP maps since the latter is more restricted but not necessary in evaluating quantum capacity.

Proposition 2 (General flag approximate degradable bound) *Let $\mathcal{N}$ be a quantum channel with CP map decomposition $\mathcal{N} = \sum_{j=0}^k \mathcal{N}_j$. The quantum capacity of $\mathcal{N}$ is bounded by*

$$Q(\mathcal{N}) \le \inf_{\sigma_0,\sigma_1} Q^{(1)}(\widehat{\mathcal{N}}) + \eta(\widehat{\mathcal{N}})\log(d_E - 1) + h\left(\eta(\widehat{\mathcal{N}})\right) + 2\eta(\widehat{\mathcal{N}})\log d_E + g\left(\eta(\widehat{\mathcal{N}})\right).$$

where $\widehat{\mathcal{N}}(\cdot) = \sum_{j=0}^{k} \mathcal{N}_j(\cdot) \otimes \sigma_j$ *and* d_E *is the dimension of the output system of* $\widehat{\mathcal{N}}^c$*. Note that this bound is never worse than the previous continuity bound [19] since the later is the case* $\sigma_0 = \sigma_1$*.*

Proof The key idea here is to consider the approximate degradable flag channels. By evaluating the degradability parameter of the flag channel and applying the continuity upper bound on the quantum capacity, we have

$$Q(\mathcal{N}) \leq \inf_{\sigma_0,\sigma_1} Q(\widehat{\mathcal{N}}) \tag{15}$$

$$\leq \inf_{\sigma_0,\sigma_1} Q^{(1)}(\widehat{\mathcal{N}}) + \varepsilon \log(d_E - 1) + h(\varepsilon) + 2\varepsilon \log d_E + g(\varepsilon), \tag{16}$$

where ε in Eq. (16) is the degradability parameter determined by $\eta(\widehat{\mathcal{N}})$. The first inequality is due to the fact that $\widehat{\mathcal{N}}$ is an extension of the original channel $\mathcal{N}$, which allows Bob to obtain extra information from the environment. The second inequality is due to Eq. (12). ■

Clearly, computing the above bound is extremely hard. However, we could restrict the number of flags and parametrize the flag states and then optimize these parameters. In particular, for the case of two pure flag states, we are able to establish the following upper bound.

Proposition 3 (Pure flag approximate degradable bound) *Let* $\mathcal{N}$ *be a quantum channel with CP map decomposition* $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1$*. The quantum capacity of* $\mathcal{N}$ *is bounded by the following optimized pure flag upper bound:*

$$Q_{pf}(\mathcal{N}) := \inf_{0\leq\alpha\leq 1} f(\mathcal{N}, \alpha), \tag{17}$$

with

$$f(\mathcal{N},\alpha) := Q^{(1)}(\widehat{\mathcal{N}}) + \eta(\widehat{\mathcal{N}}) \log(d_E - 1) + h\left(\eta(\widehat{\mathcal{N}})\right) + 2\eta(\widehat{\mathcal{N}}) \log d_E + g\left(\eta(\widehat{\mathcal{N}})\right),$$

where $\widehat{\mathcal{N}}(\cdot) = \mathcal{N}_0(\cdot) \otimes |\psi_\alpha\rangle\langle\psi_\alpha| + \mathcal{N}_1(\cdot) \otimes |0\rangle\langle 0|$*,* $|\psi\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$*, and* d_E *is the dimension of the output system of* $\widehat{\mathcal{N}}^c$*. Note that this bound is never worse than the previous continuity bound [19] since the later is just the case* $\alpha = 1$*.*

Proof As the system of the flag states is at the receiver Bob's side, then he could locally do any unitary on the system. Thus, with of loss of generality, we could fix one flag state $\sigma_1 = |0\rangle\langle 0|$ and assume that $\sigma_0 = |\psi_a\rangle\langle\psi_a|$. ■

Remark 1 Note that the objective function $f(\mathcal{N},\alpha)$ in Proposition 3 can be computed efficiently for many channels since the degradability parameter $\eta(\mathcal{N})$ can be computed via the SDP in Eq. 11. This means that we could do brute-force search on $\alpha \in [0,1]$ to find the almost optimal flag states, thus obtain the almost optimal pure flag upper bound on quantum capacity. Similarly, for the case $\sigma_0 = |e_1\rangle\langle e_1|$, $\sigma_1 = c|e_1\rangle\langle e_1| + (1-c)|e_1^\perp\rangle\langle e_1^\perp|$ considered in [23], we could also use the brute-force search approach to find the almost optimal quantum flag upper bound. We also note that the upper bound in Proposition 3 and Proposition 2 may be slightly improved by using a slightly tighter continuity bound in Theorem 7 of [19].

Furthermore, we could only optimize over the flagged channels which are degradable and obtain the following upper bound on quantum capacity. This bound can be efficiently computed since the coherent information of degradable channels can be efficiently optimized [38].

Corollary 4 (Pure flag degradable bound) *Let* $\mathcal{N}$ *be a quantum channel with CP map decomposition* $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1$*. The quantum capacity of* $\mathcal{N}$ *satisfies that*

$$Q(\mathcal{N}) \leq \widetilde{Q}_{pf}(\mathcal{N}) := \inf_{0\leq\alpha\leq 1} \left\{ Q^{(1)}(\widehat{\mathcal{N}}) | \eta(\widehat{\mathcal{N}}) = 0 \right\}, \tag{18}$$

where $\widehat{\mathcal{N}}(\cdot) = \mathcal{N}_0(\cdot) \otimes |\psi_\alpha\rangle\langle\psi_\alpha| + \mathcal{N}_1(\cdot) \otimes |0\rangle\langle 0|$ *and* $|\psi\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$*.*

Proof The key step is to restrict the degradability parameter $\eta(\widehat{\mathcal{N}})$ in the objective function $f(\mathcal{N},\alpha)$ in 3 to be 0. Then we have $f(\mathcal{N},\alpha)=Q^{(1)}(\widehat{\mathcal{N}})$. Hence, the whole optimization problem is to minimize the coherent information of the flag channel with zero degradability parameter. Note that we could also use the extendibility measures in [39] to replace the condition $\eta(\widehat{\mathcal{N}})=0$ due to the fact that the main channel is degradable if and only if its complementary channel is anti-degradable, which is equivalent to two-extendible. ■

Example 1 (Qubit depolarizing with noise parameter $p=0.1$) To see the usefulness of our approach, we first consider the qubit depolarizing channel with noise parameter $p=0.1$ as an example. Note that we denote the qubit depolarizing channel as $\mathcal{D}_p$ for simplicity and we use the following convex decomposition

$$\widehat{\mathcal{D}}_p(\rho)=(1-p)\rho\otimes|\psi_\alpha\rangle\langle\psi_\alpha|+p\,\mathrm{Tr}(\rho)\frac{I_2}{2}\otimes|0\rangle\langle 0|. \tag{19}$$

In the following Figure 3, we find that the pure flag states with parameter $\alpha=0.846$ is almost optimal. Thus, we have

$$Q(\mathcal{D}_{0.1})\leq f(\mathcal{D}_{0.1},0.846)\leq 0.45885. \tag{20}$$

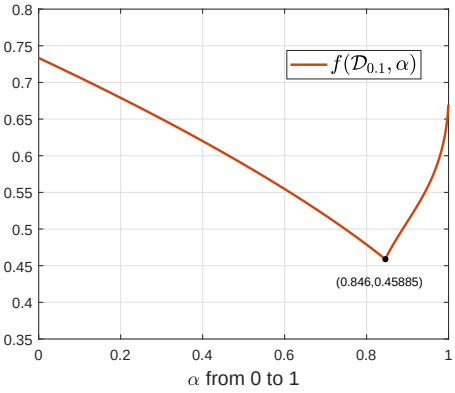


FIG. 3: Upper bounds on $Q_{\mathcal{D}_{0.1}}$ via different pure flag states $\{\sqrt{\alpha}|0\rangle+\sqrt{1-\alpha}|1\rangle,|0\rangle\}$. The solid line depicts the values of the the objective function $f(\mathcal{N},\alpha)$ in Proposition 3 for different α. The ground point implies that $Q(\mathcal{D}_{0.1})\leq f(\mathcal{D}_{0.1},0.846)\leq 0.45885$.

D. Applications to the depolarizing channel and its generalization

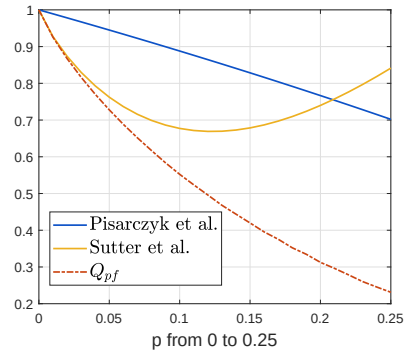
To exploit the power of our approach, we numerically compute the optimized pure flag upper bounds for several fundamental channels with interest and compare them with previously best-known bounds.

As we already remarked that our bound is never worse than the bound via approximate degradable channels, we find that our bound can strictly improve the previous bound in [19]. In Figure. 1, we consider the case of low noise regime, where we choose $3p/4\in[0,0.02]$. It is surprising to see that when the depolarizing noise is under the threshold 2%, the difference between the channel's coherent information and quantum capacity is smaller than 0.0123, which indicates that super-additivity and degenerate codes have negligible benefit for the quantum capacity in this case.

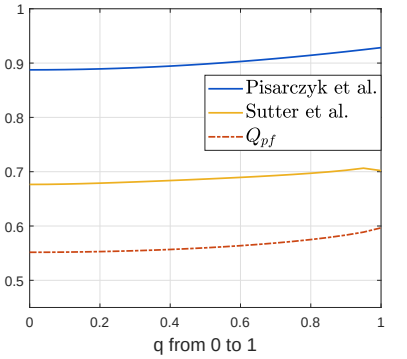
We also consider the regime of intermediate-noise in Figure 2 where $3p/4\in[0,0.15]$ and show that our bound outperforms previous bounds obtained via unoptimized flag states [23] as well as the bound obtained via degradable and anti-degradable decompositions [22].

Furthermore, we consider the shifted depolarizing channel $\mathcal{N}_{p,q}(\rho)=(1-p)\rho+p(I+qZ)/2$ [40], which generalizes the well-studied depolarizing channel. This channel outputs either the state $(I+qZ)/2$ shifted from the maximally mixed state with probability p or the noiseless input state with probability $1-p$. Figure 4 compares the upper bound of the quantum capacity derived in Proposition 3 with previously known upper bound [40]: $Q(\mathcal{N}_{p,q})\leq\log_2\left(1-p/4+\frac{1}{2}\sqrt{1-2p+p^2+q^2p^2/4}+\frac{1}{2}|p/2-\sqrt{1-2p+p^2+q^2p^2/4}|\right)$.

For quantum capacity, we only need to focus on the interval $p \in [0, 1/4]$ as the channel is anti-degradable for $p \geq 1/4$. Via numerical analysis, we find that our bounds significantly outperform the previously known bound [40] for the shift depolarizing channel.



(a) Shifted depolarizing channel $\mathcal{N}_{p,0.1}$



(b) Shifted depolarizing channel $\mathcal{N}_{0.1,q}$

FIG. 4: Upper and lower bounds on the quantum capacity $Q(\mathcal{N}_{p,q})$ of the shift depolarizing channel. The red dashed line depicts our bound Q_{pf} in Proposition 3, which improves the bound via obtained in [19, 40].

E. Application to generalized amplitude damping channel

The generalized amplitude damping (GAD) channel is one of the realistic sources of noise in superconducting quantum processor [41]. It can be viewed as the qubit analogue of the bosonic thermal channel and can be used to model lossy processes with background noise for low-temperature systems. When $N = 0$, it reduces to the conventional amplitude damping channel. For GAD channel, some prior works have established bounds on its quantum capacity [29, 42, 43].

To be specific, the generalized amplitude damping channel is a two-parameter family of channels described as follows:

$$\mathcal{A}_{y,N}(\rho) = A_1\rho A_1^\dagger + A_2\rho A_2^\dagger + A_3\rho A_3^\dagger + A_4\rho A_4^\dagger, \tag{21}$$

where $y, N \in [0, 1]$ and

$$A_1 = \sqrt{1-N}(|0\rangle\langle 0| + \sqrt{1-y}|1\rangle\langle 1|), \tag{22}$$
$$A_2 = \sqrt{y(1-N)}|0\rangle\langle 1|, \tag{23}$$
$$A_3 = \sqrt{N}(\sqrt{1-y}|0\rangle\langle 0| + |1\rangle\langle 1|), \tag{24}$$
$$A_4 = \sqrt{yN}|1\rangle\langle 0|. \tag{25}$$

To apply our bounds, we first decompose the GAD channel as the following convex combination of two degradable channels:

$$\mathcal{A}_{y,N} = \sqrt{1-N}\mathcal{A}^1_{y,N}(\rho) + \sqrt{N}\mathcal{A}^2_{y,N}(\rho), \tag{26}$$

with $\mathcal{A}^1_{y,N}(\rho) = A_1\rho A_1^\dagger/(1-N) + A_2\rho A_2^\dagger/(1-N)$ and $\mathcal{A}^2_{y,N}(\rho) = A_3\rho A_3^\dagger/N + A_4\rho A_4^\dagger/N$. Then, we could apply the pure flag approximate degradable bound in Proposition 3 by introducing the flagged channel

$$\widehat{\mathcal{A}}_{y,N} = \sqrt{1-N}\mathcal{A}^1_{y,N}(\rho) \otimes |\psi_\alpha\rangle\langle\psi_\alpha| + \sqrt{N}\mathcal{A}^2_{y,N}(\rho) \otimes |0\rangle\langle 0|, \tag{27}$$

where $|\psi_\alpha\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$.

By numerical optimization, we observe that the optimal value is achieved by $\alpha = 0$ for noise parameter $y \in (0, 1/2)$. By further exploration, we find that the flagged channel $\widehat{\mathcal{A}}_{y,N}$ with $|\psi_a\rangle = |1\rangle$ is degradable due to the fact that both $\mathcal{A}^1_{y,N}$ and $\mathcal{A}^2_{y,N}$ are degradable (cf. Lemma 4 of [21]). In this case, we derive the following bound on the quantum capacity of GAD channel.

Proposition 5 *For the GAD channel with noise parameter $y \in (0, 1/2)$ and $N \in (0, 1)$, it holds that*

$$Q(\mathcal{A}_{y,N}) \leq Q(\widehat{\mathcal{A}}_{y,N}) = \max_{p\in[0,1]} I_{\mathrm{c}}\left(p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \widehat{\mathcal{A}}_{\gamma,N}\right), \tag{28}$$

where $I_{\mathrm{c}}(\rho, \mathcal{N}) \equiv H(\mathcal{N}(\rho)) - H(\mathcal{N}^{c}(\rho))$.

Proof Since $\widehat{\mathcal{A}}_{y,N}$ is the flagged channel of $\mathcal{A}_{y,N}$, we have

$$Q(\mathcal{A}_{y,N}) \leq Q(\widehat{\mathcal{A}}_{y,N}). \tag{29}$$

Then, we have

$$\begin{aligned} Q(\mathcal{A}_{y,N}) &\leq Q(\widehat{\mathcal{A}}_{y,N}) && (30)\\ &= Q^{(1)}(\widehat{\mathcal{A}}_{y,N}) && (31)\\ &= \max_{p\in[0,1]} I_{\mathrm{c}}\left(p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \widehat{\mathcal{A}}_{\gamma,N}\right). && (32) \end{aligned}$$

The first equality holds since $\widehat{\mathcal{A}}_{y,N}$ is degradable. The second inequality is due to fact that the coherent information for degradable channels is a concave function, which implies that diagonal input states outperform non-diagonal states [17]. ■

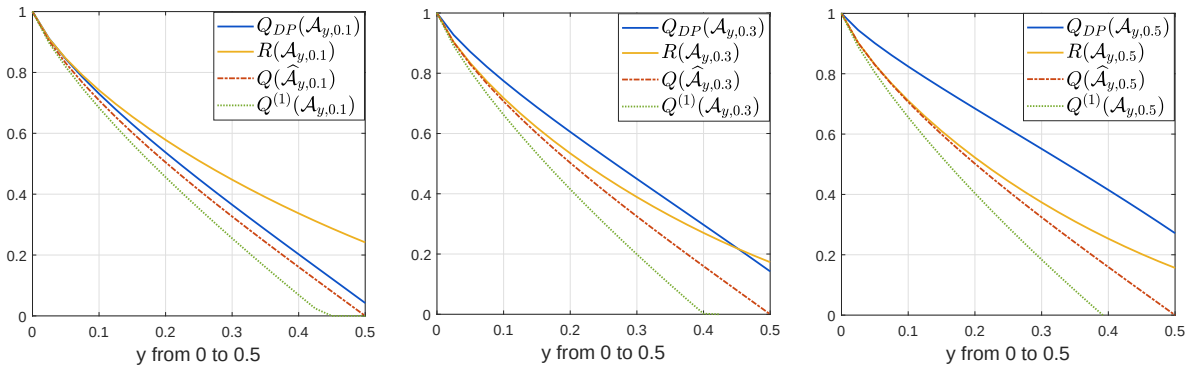
In Figure 5, we compare our bound with the previously best known bounds obtained in [29] on the quantum capacity of the GAD channel, thus reducing the gap between the lower and upper bounds. We mainly compare our bound with the data-processing bound [29] defined as

$$Q(\mathcal{A}_{\gamma,N}) \leq Q_{DP}(\mathcal{A}_{\gamma,N}) \equiv Q(\mathcal{A}_{y(1-N)/(1-yN),0}) \tag{33}$$

and the Rains information [25] defined as follows:

$$R(\mathcal{N}_{A'\to B}) \equiv \max_{\rho_A\in\mathcal{S}(A)} \min_{\sigma_{AB}\in \mathrm{PPT}'(A:B)} D\left(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \sigma_{AB}\right), \tag{34}$$

where $\mathrm{PPT}'(A:B) \equiv \{\sigma_{AB} | \sigma_{AB} \geq 0, \|\sigma_{AB}^{T_B}\|_1 \leq 1\}$, T_B is the partial transpose on system B, and $D(\rho\|\sigma) \equiv \mathrm{Tr}\left[\rho\left(\log_2\rho - \log_2\sigma\right)\right]$ is the quantum relative entropy. Rains information can be seemed as the channel version of the Rains bound [44], whose variants have many applications in quantum communication. For readers interested in the Rains bound, we refer to [44–49] for more information. Codes for computing our bound for the GAD channel can be found in [50]. The codes for computing the coherent information and the Rains information of the GAD channel are from [29].



(a) $\mathcal{A}_{y,0.1}$ (b) $\mathcal{A}_{y,0.3}$ (c) $\mathcal{A}_{y,0.5}$

FIG. 5: Upper and lower bounds on the quantum capacity $Q(\mathcal{A}_{y,N})$ of the generalized amplitude damping (GAD) channel. The green dotted line depicts the lower bound. The red dashed line depicts our bound in Eq. (28), which improves the bounds via data-processing approach and Rains information obtained in [29].

III. EXTENSION TO PRIVATE COMMUNICATION

A. New upper bounds on private capacity

The private capacity of a quantum channel is defined as the optimal rate at which classical information can be transmitted faithfully and privately from the sender Alice to the receiver Bob. The privacy here means a third party Eve who has access to the channel's environment cannot obtain anything useful about the information that Alice sends to Bob. The private capacity theorem states that the private capacity of a noisy quantum channel is given by its regularized private information [30, 51]

$$P(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} P^{(1)}\left(\mathcal{N}^{\otimes n}\right) = \sup_{n\in\mathbb{N}} \frac{1}{n} P^{(1)}\left(\mathcal{N}^{\otimes n}\right), \tag{35}$$

where $P^{(1)}(\mathcal{N}) = \max_E \left[\chi(E,\mathcal{N}) - \chi\left(E,\mathcal{N}^c\right)\right]$ is the private information with the maximization taken over all possible ensembles $E = \{p_j, \rho_j\}$ and $\chi(E,\mathcal{N}) = H\left(\sum_i p_i \mathcal{N}\left(\rho_i\right)\right) - \sum_i p_i H\left(\mathcal{N}\left(\rho_i\right)\right)$ is the Holevo information of the ensemble. The private capacity of a quantum channel is also extremely difficult to solve and only some weak converse or strong converse bounds were established in the past two decades (see, e.g., [52–56]).

Previously the upper bound on the private capacity of an ε-degradable channel $\mathcal{N}$ was established as follows [19, 57]:

$$P(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + 2\varepsilon \log(d_E - 1) + 8\varepsilon \log d_E + 2h\left(\varepsilon\right) + 4g\left(\varepsilon\right), \tag{36}$$

where $\varepsilon = \eta(\mathcal{N})$ and d_E is the dimension of the environment channel.

Using similar idea in Proposition 3 and Corollary 4, we obtain the following upper bounds on the private capacity of a quantum channel.

Proposition 6 *Let $\mathcal{N}$ be a quantum channel with CP map decomposition $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1$. The private capacity of $\mathcal{N}$ is bounded by the following optimized pure flag upper bound:*

$$P_{pf}(\mathcal{N}) := \inf_{0\leq\alpha\leq 1} \gamma(\mathcal{N}, \alpha), \tag{37}$$

with $\gamma(\mathcal{N},\alpha) := Q^{(1)}(\widehat{\mathcal{N}}) + 2\eta(\widehat{\mathcal{N}}) \log(d_E - 1) + 8\eta(\widehat{\mathcal{N}}) \log d_E + 2h\left(\eta(\widehat{\mathcal{N}})\right) + 4g\left(\eta(\widehat{\mathcal{N}})\right)$, where $\widehat{\mathcal{N}}(\cdot) = \mathcal{N}_0(\cdot) \otimes |\psi_\alpha\rangle\langle\psi_\alpha| + \mathcal{N}_1(\cdot) \otimes |0\rangle\langle 0|$, $|\psi\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$, and d_E is the dimension of the output system of $\widehat{\mathcal{N}}^c$.

Here we remark that the upper bound in Proposition 6 for private capacity may be slightly improved by using a slightly tighter continuity bound in [19, 58].

Corollary 7 *Let $\mathcal{N}$ be a quantum channel with CP map decomposition $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1$. The private capacity of $\mathcal{N}$ satisfies that*

$$P(\mathcal{N}) \leq \widetilde{P}_{pf}(\mathcal{N}) := \inf_{0\leq\alpha\leq 1} \left\{ Q^{(1)}(\widehat{\mathcal{N}}) \middle| \eta(\widehat{\mathcal{N}}) = 0 \right\}, \tag{38}$$

where $\widehat{\mathcal{N}}(\cdot) = \mathcal{N}_0(\cdot) \otimes |\psi_\alpha\rangle\langle\psi_\alpha| + \mathcal{N}_1(\cdot) \otimes |0\rangle\langle 0|$ and $|\psi\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$.

B. Applications to depolarizing channels and BB84 channels

Example 2 (Private capacity of low-noise depolarizing channel) In Figure 6, we compare our bound in and the above upper bound on the private capacity for the low-noise depolarizing channel.

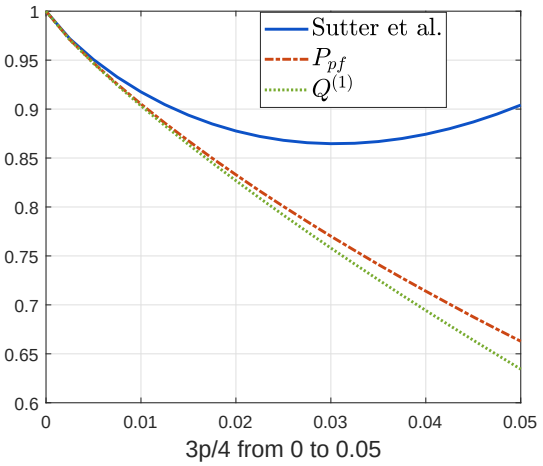


FIG. 6: Upper and lower bounds on the private capacity $P(\mathcal{D}_p)$ of the low-noise qubit depolarizing channel for the interval $3p/4 \in [0, 0.05]$. The channel's coherent information $Q^{(1)}$ (green, dotted) provides a lower bound on the private capacity $P(\mathcal{D}_p)$. The red dashed line depicts our bound P_{pf} in Proposition 6, which improves the bound via approximate degradable channels obtained in [19].

Example 3 (Private capacity of the BB84 channel) Bennett-Brassard quantum key distribution [59] is the most widely studied and practically applied quantum cryptography protocol. It is thus desirable to obtain efficiently computable bounds on the achievable key rate. Here we apply our bounds and evaluate the secret key rate of this protocol. To start the analysis, we first introduce the so-called BB84 channel: a qubit Pauli channel with independent bit-flip and phase-flip error probability where $p_X \in [0, 1/2]$ denotes the bit flip and $p_Z \in [0, 1/2]$ the phase flip probability. Due to its relevance for the BB84 protocol, this channel is known as the BB84 channel in the literature [21]. Further information about the BB84 channel and its operational justification can be found in [60].

More formally the BB84 channel is a CPTP map from $\mathcal{L}(A)$ to $\mathcal{L}(B)$:

$$\mathcal{B}_{p_X,p_Z}(\rho) = (1 - p_X - p_Z + p_X p_Z)\rho + (p_X - p_X p_Z)X\rho X + (p_Z - p_Z p_X)Z\rho Z + p_X p_Z Y\rho Y, \tag{39}$$

with $d_A = d_B = 2$. It is easy to verify that a Bell state maximizes the coherent information and then the channel coherent information of the BB84 channel is given by

$$Q^{(1)}(\mathcal{B}_{p_X,p_Z}) = 1 - h(p_X)0 - h(p_Z). \tag{40}$$

In particular, we will focus on the case where $p_X = p_Z = p$. In this case, Smith and Smolin [21] showed that

$$Q\left(\mathcal{B}_{p,p}\right) \leq h\left(\frac{1}{2} - 2p(1-p)\right) - h(2p(1-p)). \tag{41}$$

To apply our bound in Proposition 6, we first introduce the following flagged channel of the BB84 channel:

$$\widehat{\mathcal{B}}_{p,p}(\rho) = (1-p)^2\rho \otimes |\psi_\alpha\rangle\langle\psi_\alpha| + (p - p^2)X\rho X \otimes |0\rangle\langle 0| + (p - p^2)Z\rho Z \otimes |0\rangle\langle 0| + p^2 Y\rho Y \otimes |0\rangle\langle 0|, \tag{42}$$

where $|\psi_\alpha\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$. Then we could use brute-force search to optimize $\inf_{0\leq\alpha\leq 1}\gamma(\widehat{\mathcal{B}}_{p,p}, \alpha)$. In Figure. 7, we numerically implement our bound to the BB84 channel and compare our bound with previously best-known bounds.

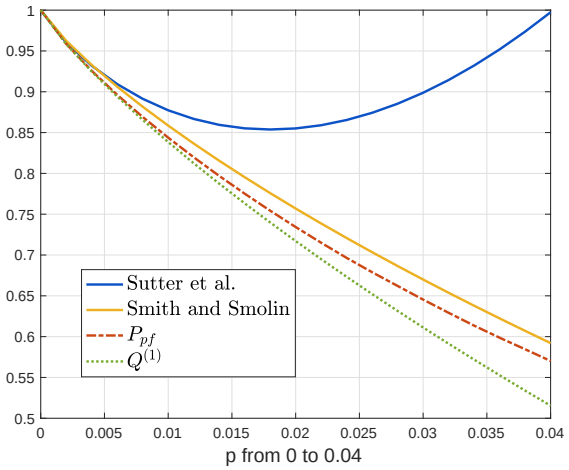


FIG. 7: Upper and lower bounds on the private capacity $P(\mathcal{B}_{p,p})$ of the low-noise BB84 channel with $p_X = p_Y = p \in [0, 0.04]$. The channel's coherent information $Q^{(1)}$ (green, dotted) provides a lower bound on the private capacity $P(\mathcal{B}_{p,p})$. The red dashed line depicts our bound P_{pf} in Proposition 6, which improves previously best known upper bounds via approximate degradable channels [19] and additive extensions [21].

IV. DISCUSSIONS

To summarize, we have seen that the concept of approximate degradable channels and the idea of flagged or extended channels can be combined to a more robust and general notion of approximate degradable flagged channels. Based on the optimization of approximate degradable flagged channels, our bounds have utilized the approximately preserved beneficial additivity properties of the approximate degradable channels as well as the flexibility or extra space provided by the flagged or extended channels. Our results have provided powerful and efficiently computable fundamental limits for quantum and private communication via quantum channels of theoretical and practical interest, including the depolarizing channel, the BB84 channel, and the generalized amplitude damping channel. In particular, our improved upper bounds on the quantum and private capacities of these channels have deepened our understanding of the ultimate limit of quantum communication. Note that codes for computing the bounds in this work can be found in [50].

One interesting future direction is to explore the interaction between flagged channels and the degradable and anti-degradable decomposition of channels [22], where the latter provides the state-of-art upper bounds on the quantum capacity of high-noise depolarizing channels. It is of great interest to have further analytical study on the depolarizing channel using our approach with a similar spirit of [8]. Another direction is to apply the main idea of this work to classical communication over quantum channels. For the classical capacity, there are several known upper bounds [16, 61–65] but the classical capacity of the amplitude damping channel is still open. It will be interesting to consider the interaction between extended channels and the approximate additivity of the Holevo information. The concept of approximate degradable flagged channels may also be useful in network information theory due to the nice features of degradable broadcast channels [66].

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