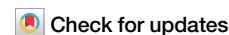


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Physical implementability for reversible magic state manipulation

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Magic states are essential for achieving universal quantum computation, yet the reversibility and efficiency of exact magic state manipulations remain less explored. Here we show that magic states in odd dimensions can be exactly and reversibly transformed under operations that preserve both the trace of states and positivity of discrete Wigner representation. Using the stochastic formalism, we demonstrate that magic mana emerges as the unique measure for such reversible magic state transformations. We propose the concept of physical implementability for characterizing the hardness and cost of maintaining reversibility. Our findings show that, analogous to the entanglement theory, going beyond the positivity constraint enables an exact reversible theory of magic manipulation. Physical implementability for quantum resource transformation provides a perspective for understanding and quantifying quantum resources, contributing to an operational framework for understanding the cost of reversible quantum resource manipulation.

Within the realm of quantum information science, the extent to which we can manipulate various quantum resources directly influences their practical utility. This challenge is often framed as the resource convertibility problem, focusing on the establishment of unique measures for transforming one resource state into another under operational restrictions. Drawing on the principles of thermodynamics¹, the exploration of quantum resource reversibility has become pivotal for enhancing the manipulation of quantum resources. This subject of study encompasses a range of pioneering efforts, from initial work that defines how transformations are characterized by local operations and classical communication in the realm of pure-state entanglement², to investigations into the handling of quantum coherence under strictly incoherent operations³. Recent efforts also expanded these conditions across diverse contexts^{4–9}.

Due to the gap in the generalized quantum Stein's lemma¹⁰, the reversibility of quantum entanglement and general quantum resources under asymptotically resource non-generating operations remains a fundamental open problem. Recent remarkable works by Hayashi and Yamasaki¹¹ and Lami¹² have proved the generalized quantum Stein's lemma, showing that the regularized relative entropy of resources fully captures the asymptotic convertibility between resource states using free operations, within errors that vanish asymptotically. This raises the question of whether the operations identified are unique, or if there exists another set of

operations that can also guarantee the reversibility of resources and lead to a single-letter resource measure.

It was shown that relaxing the trace-preserving⁹ or completely positive⁸ conditions of quantum operations can restore the reversible transformations of entanglement. Specifically, Regula and Lami⁹ demonstrate that it is possible to reversibly interconvert all states in general quantum resource theories, provided that probabilistic protocols are allowed. Meanwhile, Wang et al.⁸ show that entanglement non-generating maps enable reversible exact transformations of entangled states, leading to an exact characterization of reversible entanglement transformations governed by logarithmic negativity. Recent works^{13,14} have also considered generalized ways beyond quantum channels to virtually transform quantum resources, with a primary focus on shadow information or the measurement statistics of quantum resources. Within the framework of quantum resource theory, these theoretical developments facilitate the emergence of different insights into the problem of identifying which operational properties must be compromised to attain reversibility. Furthermore, such exploration could provide angles for quantifying and manipulating quantum resources.

The advancement of quantum information processing underscores the importance of manipulating quantum non-stabilizer resources, colloquially termed quantum magic resources, which is pivotal for fault-tolerant

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quantum computation (FTQC). The significance of quantum magic resources arises from the constraints of stabilizer circuits, where stabilizer circuits are composed of Clifford gates subject to efficient classical simulation according to the Gottesman-Knill Theorem¹⁵. To achieve universal quantum computation and unlock potential quantum advantage^{16,17}, it is necessary to incorporate and manipulate magic states. The resource theory of magic states^{18–34} plays a crucial role in quantum computing with applications in FTQC, classical simulation of quantum circuits, and gate synthesis, yet the reversibility and asymptotic transformation rates of these states remain less explored. Understanding the extent to which magic state manipulations can be reversed and the associated costs is of fundamental importance. Investigating the reversibility of magic state manipulation will not only deepen our comprehension of these essential quantum resources but also offer perspectives on quantifying the magic resources in quantum systems. Moreover, studying the magic state transformation rates will shed light on the scalability and efficiency of quantum computing protocols that rely heavily on these states.

In this paper, we introduce a reversible theory of exact magic manipulation, drawing a parallel between magic state manipulation and thermodynamics. We firstly show that magic can be manipulated exactly and reversibly if the allowable transformations preserve the positivity of the Wigner function^{35–37} in odd dimensions³⁸, referred to as PWP quasi-operations (PWPq operations). The key result is that magic mana precisely characterizes transformations of odd dimension magic states under PWPq operations (cf. Theorem 4), i.e.,

$$\rho \xrightarrow{\text{PWPq}} \sigma \iff \mathcal{M}(\rho) \geq \mathcal{M}(\sigma), \quad (1)$$

where $\mathcal{M}(\rho)$ represents the magic mana, defined as the logarithm of the absolute sum of the negative elements of the Wigner function¹⁸. This equation is particularly constructed via a representation of magic states called the stochastic Wigner representation. We further propose the concept of the physical implementability of PWPq operations. This quantity can offer a more detailed characterization of the hardness associated with maintaining reversibility between magic states even with identical magic mana. To the best of our knowledge, these results together constitute the first class of transformations that guarantee the exact reversibility of magic state manipulation with quantitative analysis of the physical costs.

Results and discussion

Stochastic formalism of magic state transformation

We firstly introduce the stochastic formalism of the Wigner representation and their corresponding properties. The idea of the stochastic formalism of Wigner representation has been employed to investigate the resource theory of quantum magic channels^{21,39}. Here, we delve deeper into additional properties of the formalism.

Definition 1. The *stochastic Wigner representation* of an odd dimensional operator $X \in \mathbb{C}^{d \times d}$ is defined as $W_X := \frac{1}{d} \sum_{\mathbf{u}} \text{Tr}[A_{\mathbf{u}} X] |\mathbf{u}\rangle$ for $\mathbf{u} \in \mathbb{Z}_d \times \mathbb{Z}_d$. Correspondingly, the *stochastic Wigner representation* of a linear map $\mathcal{N}_{A \rightarrow B}$ is defined as

$$W_{\mathcal{N}} := \frac{1}{d_B} \sum_{\mathbf{u}, \mathbf{v}} \text{Tr}[(A_{\mathbf{u}}^{\mathbf{u}})^T \otimes A_{\mathbf{v}}^{\mathbf{v}}] \mathcal{J}_{\mathcal{N}} |\mathbf{v}\rangle \langle \mathbf{u}|. \quad (2)$$

The following lemma characterizes the properties of the stochastic Wigner representation.

Lemma 1. Let $\mathcal{N}_{A \rightarrow B}$ be a linear map, and X be a matrix.

1. $W_{\mathcal{N}} W_X = W_{\mathcal{N}(X)}$.
2. W_X is real if and only if X is Hermitian.
3. $W_{\mathcal{N}}$ is real if and only if $\mathcal{N}$ is Hermitian-preserving.
4. X is positive semidefinite if and only if $\langle W_X, W_{|\psi\rangle\langle\psi|} \rangle \geq 0$ for every state $|\psi\rangle$.

Proof.

1. The result follows by the definition of stochastic Wigner representation and the properties of phase-space point operators (see Supplementary Note 2):

$$W_{\mathcal{N}} W_X \quad (3)$$

$$= \sum_{\mathbf{u}, \mathbf{v}} \mathcal{W}_{\mathcal{N}}(\mathbf{v}|\mathbf{u}) \mathcal{W}_X(\mathbf{u}|\mathbf{v}) \quad (4)$$

$$= \frac{1}{d_A d_B} \sum_{\mathbf{u}, \mathbf{v}} \text{Tr} \left[\left((A_{\mathbf{u}}^{\mathbf{u}})^T \otimes A_{\mathbf{v}}^{\mathbf{v}} \right) \mathcal{J}_{\mathcal{N}} \right] \text{Tr} [A_{\mathbf{u}}^{\mathbf{u}} X] |\mathbf{v}\rangle \quad (5)$$

$$= \frac{1}{d_B} \sum_{\mathbf{v}} \text{Tr} \left[A_{\mathbf{v}}^{\mathbf{v}} \cdot \mathcal{N} \left(\frac{1}{d_A} \sum_{\mathbf{u}} \text{Tr} [A_{\mathbf{u}}^{\mathbf{u}} X] A_{\mathbf{u}}^{\mathbf{u}} \right) \right] |\mathbf{v}\rangle \quad (6)$$

$$= \frac{1}{d_B} \sum_{\mathbf{v}} \text{Tr} [A_{\mathbf{v}}^{\mathbf{v}} \cdot \mathcal{N}(X)] |\mathbf{v}\rangle = W_{\mathcal{N}(X)}. \quad (7)$$

2. The statement trivially follows by the fact that phase-space point operators are Hermitian.
3. It suffices to prove the backward direction. Suppose $W_{\mathcal{N}}$ is real. Decompose the Choi representation of $\mathcal{N}$ as $\mathcal{J}_{\mathcal{N}} = H + S$ for $H, S = (\mathcal{J}_{\mathcal{N}} \pm \mathcal{J}_{\mathcal{N}}^{\dagger})/2$, respectively. Observe that H and iS are Hermitian. Then denote point $\mathbf{u}'$ such that $A_{\mathbf{u}'}^{\mathbf{u}'} = (A_{\mathbf{u}}^{\mathbf{u}})^T$. We have $\text{Tr}[(A_{\mathbf{u}}^{\mathbf{u}})^T \otimes A_{\mathbf{v}}^{\mathbf{v}}] \mathcal{J}_{\mathcal{N}} = \text{Tr}[A_{\mathbf{u}' \oplus \mathbf{v}} H] - i \text{Tr}[A_{\mathbf{u}' \oplus \mathbf{v}} (iS)]$. Since $A_{\mathbf{u}' \oplus \mathbf{v}}$ is Hermitian, $\text{Tr}[A_{\mathbf{u}' \oplus \mathbf{v}} H], \text{Tr}[A_{\mathbf{u}' \oplus \mathbf{v}} (iS)] \in \mathbb{R}$. Since $W_{\mathcal{N}}$ is real, $\text{Tr}[A_{\mathbf{u}' \oplus \mathbf{v}} (iS)] = 0 \Rightarrow iS = 0 \Rightarrow \mathcal{J}_{\mathcal{N}} = \mathcal{J}_{\mathcal{N}}^{\dagger}$. That is, $\mathcal{J}_{\mathcal{N}}$ is Hermitian and hence $\mathcal{N}$ is HP.
4. For every matrix Y ,

$$\text{Tr}[XY] = \text{Tr} \left[\sum_{\mathbf{u}, \mathbf{v}} \mathcal{W}_X(\mathbf{u}) \mathcal{W}_Y(\mathbf{v}) A_{\mathbf{u}} A_{\mathbf{v}} \right] \quad (8)$$

$$= d \sum_{\mathbf{u}} \mathcal{W}_X(\mathbf{u}) \mathcal{W}_Y(\mathbf{u}) = d W_X^{\dagger} W_Y. \quad (9)$$

Then the statement follows by the definition of positive semi-definite,

$$X \succcurlyeq 0 \iff \langle \psi | X | \psi \rangle = \text{Tr}[X |\psi\rangle\langle\psi|] \geq 0 \quad \forall |\psi\rangle \quad (10)$$

$$\iff \langle W_X, W_{|\psi\rangle\langle\psi|} \rangle \geq 0 \quad \forall |\psi\rangle. \quad (11)$$

■

Reversibility of exact magic manipulation

In this section, we venture beyond the confines of the quantum realm to thoroughly characterize the fundamental limits of magic transformation. We establish that the monotonicity of magic mana aligns with the existence of a set of non-physical operations, called the PWP quasi-operations (PWPq operations). As shown in Fig. 1, these operations are defined as the Hermitian-preserving and trace-preserving maps that preserves the positivity of their Wigner functions. The underlying intuition is that PWPq operations are tightly related with the stochastic transformation of quasi-probability distributions, as shown in the following statement.

Proposition 2. $\mathcal{N}_{A \rightarrow B}$ is a PWPq operation if and only if $W_{\mathcal{N}}$ is a column stochastic matrix.

Proof. Lemma 1 states that $W_{\mathcal{N}}$ is real if and only if $\mathcal{N}$ is Hermitian-preserving; Equation (2) implies that $W_{\mathcal{N}}$ is non-negative if and only if $\mathcal{N}$ is

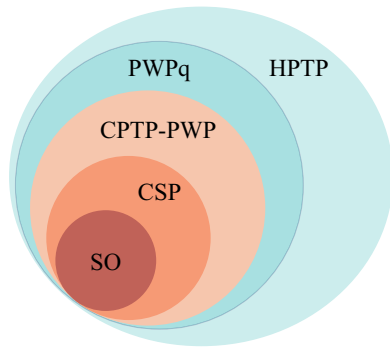


Fig. 1 | Schematic hierarchy of operation classes. The plot illustrates the inclusions among stabilizer operations (SO), completely stabilizer-preserving operations (CSP), completely positive, trace-preserving and positive-Wigner-preserving maps (CPTP-PWP), Hermitian-preserving, trace-preserving and positive-Wigner-preserving maps (PWPq), and Hermitian-preserving and trace-preserving maps (HPTP). The main focus of this paper is exact magic manipulation under PWPq operations.

positive-Wigner-preserving; one also observes that $\mathbf{1}^T W_{\mathcal{N}} = \mathbf{1}^T$ for all $\mathbf{u}$ if and only if $\mathcal{N}$ is trace-preserving, since

$$\text{Tr}[\mathcal{N}(A_A^u)] = \text{Tr}\left[\left((A_A^u)^T \otimes I_B\right)\mathcal{J}_{\mathcal{N}}\right] \quad (12)$$

$$= \sum_{\mathbf{v}} \frac{1}{d_B} \text{Tr}\left[\left((A_A^u)^T \otimes A_B^v\right)\mathcal{J}_{\mathcal{N}}\right] \quad (13)$$

$$= \sum_{\mathbf{v}} \langle \mathbf{v} | W_{\mathcal{N}} | \mathbf{u} \rangle = \mathbf{1}^T W_{\mathcal{N}} | \mathbf{u} \rangle \quad (14)$$

and $\{A_A^u\}_{\mathbf{u}}$ forms a basis of $\mathcal{L}(\mathcal{H}_A)$. We therefore conclude that $W_{\mathcal{N}}$ is real, non-negative and its columns sum to 1 if and only if $\mathcal{N}$ is HP, PWP and TP. In other words, $W_{\mathcal{N}}$ is a column stochastic matrix if and only if $\mathcal{N}$ is PWPq. ■

In the context of the Wigner representation, quantum states can be depicted as quasi-probability distributions, which allows for the possibility of stochastic manipulation through PWPq operations. Leveraging this property, we demonstrate that a quantum state ρ can be transformed to σ under PWPq operations if and only if the mana of ρ exceeds the magic mana of the target state σ . This finding establishes a parallel in characterizing magic manipulation to the role of logarithmic negativity in characterizing entanglement manipulation within entanglement theory^{8,40,41}. The necessary and sufficient condition for this transformation are detailed in the following theorem 3.

Theorem 3. For two odd dimensional quantum states, there exists PWPq operation $\mathcal{N}$ such that $\mathcal{N}(\rho) = \sigma$ if and only if

$$\mathcal{M}(\rho) \geq \mathcal{M}(\sigma), \quad (15)$$

where $\mathcal{M}(\cdot)$ is the mana of the state.

Proof. Lemma 1 and Proposition 2 states that $\mathcal{N}(\rho) = \sigma$ for a HPWP-PWP operation $\mathcal{N}$ if and only if $W_{\mathcal{N}} W_{\rho} = W_{\sigma}$ for a column stochastic matrix $W_{\mathcal{N}}$. For simplicity, denote

$$W = W_{\mathcal{N}} = \begin{bmatrix} w_1^T \\ \vdots \\ w_{d'}^T \end{bmatrix} \in \mathbb{R}^{d \times d'}, \quad (16)$$

$$\mathbf{p} = W_{\rho} = \begin{bmatrix} p_1 \\ \vdots \\ p_{d'} \end{bmatrix} \in \mathbb{R}^{d'}, \text{ and } \mathbf{q} = W_{\sigma} \in \mathbb{R}^d. \quad (17)$$

Then one can translate the existence of such $\mathcal{N}$ mapping ρ to σ as the existence of $\mathbf{w} \in \mathbb{R}^{dd'}$ satisfying

$$\begin{bmatrix} p_1 I & p_2 I & \cdots & p_{d'} I \\ \mathbf{1}^T & 0 & \cdots & 0 \\ 0 & \mathbf{1}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{d'} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad (18)$$

or in short $A\mathbf{w} = \mathbf{b}$, where I is the d -dimensional identity matrix and $\mathbf{1}$ is the vector of d ones. From Farkas' Lemma it follows that Equation (18) holds if and only if for all $\mathbf{r} \in \mathbb{R}^d$ and $\mathbf{t} \in \mathbb{R}^{d'}$, $A^T \begin{bmatrix} \mathbf{r} \\ -\mathbf{t} \end{bmatrix} \geq 0 \Rightarrow \mathbf{b}^T \begin{bmatrix} \mathbf{r} \\ -\mathbf{t} \end{bmatrix} \geq 0$, which is simplified as

$$\text{each } t_j \leq \min_{1 \leq k \leq d} p_j r_k \Rightarrow \sum_{j=1}^{d'} t_j \leq \mathbf{r}^T \mathbf{q}, \quad (19)$$

when substituting A^T and $\mathbf{b}$. Since $\mathbf{t}$ is free, by pushing each t_j towards its upper bound, above equation is equivalent to

$$\forall \mathbf{r} \in \mathbb{R}^d, \sum_{j=1}^{d'} \min_{1 \leq k \leq d} p_j r_k \leq \mathbf{r}^T \mathbf{q}. \quad (20)$$

Considering the left hand equals to $p_+ r_{\min} + (1 - p_+) r_{\max}$ only depending on p_+ , $r_{\min}$ and $r_{\max}$, where p_+ , $1 - p_+$ are the sum of the positive and negative components of $\mathbf{p}$, respectively. We could rewrite such equation as for all $r_{\min} \leq r_{\max}$

$$p_+ r_{\min} + (1 - p_+) r_{\max} \leq \mathbf{r}^T \mathbf{q}, \quad \mathbf{r} \in [r_{\min}, r_{\max}]^d, \quad (21)$$

or

$$p_+ r_{\min} + (1 - p_+) r_{\max} \leq \min_{\mathbf{r} \in [r_{\min}, r_{\max}]^d} \mathbf{r}^T \mathbf{q} \quad (22)$$

$$= q_+ r_{\min} + (1 - q_+) r_{\max}, \quad (23)$$

which is $(r_{\max} - r_{\min})(p_+ - q_+) \geq 0$ and is equivalent to

$$p_+ \geq q_+, \text{ also } \|W_{\rho}\|_1 \geq \|W_{\sigma}\|_1, \quad (24)$$

where q_+ , $1 - q_+$ are the sum of the positive and negative components of $\mathbf{q}$, respectively. Recall that $\mathcal{M}(\rho) = \log \|W_{\rho}\|_1$, $\mathcal{M}(\sigma) = \log \|W_{\sigma}\|_1$, and Equation (18) gives the existence of $\mathcal{N}$. One can therefore conclude that ρ can be transformed to σ under PWPq operations if and only if $\mathcal{M}(\rho) \geq \mathcal{M}(\sigma)$. ■

We also provide an alternative proof of Theorem 3 with matrix manipulations only. We use the same notation in Equation (16), and further assume that $\mathbf{p} = \begin{bmatrix} \mathbf{p}^+ \\ -\mathbf{p}^- \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} \mathbf{q}^+ \\ -\mathbf{q}^- \end{bmatrix}$ for $\mathbf{p}^+, \mathbf{q}^+ \geq 0$, $\mathbf{p}^-, \mathbf{q}^- > 0$. Such assumption is without loss of generality since permutation transformations are doubly stochastic (i.e., column stochastic and row stochastic).

Proof. ($\Rightarrow$) $W\mathbf{p} = \mathbf{q}$ implies $W \begin{bmatrix} \mathbf{p}^+ \\ \mathbf{0} \end{bmatrix} \geq \begin{bmatrix} \mathbf{q}^+ \\ -\mathbf{q}^- \end{bmatrix}$ and $W \begin{bmatrix} \mathbf{0} \\ \mathbf{p}^- \end{bmatrix} \geq \begin{bmatrix} -\mathbf{q}^+ \\ \mathbf{q}^- \end{bmatrix}$. Since $W, \mathbf{p}^+, \mathbf{p}^-$ have non-negative entries, $W \begin{bmatrix} \mathbf{p}^+ \\ \mathbf{0} \end{bmatrix}, W \begin{bmatrix} \mathbf{0} \\ \mathbf{p}^- \end{bmatrix} \geq 0$ and hence

$$\|\mathbf{p}^+\|_1 = \mathbf{I}^T \begin{bmatrix} \mathbf{p}^+ \\ \mathbf{0} \end{bmatrix} = \mathbf{I}^T W \begin{bmatrix} \mathbf{p}^+ \\ \mathbf{0} \end{bmatrix} \quad (25)$$

$$\geq \mathbf{I}^T \begin{bmatrix} \mathbf{q}^+ \\ \mathbf{0} \end{bmatrix} = \|\mathbf{q}^+\|_1. \quad (26)$$

Similar idea holds for proving $\|\mathbf{p}^-\|_1 \geq \|\mathbf{q}^-\|_1$. Then the result follows as $\mathbf{p}$ and $\mathbf{q}$ sum to one.

($\Leftarrow$) Denote $\mathbf{q}^- \in \mathbb{R}^l$. To start with, suppose $l \geq 2$. Then denote $\hat{\mathbf{q}}^-$ as the vector of the first $(l-1)$ entries of $\mathbf{q}^-$ so that

$$\mathbf{q}^- = \begin{bmatrix} \hat{\mathbf{q}}^- \\ \mathbf{I}^T(\mathbf{q}^- - \hat{\mathbf{q}}^-) \end{bmatrix}, \mathbf{q} = \begin{bmatrix} \mathbf{q}^+ \\ -\hat{\mathbf{q}}^- \\ 1 - \mathbf{I}^T(\mathbf{q}^+ - \hat{\mathbf{q}}^-) \end{bmatrix}. \quad (27)$$

Since $\|\mathbf{p}\|_1 \geq \|\mathbf{q}\|_1$ and $\|\mathbf{p}^+\|_1 - \|\mathbf{p}^-\|_1 = \|\mathbf{q}^+\|_1 - \|\mathbf{q}^-\|_1$, $\|\mathbf{p}^+\|_1 \geq \|\mathbf{q}^+\|_1$. The desired W can then be constructed as follows:

$$W = \begin{bmatrix} W^+ & 0 \\ 0 & W^- \\ \mathbf{c}^{+T} & \mathbf{c}^{-T} \end{bmatrix} \quad (28)$$

$$\text{with } W^+ = \frac{\mathbf{q}^+ \mathbf{I}^T}{\|\mathbf{p}^+\|_1}, W^- = \frac{\hat{\mathbf{q}}^- \mathbf{I}^T}{\|\mathbf{p}^-\|_1}, \text{ and} \quad (29)$$

$$\mathbf{c}^+ = \left(1 - \frac{\|\mathbf{q}^+\|_1}{\|\mathbf{p}^+\|_1}\right) \mathbf{I}, \mathbf{c}^- = \left(1 - \frac{\|\hat{\mathbf{q}}^-\|_1}{\|\mathbf{p}^-\|_1}\right) \mathbf{I}. \quad (30)$$

From above construction, observe that W is column stochastic and

$$\mathbf{c}^{+T} \mathbf{p}^+ - \mathbf{c}^{-T} \mathbf{p}^- \quad (31)$$

$$= (\|\mathbf{p}^+\|_1 - \|\mathbf{q}^+\|_1) - (\|\mathbf{p}^-\|_1 - \|\hat{\mathbf{q}}^-\|_1) \quad (32)$$

$$= 1 - (\|\mathbf{q}^+\|_1 - \|\hat{\mathbf{q}}^-\|_1) = 1 - \mathbf{I}^T(\mathbf{q}^+ - \hat{\mathbf{q}}^-). \quad (33)$$

Consequently, we have

$$W\mathbf{p} = \begin{bmatrix} W^+ & 0 \\ 0 & W^- \\ \mathbf{c}^{+T} & \mathbf{c}^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{p}^+ \\ -\mathbf{p}^- \end{bmatrix} = \begin{bmatrix} W^+ \mathbf{p}^+ \\ -W^- \mathbf{p}^- \\ \mathbf{c}^{+T} \mathbf{p}^+ - \mathbf{c}^{-T} \mathbf{p}^- \end{bmatrix} \quad (34)$$

$$= \begin{bmatrix} \mathbf{q}^+ \\ -\hat{\mathbf{q}}^- \\ 1 - \mathbf{I}^T(\mathbf{q}^+ - \hat{\mathbf{q}}^-) \end{bmatrix} = \mathbf{q}, \quad (35)$$

as required. In the case when $l = 0$, both $\mathbf{p}$ and $\mathbf{q}$ have no negative components and hence $W = W^+$; when $l = 1$ i.e., $\mathbf{q}$ has only one negative entry, one do not need to consider W^- and hence $W = \begin{bmatrix} W^+ & 0 \\ \mathbf{c}^{+T} & \mathbf{c}^{-T} \end{bmatrix}$. ■

To achieve universal quantum computation and unlock potential quantum advantages, it is essential to incorporate and manipulate a large number of identically and independently distributed copies of magic states.

Consequently, understanding the transformation rate associated with converting between different magic states becomes crucial. Building on Theorem 3, we here provide the asymptotic exact magic transformation rate between magic states and establish the reversibility of these transformations under PWPq operations in the asymptotic regime. Rigorous definition of exact transformation rate is given in Supplementary Note 1.

Theorem 4. For any two odd dimensional states ρ and σ , the asymptotic exact magic transformation rate is given by

$$R_{\text{PWPq}}(\rho \rightarrow \sigma) = \frac{\mathcal{M}(\sigma)}{\mathcal{M}(\rho)}, \quad (36)$$

which implies the reversibility of asymptotic exact magic state manipulation under PWPq operations, i.e.,

$$R_{\Omega}(\rho \rightarrow \sigma) \times R_{\Omega}(\sigma \rightarrow \rho) = 1. \quad (37)$$

Theorem 4 can be intuitively understood if we consider a scenario where m copies of a quantum state ρ and n copies of another state σ are interchangeable under PWPq operations. In this case they own equal amounts of magic mana by Theorem 3, and hence the precise exchange rate between ρ and σ is determined by the ratio $m/n = \mathcal{M}(\rho)/\mathcal{M}(\sigma)$. From this standpoint, Theorem 4 establishes that the resource cost of preparing a magic state is balanced by the resource obtain from consuming it, with the exchange rate directly linked to the magic mana of the states involved.

Physical Implementability of PWPq operations

Theorem 3 posits that under PWPq operations, quantum states with identical magic mana can undergo reversible transformations amongst themselves. For instance, consider the Strange state $|\mathbb{S}\rangle := (|1\rangle - |2\rangle)/\sqrt{2}$ and the Norrell state $|\mathbb{N}\rangle := (-|0\rangle + 2|1\rangle - |2\rangle)/\sqrt{6}$, which have the same magic mana. According to Theorem 3, these states can be transformed reversibly into one another. However, the theorem does not specify which state possesses more “magic”, nor does it address the comparative difficulty of transforming between these states. This leads to the pivotal question: how can we quantify the difficulty of transferring between quantum states when they exhibit the same magic mana? Addressing this question necessitates a more precise metric to establish a hierarchy of magic among states with identical magic mana.

To answer this question, we introduce a metric that quantifies the difficulty associated with transferring magic states under PWPq operations. According to the golden rule of quantum resource theory that each resource theory comes with a set of free states $\mathcal{F}$ and a set of free operations $\mathcal{O}$, and the minimum requirement for free operations is that they should not create resourceful states out of free states⁴² and in the similar spirit of^{43–45}, we define this quantity as follows,

Definition 2. For two quantum states ρ, σ , the *physical implementability of magic manipulations* from ρ to σ is,

$$\nu(\rho \rightarrow \sigma) = \log \min \left\{ \sum_j |c_j| \mid \mathcal{N} = \sum_j c_j \mathcal{N}_j, \sum_j c_j = 1, W_{\mathcal{N}}, W_{\mathcal{N}_j} \geq 0, \mathcal{N}_j \in \text{CPTP} \right\}. \quad (38)$$

One also considers the transformation error of the magic manipulation among these states, such that the transformation fidelity $F(\mathcal{N}(\rho), \sigma) = \text{Tr} \sqrt{\mathcal{N}(\rho)^{1/2} \sigma \mathcal{N}(\rho)^{1/2}}$ is no less than $1 - \varepsilon$. The physical implementability of ε -error state convertibility can be determined via the following SDP.

$$\min_{c, \mathcal{J}_{\mathcal{N}_{\pm}}, \Lambda} 2c - 1 \quad (39a)$$

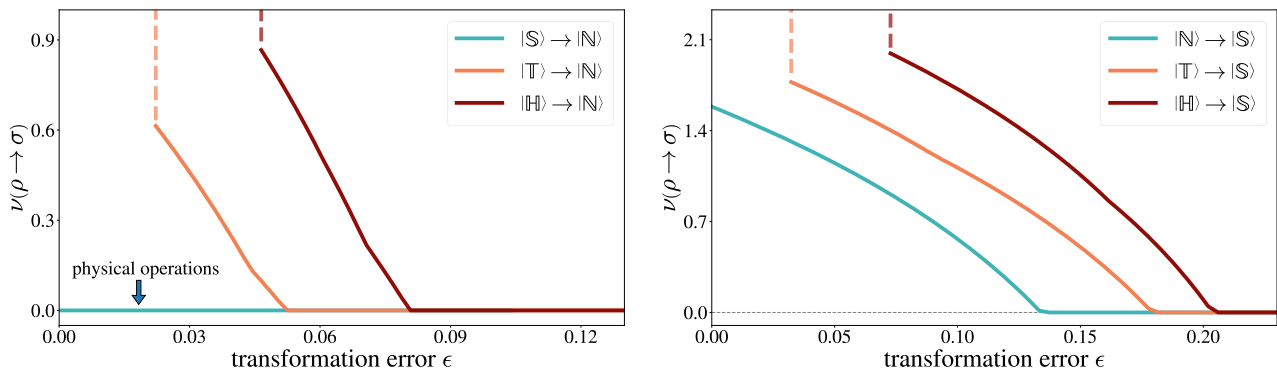


Fig. 2 | Physical implementability of state transformations with error. The states presented in the figure are $|\mathbb{S}\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$, $|\mathbb{N}\rangle = (-|0\rangle + 2|1\rangle - |2\rangle)/\sqrt{6}$, $|\mathbb{T}\rangle = (e^{2\pi i/9}|0\rangle + |1\rangle + e^{-2\pi i/9}|2\rangle)/\sqrt{3}$ and $|\mathbb{H}\rangle$ is the eigenstate of QFT_3 w.r.t. eigenvalue $+1$. Here ϵ is the allowed transformation error, and $\nu(\rho \rightarrow \sigma)$

quantifies the difficulty of physically implementing magic state conversions from ρ to σ . When $\nu > 0$, such conversion requires non-physical operations. The dashed lines indicate the infeasibility of performing such conversions, i.e., $\nu \rightarrow \infty$.

$$\text{s.t. } \mathcal{J}_{\mathcal{N}} = \mathcal{J}_{\mathcal{N}_+} - \mathcal{J}_{\mathcal{N}_-}, \mathcal{J}_{\mathcal{N}_\pm} \succcurlyeq 0 \quad (39b)$$

$$\text{Tr}_2 \mathcal{J}_{\mathcal{N}_+} = cI, \text{Tr}_2 \mathcal{J}_{\mathcal{N}_-} = (c-1)I \quad (39c)$$

$$W_{\mathcal{N}_\pm}, W_{\mathcal{N}} \geq 0 \quad (39d)$$

$$2 - \text{Tr}(\Lambda + \Lambda^\dagger) \leq 2\epsilon \quad (39e)$$

$$\begin{pmatrix} \text{Tr}_2[(\rho^T \otimes I)\mathcal{J}_{\mathcal{N}}] & \Lambda \\ \Lambda^\dagger & \sigma \end{pmatrix} \succcurlyeq 0 \quad (39f)$$

Note that the output of this SDP evaluates to 2^ν . Also, when $\nu \geq 0$, there exists a PWPq operation to transform states and $\nu = 0$ if and only if the operations are CPTP-PWP. To achieve an estimation error within an error ϵ with a probability no less than $1 - \delta$, the number of total sampling times S can be estimated by Hoeffding's inequality as $S \geq 2^{2\nu+1} \log(2/\delta)/\epsilon^2$. Therefore, ν directly quantifies the hardness of quantum state convertibility statistically. Technical details of above claims are deferred to Supplementary Note 3.

To illustrate the hardness of implementing PWPq operations physically, we consider the transformations between pairs of $|\mathbb{S}\rangle$, $|\mathbb{N}\rangle$ and other two well-known magic states,

$$|\mathbb{T}\rangle := (e^{2\pi i/9}|0\rangle + |1\rangle + e^{-2\pi i/9}|2\rangle)/\sqrt{3}, \quad (40)$$

$$|\mathbb{H}\rangle \text{ is the eigenstate of } \text{QFT}_3 \text{ w.r.t. eigenvalue } +1, \quad (41)$$

where QFT_3 is the quantum Fourier transform matrix of dimension 3.

Figure 2 demonstrates that state transformations between $|\mathbb{S}\rangle$ and $|\mathbb{N}\rangle$ via PWPq operations are feasible. The process is entirely physical when converting from $|\mathbb{S}\rangle$ to $|\mathbb{N}\rangle$ but necessitates extra physical resources when reversing the direction. Conversely, states $|\mathbb{T}\rangle$ and $|\mathbb{H}\rangle$ not only require additional resources but also incur transformation errors in their conversion to $|\mathbb{N}\rangle$ or $|\mathbb{S}\rangle$. Also, since the magic mana of the two states can be quantitatively ordered as $\mathcal{M}(|\mathbb{S}\rangle) = \mathcal{M}(|\mathbb{N}\rangle) > \mathcal{M}(|\mathbb{T}\rangle) > \mathcal{M}(|\mathbb{H}\rangle)$, these empirical observations corroborate the results detailed in Theorem 3. Despite $|\mathbb{N}\rangle$ and $|\mathbb{S}\rangle$ have the same magic mana, the reversibility between $|\mathbb{N}\rangle$ and $|\mathbb{S}\rangle$ is still physically hard to implement as suggested by $\nu(|\mathbb{N}\rangle \rightarrow |\mathbb{S}\rangle)$.

Overall, physical implementability suggests that the hardness of transforming these four states can be ranked as $|\mathbb{S}\rangle, |\mathbb{N}\rangle, |\mathbb{T}\rangle, |\mathbb{H}\rangle$. This result can be extended to a finite number of resource states by initially comparing the magic mana of these states. Then for each pair of states with identical mana, we compare their physical implementability. This approach

can heuristically suggest the desired hierarchy among the states. Furthermore, our numerical experiments indicate the transformation of certain magic states requires non-physical operations, implying the reversibility of magic state manipulation to be stemmed from the non-physicality of the operations involved.

We also note that our definition of the physical implementability of PWPq operations deviates from one previous work¹³, where the HPTP map is allowed to generate resources. Specifically, their optimization program focuses on identifying the minimal sampling overhead and resource requirements necessary for quantum state convertibility. However, we impose a restriction requiring the map to be resource non-generating in our study. This adheres to the golden rules of quantum resource theory and aids in determining the additional costs associated with converting quantum states under PWPq operations, even when the two quantum states possess identical magic mana.

Conclusions

Our work demonstrates the reversibility of exact magic state manipulation under PWPq operations. We introduce the stochastic formalism of magic state transformations, implying that these relaxed operations are essentially stochastic transitions among quasi-probability distributions. This finding allows us to identify the setting of magic transformations as one that is completely governed by a unique resource measure - magic mana, thus mirroring the parallels between the resource theories of magic states and thermodynamics. We further propose the physical implementability of magic transformation to analyze the physical cost of guaranteeing such reversibility. This quantity can offer a deeper characterization of the hardness associated with maintaining reversibility between magic states even with the identical magic mana.

The problem of investigating reversibility in quantum resource theory typically involves completely positive, trace preserving and resource non-generating operations^{4,46}, with tasks are generally framed as exact transformations or transformations with asymptotically vanishing error. However, analyzing reversibility under these strict conditions poses significant challenges. Previous work done by Bartosz and Lami⁹ demonstrates that the reversibility of general quantum resources can be recovered by expanding the set of operations into probabilistic protocols with non-vanishing probability of success. Related works on QSL^{11,12} explore the relaxation of the resource non-generating condition to asymptotic resource non-generating operations, establishing the existence of resource transformation reversibility. In the similar spirit, our work shows that relaxing the positivity constraint can enable exact reversibility in magic state manipulation. Taking together, all these findings shed light on studying the quantum resource manipulation, specifically in understanding the interplay between reversibility, success probability and positivity of allowed transformations of quantum resources.

To the best of our knowledge, our reversible theory of exact magic manipulation provides new guidance on finding the smallest subset of quantum operations that can guarantee the reversibility of asymptotic magic manipulation. It also contributes to the broader comprehension of the fundamental problem of general quantum resource transformation. The physical implementability for exact reversible magic transformations also provides an angle for understanding and exploring the quantification and manipulation of quantum resources. The stochastic formalism has been extended to qubits by specifically focusing on rebits, as outlined in recent studies^{47,48}. Given this development, extending our results to encompass qubit fault-tolerant quantum computation represents a promising and potentially fruitful direction for future research.

Methods

We use CVX^{49,50}, a package for specifying and solving convex programs in MATLAB, to perform numerical experiments on physical implementability computation.

Data availability

Original data that support the plots and other findings of this study is available from the corresponding authors upon reasonable request.

Code availability

Code used in the numerical experiments on physical implementability computation is available at GitHub⁵¹. Other code used in this study is available from the corresponding authors upon reasonable request.

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The authors are listed in alphabetical order. Yu-Ao Chen, Gilad Gour, Xin Wang, Lei Zhang, and Chenghong Zhu contributed equally to this work. Xin

Wang formulated the initial idea and the framework. All co-authors contributed to the preparation of the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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