

Robustness to Incorrect Models in Average-Cost Optimal Stochastic Control *

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Abstract

We study continuity properties of infinite-horizon average expected cost problems with respect to (controlled) transition kernels, and applications of these results to the problem of robustness of control policies designed for incorrect models applied to systems with incomplete models. We show that sufficient conditions presented in the literature for discounted-cost problems are in general not sufficient to ensure robustness for average-cost problems. However, we show that the average optimal cost is continuous in the convergences of controlled transition kernel models where convergence of models entails (i) continuous weak convergence in state and actions, and (ii) continuous setwise convergence in the actions for every fixed state variable, in addition to uniform ergodicity and regularity conditions. Using such continuity results, we establish that the mismatch error due to the application of a control policy designed for an incorrectly estimated model to the true model decreases to zero as the incorrect model approaches the true model under the stated convergence criteria. Our findings significantly relax related studies in the literature which have primarily considered the more restrictive total variation convergence criteria.

1 Introduction

1.1 Preliminaries

The paper studies continuity and robustness properties of infinite-horizon average cost problems with respect to transition probabilities. Continuity results are used

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to establish robustness of optimal control policies applied to systems with incorrect models. Before proceeding further with the problem formulation, we present the setup. Let $\mathbb{X} \subset \mathbb{R}^n$ be a Borel set in which the elements of a controlled Markov process $\{X_t, t \in \mathbb{Z}_+\}$ take values. Here and throughout the paper, $\mathbb{Z}_+$ denotes the set of non-negative integers and $\mathbb{N}$ denotes the set of positive integers. Let $\mathbb{U}$, the action space, be a Borel subset of some Euclidean space. An *admissible policy* γ is a sequence of control functions $\{\gamma_t, t \in \mathbb{Z}_+\}$, such that γ_t is measurable on the σ -algebra generated by the information variables

$$I_t = \{X_{[0,t]}, U_{[0,t-1]}\}, \quad t \in \mathbb{N}, \quad I_0 = \{X_0\},$$

where

$$U_t = \gamma_t(I_t), \quad t \in \mathbb{Z}_+, \quad (1)$$

are the $\mathbb{U}$ -valued control actions.

We define Γ to be the set of all such admissible policies.

The joint distribution of the state and control processes is determined by (1) and the following relationship:

$$\begin{aligned} & \Pr\left(X_t \in B \mid (X, U)_{[0,t-1]} = (x, u)_{[0,t-1]}\right) \\ &= \int_B \mathcal{T}(dx_t | x_{t-1}, u_{t-1}), B \in \mathcal{B}(\mathbb{X}), t \in \mathbb{N}, \end{aligned}$$

where $\mathcal{T}(\cdot | x, u)$ is a stochastic kernel (that is, a regular conditional probability measure) from $\mathbb{X} \times \mathbb{U}$ to $\mathbb{X}$.

The objective of the controller is to minimize the infinite-horizon average expected cost

$$J_\infty(\mathcal{T}, \gamma) = \limsup_{N \rightarrow \infty} \frac{1}{N} E_{x_0}^{\mathcal{T}, \gamma} \left[\sum_{t=0}^{N-1} c(X_t, U_t) \right]$$

over the set of admissible policies $\gamma \in \Gamma$, where $c : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$ is the stage-wise (Borel measurable) cost function and $E_{x_0}^{\mathcal{T}, \gamma}$ denotes the expectation with initial state x_0 and transition kernel $\mathcal{T}$ under policy γ .

To denote the explicit dependence of the optimal cost in the transition kernel, we use the notation

$$J_\infty^*(\mathcal{T}) = \inf_{\gamma \in \Gamma} J_\infty(\mathcal{T}, \gamma).$$

The focus of the paper is to address the following problems:

Problem P1: Continuity of $J_\infty^*(\mathcal{T})$ (optimal cost for controlled setup) in transition kernels. Suppose $\{\mathcal{T}_n, n \in \mathbb{N}\}$ is a sequence of transition kernels converging in some sense to $\mathcal{T}$. When does $\mathcal{T}_n \rightarrow \mathcal{T}$ imply

$$J_\infty^*(\mathcal{T}_n) \rightarrow J_\infty^*(\mathcal{T})?$$

Problem P2: Robustness of policies designed for incorrectly estimated models. Suppose $\{\mathcal{T}_n, n \in \mathbb{N}\}$ is a sequence of transition kernels converging in some sense to $\mathcal{T}$. If we design optimal policies γ_n^* for the estimated model $\mathcal{T}_n$ and apply them to the real model $\mathcal{T}$, when does $\mathcal{T}_n \rightarrow \mathcal{T}$ imply

$$J_\infty(\mathcal{T}, \gamma_n^*) \rightarrow J_\infty^*(\mathcal{T})?$$

1.2 Literature Review and Contributions

Robustness is a desired property for the optimal control of stochastic or deterministic systems when a given model does not reflect the actual system perfectly, as is usually the case in practice. This is a classical problem, so there is a very large literature on robust stochastic control and its application to learning-theoretic methods; see e.g. [11, 4, 17, 18, 6, 5, 22, 14, 2, 23]. A rather comprehensive literature review is presented in [13], and we refer the reader to the literature review therein.

In [12, 13], we studied continuity and robustness properties of partially and fully observed models under weak and total variation convergence of transition probabilities for infinite-horizon optimal stochastic control under discounted-cost criteria. We showed that the expected induced cost is robust under total variation in that it is continuous in the mismatch of transition kernels under convergence in total variation for the discounted cost setup. By imposing further assumptions on the measurement models and on the kernel itself, it was shown that the optimal discounted cost can be made continuous under weak convergence of transition kernels as well.

For the expected average-cost criteria, in this paper, we show that the sufficient conditions presented in [12, 13] for the expected discounted cost criteria may not guarantee continuity and robustness. This arises from the fact that the persistent errors between the transition probabilities for time stages in the distant future still matter unlike the discounted-cost setup, where such errors are discounted away.

A particularly relevant study which has investigated the robustness problem under the expected average-cost criterion is [9] by Hernandez-Lerma: In [9, Chapter 3], a related problem to what we study in this paper is considered: Assume the transition probabilities are estimated through time, and at every time step t the estimate is updated to some $\mathcal{T}_t$. Is it true that, as the estimates through time get closer to the true model (as $\mathcal{T}_t \rightarrow \mathcal{T}$) we can establish continuity and robustness?

Contributions. In view of the literature review reported above, in this paper we establish robustness and continuity results under much weaker conditions on the proximity and convergence properties between incorrect models and a true model: we show that the average optimal cost is continuous in the convergences of controlled transition kernel models where convergences of models entail (i) continuous weak convergence in state and actions, or (ii) continuous setwise convergence in the actions for every fixed state variable, in addition to ergodicity conditions. We show that the mismatch error due to the application of a control policy designed for an incorrectly estimated model to the true model decreases to zero as the incorrect model approaches the true model under the stated convergence criteria. In particular, compared to the results in [9] by Hernandez-Lerma, where a uniform total variation convergence over state and control action variables of the transition kernels is provided as a sufficient condition, we show that the uniform convergence on the state variable may be relaxed under the total variation convergence, and more importantly, we establish the continuity and robustness under the weak convergence and setwise convergences of the transition kernels which are typically much more relaxed notions of convergence than the total variation convergence as we discuss further in Section 2. As noted above, for the expected average-cost criteria, the conditions presented in [12, 13] for the expected discounted cost criteria do not guarantee continuity and robustness, and new techniques are needed to investigate the average-cost setup, which this paper presents.

2 Some Examples and Convergence Criteria for Transition Kernels

2.1 Convergence criteria for transition kernels

Before presenting convergence criteria for controlled transition kernels, we first review convergence of probability measures. Three important notions of convergences for sets of probability measures to be studied in the paper are weak convergence, setwise convergence and convergence under total variation. For $N \in \mathbb{N}$, a sequence $\{\mu_n, n \in \mathbb{N}\}$ in $\mathcal{P}(\mathbb{R}^N)$ is said to converge to $\mu \in \mathcal{P}(\mathbb{R}^N)$ *weakly* if

$$\int_{\mathbb{R}^N} c(x) \mu_n(dx) \rightarrow \int_{\mathbb{R}^N} c(x) \mu(dx) \quad (2)$$

for every continuous and bounded $c : \mathbb{R}^N \rightarrow \mathbb{R}$. $\{\mu_n\}$ is said to converge *setwise* to $\mu \in \mathcal{P}(\mathbb{R}^N)$ if (2) holds for all measurable and bounded $c : \mathbb{R}^N \rightarrow \mathbb{R}$.

For probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$, the *total variation* metric is given by

$$\|\mu - \nu\|_{TV} = \sup_{f: \|f\|_\infty \leq 1} \left| \int f(x) \mu(dx) - \int f(x) \nu(dx) \right|,$$

where the supremum is taken over all measurable real-valued f , such that $\|f\|_\infty = \sup_{x \in \mathbb{R}^N} |f(x)| \leq 1$. A sequence $\{\mu_n\}$ is said to converge in total variation to $\mu \in \mathcal{P}(\mathbb{R}^N)$ if $\|\mu_n - \mu\|_{TV} \rightarrow 0$. where the supremum is taken over all measurable real f such that $\|f\|_\infty = \sup_{x \in \mathbb{R}^N} |f(x)| \leq 1$. A sequence $\{\mu_n\}$ is said to converge in total variation to $\mu \in \mathcal{P}(\mathbb{R}^N)$ if $\|\mu_n - \mu\|_{TV} \rightarrow 0$. Total variation defines a stringent metric for convergence; for example, a sequence of discrete probability measures does not converge in total variation to a probability measure which admits a density function. Setwise convergence, though, induces a topology on the space of probability measures which is not metrizable [7, p. 59]. However, the space of probability measures on a complete, separable, metric (Polish) space endowed with the topology of weak convergence is itself complete, separable and metric [16]. Building on the above, we introduce the following convergence notions for (controlled) transition kernels.

Definition 1. For a sequence of transition kernels $\{\mathcal{T}_n, n \in \mathbb{N}\}$, we say that

1. $\mathcal{T}_n \rightarrow \mathcal{T}$ weakly if $\mathcal{T}_n(\cdot|x, u) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly, for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$.
2. $\mathcal{T}_n \rightarrow \mathcal{T}$ setwise if $\mathcal{T}_n(\cdot|x, u) \rightarrow \mathcal{T}(\cdot|x, u)$ setwise, for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$.
3. $\mathcal{T}_n \rightarrow \mathcal{T}$ under the total variation distance if $\mathcal{T}_n(\cdot|x, u) \rightarrow \mathcal{T}(\cdot|x, u)$ under total variation, for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$.

We also note here that relative entropy convergence, through Pinsker's inequality [8, Lemma 5.2.8], is stronger than even total variation convergence which has also been studied in robust stochastic control as reviewed earlier. Another metric for probability measures is the Wasserstein distance: For compact spaces, the Wasserstein distance of order 1, denoted by W_1 , metrizes the weak topology (see [24, Theorem 6.9]). For non-compact spaces convergence in the W_1 metric implies weak convergence (in particular this metric bounds from above the Bounded-Lipschitz metric [24, p.109]). Considering these relations, our results in this paper can be directly generalized to the relative entropy distance or the Wasserstein distance.

2.2 Examples

In the following we give some examples to show what these convergence types mean on the functional representation of the dynamics. The examples are taken

from [13] and presented here for completeness. For a more detailed discussion on the examples, the reader can refer to [13, Section 2.2].

Let a controlled model be given as

$$x_{t+1} = F(x_t, u_t, w_t),$$

where $\{w_t\}$ is an i.i.d. noise process. The uncertainty on the transition kernel for such a system may arise from lack of information on F or the i.i.d. noise process w_t or both:

- (i) Let $\{F_n\}$ denote an approximating sequence for F , so that $F_n(x, u, w) \rightarrow F(x, u, w)$ pointwise. Assume that the probability measure of the noise is known. Then, corresponding kernels $\mathcal{T}_n$ converges weakly to $\mathcal{T}$: If we denote the probability measure of w with μ , for any $g \in C_b(\mathbb{X})$ and for any $(x_0, u_0) \in \mathbb{X} \times \mathbb{U}$ using the dominated convergence theorem we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int g(x_1) \mathcal{T}_n(dx_1 | x_0, u_0) &= \lim_{n \rightarrow \infty} \int g(F_n(x_0, u_0, w)) \mu(dw) \\ &= \int g(F(x_0, u_0, w)) \mu(dw) = \int g(x_1) \mathcal{T}(dx_1 | x_0, u_0). \end{aligned}$$

- (ii) Much of the robust control literature deals with deterministic systems, where the nominal model is a deterministic perturbation of the actual model (see e.g. [25, 20]). The considered model is in the following form; $\tilde{F}(x_t, u_t) = F(x_t, u_t) + \Delta F(x_t, u_t)$, where F represents the nominal model and ΔF is the model uncertainty satisfying some norm bounds. For such deterministic systems, pointwise convergence of $\tilde{F}$ to the nominal model F , i.e. $\Delta F(x_t, u_t) \rightarrow 0$, can be viewed as weak convergence for deterministic systems by the discussion in (i). It is evident, however, that total variation convergence would be too strong for such a convergence criterion, since $\delta_{\tilde{F}(x_t, u_t)} \rightarrow \delta_{F(x_t, u_t)}$ weakly but $\|\delta_{\tilde{F}(x_t, u_t)} - \delta_{F(x_t, u_t)}\|_{TV} = 2$ for all $\Delta F(x_t, u_t) > 0$.

- (iii) Let $F(x_t, u_t, w_t) = f(x_t, u_t) + w_t$ be such that the function f is known and the probability law μ of w_t is not misspecified, such that an incorrect model μ_n is assumed. If $\mu_n \rightarrow \mu$ weakly, setwise or in total variation, then the corresponding transition kernels $\mathcal{T}_n$ converges in the same sense to $\mathcal{T}$. Observe the following,

$$\begin{aligned} &\int g(x_1) \mathcal{T}_n(dx_1 | x_0, u_0) - \int g(x_1) \mathcal{T}(dx_1 | x_0, u_0) \\ &= \int g(w_0 + f(x_0, u_0)) \mu_n(dw_0) - \int g(w_0 + f(x_0, u_0)) \mu(dw_0). \quad (3) \end{aligned}$$

- a. Suppose $\mu_n \rightarrow \mu$ weakly. If g is a continuous and bounded function then $g(\cdot + f(x_0, u_0))$ is a continuous and bounded function for all $(x_0, u_0) \in \mathbb{X} \times \mathbb{U}$. Thus, (3) goes to 0. Note that f does not need to be continuous.
 - b. Suppose $\mu_n \rightarrow \mu$ setwise. If g is a measurable and bounded function, then $g(\cdot + f(x_0, u_0))$ measurable and bounded for all $(x_0, u_0) \in \mathbb{X} \times \mathbb{U}$. Thus, (3) goes to 0.
 - c. Finally, assume $\mu_n \rightarrow \mu$ in total variation. If g is bounded, (3) converges to 0, as in item (b). A special case for this would be the following: Assume μ_n and μ admit densities h_n and h respectively, then the pointwise convergence of h_n to h implies the convergence of μ_n to μ in total variation by Scheffe's Theorem [1].
- (v) Suppose now neither F nor the probability model of w_t is known perfectly. It is assumed that w_t admits a measure μ_n and $\mu_n \rightarrow \mu$ weakly. For the function F we again have an approximating sequence $\{F_n\}$. If $F_n(x, u, w_n) \rightarrow F(x, u, w)$ for all $(x, u) \in \mathbb{X} \times \mathbb{U}$ and for any $w_n \rightarrow w$, then the transition kernel $\mathcal{T}_n$ corresponding to the model F_n converges weakly to the one of F , $\mathcal{T}$: For any $g \in C_b(\mathbb{X})$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int g(x_1) \mathcal{T}_n(dx_1 | x_0, u_0) &= \lim_{n \rightarrow \infty} \int g(F_n(x_0, u_0, w)) \mu_n(dw) \\ &= \int g(F(x_0, u_0, w)) \mu(dw) = \int g(x_1) \mathcal{T}(dx_1 | x_0, u_0). \end{aligned}$$

In the analysis above we used a generalized dominated convergence result, Lemma 3, to be presented later building on [15, Theorem 3.5] and [21, Theorem 3.5].

- (vi) Suppose that $F(x, u, \cdot) : \mathbb{W} \rightarrow \mathbb{X}$ is invertible for all fixed (x, u) and $F(x, u, w)$ is continuous and bounded on $\mathbb{X} \times \mathbb{U} \times \mathbb{W}$. We construct the empirical measures for the noise process w_t such that for every (fixed) Borel $B \subset \mathbb{W}$, and for every $n \in \mathbb{N}$, the empirical occupation measures are

$$\mu_n(B) = \frac{1}{n-1} \sum_{i=1}^n \mathbb{1}_{\{F_{x_{i-1}, u_{i-1}}^{-1}(x_i) \in B\}}$$

where $F_{x_{i-1}, u_{i-1}}^{-1}(x_i)$ denotes the inverse of $F(x_{i-1}, u_{i-1}, w) : \mathbb{W} \rightarrow \mathbb{X}$ for given (x_{i-1}, u_{i-1}) . Using the noise measurements, we construct the empirical transition kernel estimates for any (x_0, u_0) and Borel B as

$$\mathcal{T}_n(B | x_0, u_0) = \mu_n(F_{x_0, u_0}^{-1}(B)).$$

We have $\mu_n \rightarrow \mu$ weakly with probability one ([3], Theorem 11.4.1). This also implies that the transition kernels are such that $\mathcal{T}_n(\cdot|x_n, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly for any $(x_n, u_n) \rightarrow (x, u)$. To see that observe the following for $h \in C_b(\mathbb{X})$,

$$\begin{aligned} & \int h(x_1) \mathcal{T}_n(dx_1|x_n, u_n) - \int h(x_1) \mathcal{T}(dx_1|x, u) \\ &= \int h(f(x_n, u_n, w)) \mu_n(dw) - \int h(f(x, u, w)) \mu(dw) \rightarrow 0. \end{aligned}$$

For the last step, we used that $\mu_n \rightarrow \mu$ weakly and $h(f(x_n, u_n, w))$ continuously converge to $h(f(x, u, w))$ i.e. $h(f(x_n, u_n, w_n)) \rightarrow h(f(x, u, w))$ for some $w_n \rightarrow w$ since f and h are continuous functions.

3 Continuity under Convergence of Transition Kernels

3.1 Some differences with the infinite horizon discounted problem

In [13], we studied continuity of infinite horizon discounted cost problem under the convergence of transition kernels. In the following, we first show that the sufficient conditions presented in [13] to guarantee the continuity may not be sufficient for the infinite horizon average cost setup.

Let the infinite horizon discounted cost under a policy γ and the optimal cost be defined by

$$J_\beta(\mathcal{T}, \gamma) := E_P^{\mathcal{T}, \gamma} \left[\sum_{t=0}^{\infty} \beta^t c(X_t, U_t) \right], J_\beta^*(\mathcal{T}) := \inf_{\gamma \in \Gamma} J_\beta(\mathcal{T}, \gamma).$$

The following theorem gives sufficient conditions to guarantee the continuity of optimal discounted cost function under weak convergence of transition kernels.

Theorem 1. [13, Theorem 5.2] *Assume the following assumptions hold:*

- (i) $\mathcal{T}_n(\cdot|x_n, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly for every $x \in \mathbb{X}$ and $u \in \mathbb{U}$ and $(x_n, u_n) \rightarrow (x, u)$,
- (ii) $\mathcal{T}(\cdot|x, u)$ is weakly continuous in (x, u) ,
- (iii) The cost function $c(x, u)$ is continuous and bounded in $\mathbb{X} \times \mathbb{U}$,
- (iv) The action space $\mathbb{U}$ is compact.

Then $J_\beta(\mathcal{T}_n, \gamma_n^*) \rightarrow J_\beta(\mathcal{T}, \gamma^*)$, for any initial state x_0 , as $n \rightarrow \infty$.

In the following, we show that these conditions in Theorem 1 may not be sufficient for continuity for average cost problems. The following example shows that even for a control-free setup even if the assumptions of the above theorem hold, infinite horizon average cost function is not continuous. This example also covers the controlled case via using trivial control.

Example 1. Assume that $x_0 = 0$ and the transition kernels are given by

$$\mathcal{T}(\cdot|x) = \delta_x(\cdot), \quad \mathcal{T}_n(\cdot|x) = \delta_{x+\frac{1}{n}}(\cdot).$$

The cost function is given as

$$c(x) = \begin{cases} |x| & \text{if } |x| \leq 1, \\ 1 & \text{if } |x| > 1. \end{cases}$$

Notice that $\mathcal{T}_n(\cdot|x_n) \rightarrow \mathcal{T}(\cdot|x)$ weakly for any $x_n \rightarrow x$ and $\mathcal{T}(\cdot|x)$ is weakly continuous.

It is easy to see that the cost for $\mathcal{T}$ is 0 as the state always stays at 0 that is $J_\infty(\mathcal{T}) = 0$. The cost for $\mathcal{T}_n$ can be calculated as follows:

$$J_\infty(\mathcal{T}_n) = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{k=0}^n \frac{k}{n} + \sum_{k=n+1}^N 1 \right) = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\frac{n+1}{2} + N - n - 1 \right) = 1 \neq 0.$$

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3.2 Ergodicity properties of controlled Markov chains

For the average cost problems, as we will see later in this section, ergodicity properties of the state process play an important role for continuity and robustness. In the following result from [10], we summarize some of these ergodicity properties and provide some relations between them.

In the following we will denote the set of all stationary policies by Γ_s . For the transitions under some stationary policy γ , we will use the following notation: $\mathcal{T}(\cdot|x, \gamma) := \mathcal{T}(\cdot|x, \gamma(x))$.

We also define the t -step transition kernel $\mathcal{T}^t(\cdot|x, \gamma)$ in an iterative fashion as follows:

$$\mathcal{T}^t(\cdot|x, \gamma) := \int \mathcal{T}(\cdot|x_{t-1}, \gamma) \mathcal{T}^{t-1}(dx_{t-1}|x, \gamma),$$

where $\mathcal{T}^1(\cdot|x, \gamma) = \mathcal{T}(\cdot|x, \gamma)$.

We will use the following ergodicity condition for our results.

Assumption 1. *For every stationary policy γ , the transition kernels $\mathcal{T}$ and $\mathcal{T}_n$ lead to positive Harris recurrent chains and in particular admit invariant measures π_γ and π_γ^n , and for these invariant measures uniformly for every initial point $x \in \mathbb{X}$ we have:*

$$\lim_{t \rightarrow \infty} \sup_{\gamma \in \Gamma} \|\mathcal{T}^t(\cdot|x, \gamma) - \pi_\gamma(\cdot)\|_{TV} = 0$$

$$\lim_{t \rightarrow \infty} \sup_n \sup_{\gamma \in \Gamma} \|\mathcal{T}_n^t(\cdot|x, \gamma) - \pi_\gamma^n(\cdot)\|_{TV} = 0.$$

For the results on this paper, we will make use of Assumption 1. However, using the following theorem, alternative conditions can also be used.

Theorem 2. [10, Theorem 3.2]

- a. *There exists a state $x^* \in \mathbb{X}$ and a number $\beta > 0$ such that*

$$\mathcal{T}(\{x^*\}|x, \gamma) \geq \beta, \quad \text{for all } x \in \mathbb{X}, \gamma \in \Gamma_s.$$

- b. *There exists a positive integer t and a non-trivial measure μ on $\mathbb{X}$ such that*

$$\mathcal{T}^t(\cdot|x, \gamma) \geq \mu(\cdot) \quad \text{for all } x \in \mathbb{X}, \gamma \in \Gamma_s.$$

- c. *For each $\gamma \in \Gamma_s$, the transition kernel $\mathcal{T}(dy|x, \gamma)$ has a density $p(y|x, \gamma)$ with respect to a sigma-finite measure m on $\mathbb{X}$, and there exist $\epsilon > 0$ and $C \in \mathcal{B}(\mathbb{X})$ such that $m(C) > 0$ and*

$$p(y|x, \gamma) \geq \epsilon \quad \text{for all } y \in C, x \in \mathbb{X}, \gamma \in \Gamma_s.$$

- d. *For each $\gamma \in \Gamma_s$, $\mathcal{T}(dy|x, \gamma)$ has a density $p(y|x, \gamma)$ with respect to a sigma-finite measure m on $\mathbb{X}$, and $p(y|x, \gamma) \geq p_0(y)$ for all $x, y \in \mathbb{X}$, $\gamma \in \Gamma_s$, where p_0 is a non-negative measurable function with $\int p_0(y)m(dy) > 0$.*

- e. *There exists a positive integer t and a measure μ on $\mathbb{X}$ such that*

$$\mu(\mathbb{X}) < 2 \text{ and } \mathcal{T}^t(\cdot|x, \gamma) \leq \mu(\cdot) \text{ for all } x \in \mathbb{X}, \gamma \in \Gamma_s.$$

- f. *There exists a positive integer t and a positive number $\beta < 1$ such that*

$$\|\mathcal{T}^t(\cdot|x, \gamma) - \mathcal{T}^t(\cdot|x', \gamma)\|_{TV} \leq 2\beta \text{ for all } x, x' \in \mathbb{X}, \gamma \in \Gamma_s.$$

- g. *There exists a positive integer t and a positive number β for which the following holds: For each $\gamma \in \Gamma_s$, there is a probability measure μ_γ on $\mathbb{X}$ such that*

$$\mathcal{T}^t(\cdot|x, \gamma) \geq \beta \mu_\gamma(\cdot) \text{ for all } x \in \mathbb{X}.$$

- h. There exist positive numbers c and β , with $\beta < 1$, for which the following holds: For each $\gamma \in \Gamma_s$, there is a probability measure p_γ on $\mathbb{X}$ such that

$$\|\mathcal{T}^t(\cdot|x, \gamma) - p_\gamma(\cdot)\|_{TV} \leq c\beta^t \text{ for all } x \in \mathbb{X}, t \in \mathbb{N}.$$

- i. The state process is uniformly ergodic such that $\lim_{t \rightarrow \infty} \|\mathcal{T}^t(\cdot|x, \gamma) - p_\gamma(\cdot)\|_{TV} = 0$ uniformly in $x \in \mathbb{X}$ and $\gamma \in \Gamma_s$.

The conditions above are related as follows:

$$\begin{aligned} a &\rightarrow b \\ e &\rightarrow f \\ c &\rightarrow d \rightarrow b \rightarrow f \leftrightarrow g \leftrightarrow h \leftrightarrow i. \end{aligned}$$

◇

Remark 1. Notice that condition (i) is the same as Assumption 1, thus one can make use of different assumptions through the relations provided in Theorem 2.

3.3 Optimality of stationary policies

For our continuity and robustness results, it will prove to be instrumental to work with stationary policies. This will be without any loss under mild conditions to be presented in this subsection.

An approach for average cost problems is to make use of average cost optimality equation (ACOE). To work with ACOE one usually needs contraction properties on the transition kernel. The following result provides further alternative sufficient conditions on existence of optimal policies (which turn out to be stationary) for infinite horizon average cost problems.

Assumption 2.

- (A) The condition f in Theorem 2 (or any other suitable condition among a-i) holds,
- (B) The action space $\mathbb{U}$ is compact,
- (C) $c(x, u)$ is bounded and continuous in (x, u) ,
- (C') $c(x, u)$ is bounded and continuous in u for every fixed x ,
- (D) $\mathcal{T}(\cdot|x, u)$ is weakly continuous in (x, u) ,
- (D') $\mathcal{T}(\cdot|x, u)$ is setwise continuous in u for every x .

Theorem 3. [9, Corollary 3.6] Suppose Assumption 2 A, B, and, either C and D, or C' and D', hold. Then $J_\infty(\mathcal{T}, \gamma)$ admits an optimal stationary policy. $\diamond$

For the rest of the paper, we will assume that the optimal policies can be selected from those which are stationary with suitable assumptions and we will denote the family of stationary policies by Γ .

3.4 Approximation of infinite horizon cost by finite horizon cost

We denote the t -step finite horizon cost function under a stationary policy γ and a transition model $\mathcal{T}$ by $J_t(\mathcal{T}, \gamma)$ and the corresponding optimal cost is denoted by $J_t^*(\mathcal{T})$, i.e.

$$J_t(\mathcal{T}, \gamma) = \sum_{i=0}^{t-1} E_\gamma^\mathcal{T}[c(X_i, U_i)]$$

$$J_t^*(\mathcal{T}) = \inf_{\gamma \in \Gamma} J_t(\mathcal{T}, \gamma).$$

The following result shows that the infinite horizon average cost induced by a stationary policy can be approximated by a finite cost under the same stationary policies with proper ergodicity conditions.

Lemma 1. Under Assumption 1, if the cost function c is bounded then for every initial state we have

$$\sup_{\gamma \in \Gamma} \left| \frac{J_t(\mathcal{T}, \gamma)}{t} - J_\infty(\mathcal{T}, \gamma) \right| \rightarrow 0,$$

$$\sup_{\gamma \in \Gamma} \sup_n \left| \frac{J_t(\mathcal{T}_n, \gamma)}{t} - J_\infty(\mathcal{T}_n, \gamma) \right| \rightarrow 0.$$

Proof. Under the assumption, we have that

$$J_\infty(\mathcal{T}, \gamma) = \int c(x, \gamma(x)) \pi^\gamma(dx).$$

Thus, we can write

$$\begin{aligned} \left| \frac{J_t(\mathcal{T}, \gamma)}{t} - J_\infty(\mathcal{T}, \gamma) \right| &= \left| \frac{1}{t} \sum_{i=0}^{t-1} E_\gamma^\mathcal{T}[c(X_i, U_i)] - \int c(x, \gamma(x)) \pi^\gamma(dx) \right| \\ &\leq \frac{1}{t} \sum_{i=0}^{t-1} \left| \int c(x_i, \gamma(x_i)) \mathcal{T}^i(dx_i | x_0, \gamma) - \int c(x, \gamma(x)) \pi^\gamma(dx) \right| \\ &\leq \frac{1}{t} \sum_{i=0}^{t-1} \|\mathcal{T}^i(\cdot | x_0, \gamma) - \pi^\gamma\|_{TV} \|c\|_\infty. \end{aligned}$$

We now fix an $\epsilon > 0$ and choose a $t_\epsilon < \infty$ such that $\|\mathcal{T}^i(\cdot|x_0, \gamma) - \pi^\gamma\|_{TV} < \epsilon$ for all $i > t_\epsilon$. We also choose another T_ϵ with $\frac{2t_\epsilon}{t} < \epsilon$ for all $t > T_\epsilon$. With this setup, we have

$$\begin{aligned} \frac{1}{t} \sum_{i=0}^{t-1} \|\mathcal{T}^i(\cdot|x_0, \gamma) - \pi^\gamma\|_{TV} &\leq \frac{1}{t} \sum_{i=0}^{t_\epsilon-1} \|\mathcal{T}_\gamma^i(\cdot|x_0) - \pi^\gamma\|_{TV} + \frac{1}{t} \sum_{i=t_\epsilon}^t \|\mathcal{T}_\gamma^i(\cdot|x_0) - \pi^\gamma\|_{TV} \\ &\leq \frac{2t_\epsilon}{t} + \epsilon \leq 2\epsilon, \quad \forall t > T_\epsilon. \end{aligned}$$

We have shown that for any fixed $\epsilon > 0$, we can choose a $T_\epsilon < \infty$, independent of γ , such that

$$\left| \frac{J_t(\mathcal{T}, \gamma)}{t} - J_\infty(\mathcal{T}, \gamma) \right| < \epsilon, \quad \forall t > T_\epsilon.$$

Hence the result is complete for $\mathcal{T}$. For $\mathcal{T}_n$ the result follows from the same steps since we can again choose such t_ϵ and T_ϵ due to the uniformity over n and γ in Assumption 1. $\square$

The next result from [9, Corollary 4.11] shows that the optimal infinite horizon cost can be approximated by an optimal finite horizon cost induced by the same transition kernel.

Lemma 2. *Suppose the cost function c is bounded and one the following assumptions holds.*

- *Assumption 2 A, B, C, D or A, B, C', D' (for $\mathcal{T}$ and $\mathcal{T}_n$).*

Then, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \left| J_\infty^*(\mathcal{T}) - \frac{J_t^*(\mathcal{T})}{t} \right| &\rightarrow 0, \\ \lim_{t \rightarrow \infty} \sup_n \left| J_\infty^*(\mathcal{T}_n) - \frac{J_t^*(\mathcal{T}_n)}{t} \right| &\rightarrow 0. \end{aligned}$$

3.5 Continuity under the convergence of transition kernels

Theorem 4. *We have that*

$$|J_\infty^*(\mathcal{T}_n) - J_\infty^*(\mathcal{T})| \rightarrow 0$$

under

- c1. Under Assumption 2 A, B, C and D if $\mathcal{T}_n(\cdot|x_n, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly for any $(x_n, u_n) \rightarrow (x, u)$.
- c2. Under Assumption 2 A, B, C' and D' if $\mathcal{T}_n(\cdot|x, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ setwise for any $u_n \rightarrow u$ for every fixed x .

Proof. We use the following bound:

$$|J_\infty^*(\mathcal{T}_n) - J_\infty^*(\mathcal{T})| \leq |J_\infty^*(\mathcal{T}_n) - \frac{J_t^*(\mathcal{T}_n)}{t}| + |\frac{J_t^*(\mathcal{T}_n)}{t} - \frac{J_t^*(\mathcal{T})}{t}| + |\frac{J_t^*(\mathcal{T})}{t} - J_\infty^*(\mathcal{T})|.$$

The first and the last terms above can be made arbitrarily small by choosing t large enough uniformly over n using Lemma 2 under suitable assumptions. For the second term, we can use continuity results for finite time problems for the fixed t as the assumptions cover the requirements of ([13, Theorem 4.2] or [13, Section 5.3]). $\square$

Remark 2. We note that any condition set provided for continuity of the optimal cost function under setwise convergence of the transition kernels is also a sufficient set of conditions for the continuity of the optimal cost function under total variation convergence of the transition kernels.

4 Robustness to Incorrect Controlled Transition Kernel Models

In this section, we investigate robustness for infinite horizon average cost problems. We first restate the problem: Consider a MDP with transition kernel $\mathcal{T}_n$, and assume that an optimal control policy for this MDP under the average cost criterion is γ_n^* that is

$$\inf_{\gamma \in \Gamma} J_\infty(\mathcal{T}_n, \gamma) = J_\infty(\mathcal{T}_n, \gamma_n^*).$$

Now, consider another MDP with transition kernel $\mathcal{T}$ whose the optimal cost denoted by $J_\infty^*(\mathcal{T})$. The question we ask is the following: if the controller does not know the true transition kernel $\mathcal{T}$ and calculates an optimal policy assuming the transition kernel is $\mathcal{T}_n$, then the incurred cost by this policy is

$$J_\infty(\mathcal{T}, \gamma_n^*).$$

The focus of this section is to find sufficient conditions such that as $\mathcal{T}_n \rightarrow \mathcal{T}$, we have

$$J_\infty(\mathcal{T}, \gamma_n^*) \rightarrow J_\infty(\mathcal{T}, \gamma^*).$$

The first issue with this question is the following one: assume that the MDP with kernel $\mathcal{T}_n$ admits two different optimal policies γ_n^1 and γ_n^2 . Although, the cost incurred by these policies under the kernel $\mathcal{T}_n$ are the same, under the kernel $\mathcal{T}$, they may cause different cost values. That is, even though we have that

$$J_\infty(\mathcal{T}_n, \gamma_n^1) = J_\infty(\mathcal{T}_n, \gamma_n^2) = J_\infty^*(\mathcal{T}_n),$$

we may have

$$J_\infty(\mathcal{T}, \gamma_n^1) \neq J_\infty(\mathcal{T}, \gamma_n^2).$$

An example is as follows:

Consider a system with state space $\mathbb{X} = [-1, 1]$, control action space $\mathbb{U} = \{-1, 0, 1\}$, the cost function $c(x, u) = (x - u)^2$ and the transition models given as

$$\begin{aligned}\mathcal{T}_n(\cdot | x, u) &= \frac{1}{2}\delta_1(\cdot) + \frac{1}{2}\delta_{-1}(\cdot) \\ \mathcal{T}(\cdot | x, u) &= \delta_0(\cdot)\end{aligned}$$

Notice that two optimal policies for $\mathcal{T}_n$ are

$$\gamma_n^1(x) = \begin{cases} 1 & \text{if } x = 1, \\ -1 & \text{if } x = -1, \\ 0 & \text{else.} \end{cases} \quad \gamma_n^2(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

However, if the initial point is $x_0 = 0$, we have that $J_\infty(\mathcal{T}, \gamma_n^1) = 0 \neq 1 = J_\infty(\mathcal{T}, \gamma_n^2)$.

In what follows, we show that under total variation convergence of $\mathcal{T}_n \rightarrow \mathcal{T}$, this issue does not cause a problem so that we have $J_\infty(\mathcal{T}, \gamma_n^*) \rightarrow J_\infty(\mathcal{T}, \gamma^*)$ for any stationary optimal policy γ_n^* . However, under weak or convergence of the transition models, we establish the same result under some particularly constructed optimal policies γ_n^* , namely we focus on the policies that solve the average cost optimality equation (ACOE).

4.1 Robustness under weak convergence of transition kernels

The Average Cost Optimality Equation (ACOE)

Now, we discuss the average cost optimality equation, and we use it for analyzing robustness properties of the MDPs under weak convergence of the transition kernels.

Define the operator $T : B(\mathbb{X}) \rightarrow B(\mathbb{X})$ where $B(\mathbb{X})$ denotes the set of bounded and measurable functions on $\mathbb{X}$ such that for $v \in B(\mathbb{X})$

$$Tv(x) := \inf_{u \in \mathbb{U}} \left(c(x, u) + \int_{\mathbb{X}} v(y) \mathcal{T}(dy|x, u) \right). \quad (4)$$

We define the span semi-norm of a function $v \in B(\mathbb{X})$ by

$$sp(v) := \sup_x v(x) - \inf_x v(x).$$

One can show that the operator defined in (4) is a contraction in $B(\mathbb{X})$ (space of all bounded measurable functions) under the span-norm with Assumption 1 or any suitable one from Assumption 2 [9, Lemma 3.5]. Hence, according to the Banach fixed point theorem there exists a fixed point $v^* \in B(\mathbb{X})$ such that $sp(Tv^* - v^*) = 0$. By the definition of the span-norm, $Tv^*(x) - v^*(x) = j^*$ for a constant j^* for all $x \in \mathbb{X}$. That is

$$j^* + v^*(x) = \inf_{u \in \mathbb{U}} \left(c(x, u) + \int_{\mathbb{X}} v^*(y) \mathcal{T}(dy|x, u) \right). \quad (5)$$

This constant j^* is the optimal infinite horizon average cost and the equation (5) is called the average cost optimality equation (ACOE).

Now, we formalize these observations:

Theorem 5. *Suppose the cost function c is bounded, under Assumption 1, there exists a $\beta < 1$ such that the following holds:*

- (i) $sp(Tv - Tw) \leq \beta sp(u - w)$, for any $v, w \in B(\mathbb{X})$ where T is the operator defined in (4).
- (ii) Since T is a contraction under the span norm, it admits a fixed point $v^* \in B(\mathbb{X})$ such that

$$j^* + v^*(x) = \inf_{u \in \mathbb{U}} \left(c(x, u) + \int_{\mathbb{X}} v^*(y) \mathcal{T}(dy|x, u) \right),$$

for some constant j^* .

- (iii) For any initial point $x_0 \in \mathbb{X}$, the constant j^* defined in (ii), is the optimal infinite horizon average cost for the kernel $\mathcal{T}$, that is

$$j^* = J_{\infty}^*(\mathcal{T}, x_0) = \inf_{\gamma \in \Gamma} J_{\infty}(\mathcal{T}, \gamma, x_0)$$

for every $x_0 \in \mathbb{X}$.

(iv) If there exists a policy $\gamma^* \in \Gamma$, satisfying the ACOE, then this stationary policy is an optimal policy for the average infinite horizon cost problem, that is, if γ^* satisfies the following,

$$j^* + v^*(x) = c(x, \gamma^*(x)) + \int_{\mathbb{X}} v^*(y) \mathcal{T}(dy|x, \gamma^*(x)),$$

$$\text{then } J_\infty(\mathcal{T}, \gamma^*, x_0) = J_\infty^*(\mathcal{T}, x_0).$$

We now state the main result of this section.

Theorem 6. We have that $J_\infty(\mathcal{T}, \gamma_n^*, x) \rightarrow J_\infty^*(\mathcal{T}, x)$ for any $x \in \mathbb{X}$, where γ_n^* is the optimal policy for the transition kernel $\mathcal{T}_n$ that satisfies the ACOE, under Assumption 2 A, B, C and D if $\mathcal{T}_n(\cdot|x_n, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly for any $(x_n, u_n) \rightarrow (x, u)$.

Proof. Consider the following two ACOE for the kernels $\mathcal{T}_n$ and $\mathcal{T}$ with their fixed point v_n^* and v^*

$$j_n^* + v_n^*(x) = \inf_{u \in \mathbb{U}} \left[c(x, u) + \int v_n^*(y) \mathcal{T}_n(dy|x, u) \right] \quad (6)$$

$$j^* + v^*(x) = \inf_{u \in \mathbb{U}} \left[c(x, u) + \int v^*(y) \mathcal{T}(dy|x, u) \right] \quad (7)$$

In Appendix A we establish that for all $x_n \rightarrow x$

$$v_n^*(x_n) - v^*(x) \rightarrow c \quad (8)$$

for some constant $|c| < \infty$.

Since $\mathbb{U}$ is compact, for every $x_n \rightarrow x$, $\gamma_n(x_n)$ has a convergent subsequence which converges to say some $u^* \in \mathbb{U}$. If we take the limit along this subsequence for (6), using the assumptions that $\mathcal{T}_n(\cdot|x_n, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly, the fact that $\lim_{n \rightarrow \infty} (v_n^*(x_n) - v^*(x)) = c$, and that $j_n^* \rightarrow j^*$ (continuity results Theorem 4) we get

$$\begin{aligned} & \lim_k \left(j_{n_k}^* + v_{n_k}^*(x_{n_k}) \right) \\ &= \lim_k c(x, \gamma_{n_k}^*(x_{n_k})) + \int v_{n_k}^*(y) \mathcal{T}_{n_k}(dy|x_{n_k}, \gamma_{n_k}^*(x_{n_k})) \\ &= j^* + v^*(x) + c = c(x, u^*) + \int v^*(y) \mathcal{T}(dy|x, u^*) + c. \end{aligned}$$

Therefore, u^* satisfies the ACOE for the kernel $\mathcal{T}$ and thus, any convergent subsequence of $\gamma_n^*(x_n)$ is an optimal action for x for the kernel $\mathcal{T}$.

Now consider the following operator $\hat{T}_n$, for the kernel $\mathcal{T}$ and the policy γ_n^* which is optimal for $\mathcal{T}_n$

$$\hat{T}_n \hat{v}_n(x) = c(x, \gamma_n^*(x)) + \int \hat{v}_n(y) \mathcal{T}(dy|x, \gamma_n^*(x)). \quad (9)$$

One can show that this operator is also a contraction under span semi-norm and admits a fixed point $\hat{v}_n^*$ such that

$$\hat{j}_n + \hat{v}_n^*(x) = c(x, \gamma_n^*(x)) + \int \hat{v}_n^*(y) \mathcal{T}(dy|x, \gamma_n^*(x))$$

where $\hat{j}_n = J_\infty(\mathcal{T}, \gamma_n^*, x)$ for all x . Hence, we need to show that $\hat{j}_n \rightarrow j^*$ to complete the proof. To show this, in Appendix B, we prove that

$$\lim_{n \rightarrow \infty} \hat{v}_n^*(x_n) - v^*(x) = \hat{c}, \quad (10)$$

for any $x_n \rightarrow x$ for some constant $\hat{c} < \infty$.

Now, assume that $\lim_n \hat{j}_n \neq j^*$ and that there exists a subsequence $\hat{j}_{n_k}$ and an $\epsilon > 0$ such that $|\hat{j}_{n_k} - j^*| > \epsilon$ for every k . We will show that this cannot hold, by establishing the existence of a further subsequence $\hat{j}_{n_{k_l}}$ which converges to j^* in the following.

We first note that $\lim_{n \rightarrow \infty} \hat{v}_n^*(x_n) - v^*(x) = \hat{c}$. Hence, [15, Theorem 3.5] (or [21, Theorem 3.5]) yields that $\int \hat{v}_{n_{k_l}}^*(y) \mathcal{T}(dy|x, \gamma_{n_{k_l}}^*(x)) \rightarrow \int v^*(y) \mathcal{T}(dy|x, u^*) + \hat{c}$ where $\hat{c}$ also satisfies $(\hat{v}_{n_{k_l}}^*(x) - v^*(x)) \rightarrow \hat{c}$.

Therefore, taking the limit along this subsequence, we get

$$\begin{aligned} & \lim_{l \rightarrow \infty} \hat{j}_{n_{k_l}} \\ &= \lim_{l \rightarrow \infty} c(x, \gamma_{n_{k_l}}^*(x)) + \int \hat{v}_{n_{k_l}}^*(y) \mathcal{T}(dy|x, \gamma_{n_{k_l}}^*(x)) - \hat{v}_{n_{k_l}}^*(x) \\ &= c(x, u^*) + \int v^*(y) \mathcal{T}(dy|x, u^*) - v^*(x) = j^*. \end{aligned}$$

This contradicts to $|\hat{j}_{n_k} - j^*| > \epsilon$, hence we conclude that $\hat{j}_n \rightarrow j^*$. □ □

4.2 Robustness under setwise convergence of transition kernels

Theorem 7. *We have that $J_\infty(\mathcal{T}, \gamma_n^*, x) \rightarrow J_\infty^*(\mathcal{T}, x)$ for any $x \in \mathbb{X}$, where γ_n^* is the optimal policy for the transition kernel $\mathcal{T}_n$ that satisfies the ACOE, under Assumption 2 A, B, C and D if $\mathcal{T}_n(\cdot|x, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly for any $u_n \rightarrow u$.*

Proof. The proof follows the similar steps as in the proof of Theorem 6. Consider again the following two ACOE for the kernels $\mathcal{T}_n$ and $\mathcal{T}$ with their fixed point v_n^* and v^*

$$j_n^* + v_n^*(x) = \inf_{u \in \mathbb{U}} \left[c(x, u) + \int v_n^*(y) \mathcal{T}_n(dy|x, u) \right] \quad (11)$$

$$j^* + v^*(x) = \inf_{u \in \mathbb{U}} \left[c(x, u) + \int v^*(y) \mathcal{T}(dy|x, u) \right]. \quad (12)$$

With the same argument in Appendix A using [13, Theorem 4.8] one show that

$$v_n^*(x) - v^*(x) \rightarrow c$$

for some constant $|c| < \infty$ for all x .

Since $\mathbb{U}$ is compact, for every x , $\gamma_n(x)$ has a convergent subsequence which converges to say some $u^* \in \mathbb{U}$. If we take the limit along this subsequence for (11), using the assumptions that $\mathcal{T}_n(\cdot|x, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ setwise, the fact that $\lim_{n \rightarrow \infty} (v_n^*(x) - v^*(x)) = c$, and that $j_n^* \rightarrow j^*$ (continuity results Theorem 4) we get

$$\begin{aligned} \lim_k j_{n_k}^* + v_{n_k}^*(x) &= \lim_k c(x, \gamma_{n_k}^*(x)) + \int v_{n_k}^*(y) \mathcal{T}_{n_k}(dy|x, \gamma_{n_k}^*(x)) \\ &= j^* + v^*(x) + c = c(x, u^*) + \int v^*(y) \mathcal{T}(dy|x, u^*) + c. \end{aligned}$$

Therefore, u^* satisfies the ACOE for the kernel $\mathcal{T}$ and thus, any convergent subsequence of $\gamma_n^*(x)$ is an optimal action for x for the kernel $\mathcal{T}$.

Now consider the operator $\hat{T}_n$ again,

$$\hat{T}_n \hat{v}_n(x) = c(x, \gamma_n^*(x)) + \int \hat{v}_n(y) \mathcal{T}(dy|x, \gamma_n^*(x)). \quad (13)$$

We write

$$\hat{j}_n + \hat{v}_n^*(x) = c(x, \gamma_n^*(x)) + \int \hat{v}_n^*(y) \mathcal{T}(dy|x, \gamma_n^*(x))$$

where $\hat{j}_n = J_\infty(\mathcal{T}, \gamma_n^*, x)$ for all x . Hence, we need to show that $\hat{j}_n \rightarrow j^*$ to complete the proof. By replicating the same arguments in Appendix B for setwise convergence and using [19, Theorem 20] (setwise convergence with varying functions), one can prove that

$$\lim_{n \rightarrow \infty} \hat{v}_n^*(x) - v^*(x) = \hat{c},$$

for any x for some constant $\hat{c} < \infty$.

Now, assume that $\lim_n \hat{j}_n \neq j^*$ and that there exists a subsequence $\hat{j}_{n_k}$ and an $\epsilon > 0$ such that $|\hat{j}_{n_k} - j^*| > \epsilon$ for every k . We first note that $\lim_{n \rightarrow \infty} \hat{v}_n^*(x) - v^*(x) = \hat{c}$. Hence, [19, Theorem 20] yields that $\int \hat{v}_{n_{k_l}}^*(y) \mathcal{T}(dy|x, \gamma_{n_{k_l}}^*(x)) \rightarrow \int v^*(y) \mathcal{T}(dy|x, u^*) + \hat{c}$ where $\hat{c}$ also satisfies $(\hat{v}_{n_{k_l}}^*(x) - v^*(x)) \rightarrow \hat{c}$.

Therefore, taking the limit along this subsequence, we get

$$\begin{aligned} \lim_{l \rightarrow \infty} \hat{j}_{n_{k_l}} &= \lim_{l \rightarrow \infty} c(x, \gamma_{n_{k_l}}^*(x)) + \int \hat{v}_{n_{k_l}}^*(y) \mathcal{T}(dy|x, \gamma_{n_{k_l}}^*(x)) - \hat{v}_{n_{k_l}}^*(x) \\ &= c(x, u^*) + \int v^*(y) \mathcal{T}(dy|x, u^*) - v^*(x) = j^*. \end{aligned}$$

This contradicts to $|\hat{j}_{n_k} - j^*| > \epsilon$, hence we conclude that $\hat{j}_n \rightarrow j^*$. $\square$

4.3 Robustness under total variation convergence of transition kernels

In this section, we will show that for any stationary policy γ_n^* that is optimal for $\mathcal{T}_n$, as $\mathcal{T}_n \rightarrow \mathcal{T}$ in total variation, we have $J_\infty(\mathcal{T}, \gamma_n^*) \rightarrow J_\infty^*(\mathcal{T})$ under proper conditions.

We note that, since setwise convergence is less stringent than total variation, it is not surprising that we can establish robustness under total variation convergence of the kernels as well. However, in the previous section, we showed convergence only when the policies were restricted to be among those which solves the ACOE for every point in the state space; in the analysis below, the result will be more general and the policies considered are just required to be optimal, without the requirement that they solve the ACOE for every $x \in \mathbb{X}$. This is not vacuous, as the following example shows.

Example 2. Consider $\mathbb{X} = [-1, 1]$ and $\mathbb{U} = \{0, 1, 2\}$.

Let the kernels be given in the following form for $n \geq 1$:

$$\begin{aligned} \mathcal{T}_n(\cdot|x, u) &= \left(\frac{1}{2} \delta_{\frac{1}{n}}(\cdot) + \frac{1}{2} \delta_{-\frac{1}{n}}(\cdot) \right) \mathbb{1}_{\{x \geq \frac{1}{n}\}} \\ &\quad + \left(\frac{1}{2} \delta_{\frac{1}{n}}(\cdot) + \frac{1}{2} \delta_{-\frac{1}{n}}(\cdot) \right) \mathbb{1}_{\{x \leq -\frac{1}{n}\}} \\ &\quad + \left(\frac{1}{3} \delta_{\frac{1}{n}}(\cdot) + \frac{1}{3} \delta_{-\frac{1}{n}}(\cdot) + \frac{1}{3} \delta_0(\cdot) \right) \mathbb{1}_{\{-\frac{1}{n} < x < \frac{1}{n}\}} \\ \mathcal{T}(\cdot|x, u) &= \delta_0(\cdot). \end{aligned}$$

The cost function is given by:

$$c(x, u) = \begin{cases} (x)\mathbb{1}_{x \geq 0} + 0\mathbb{1}_{x < 0} & \text{if } u = 0, 1 \\ 3 & \text{if } u = 2. \end{cases}$$

One can show that this setup satisfies Assumption 2 A, B, C and D.

With this setup, one (among many others) optimal policy for $\mathcal{T}_n$ when the initial state is $x = -1$ is given by;

$$\gamma_n^*(x) = \begin{cases} 1 & \text{if } x \leq -\frac{1}{n} \\ 0 & \text{if } x \geq \frac{1}{n} \\ 2 & \text{otherwise.} \end{cases}$$

When the initial state is -1 , the cost under this policy is

$$J_\infty(\mathcal{T}_n, \gamma_n^*) = \lim_{N \rightarrow \infty} \frac{1}{N} N \frac{1}{2} \frac{1}{n} = \frac{1}{2n} \rightarrow 0.$$

Therefore the policy γ_n^* is indeed optimal for $\mathcal{T}_n$ for large n . An optimal policy for $\mathcal{T}$ is given by $\gamma^*(x) = 1$. Thus, the average cost values can be calculated as:

$$\begin{aligned} J_\infty(\mathcal{T}, \gamma_n^*) &= \lim_{N \rightarrow \infty} \sum_{t=0}^{N-1} E[c(X_t, \gamma_n^*(X_t))] \\ &= \lim_{N \rightarrow \infty} \sum_{t=1}^{N-1} c(0, \gamma_n^*(0)) = \lim_{N \rightarrow \infty} \sum_{t=1}^{N-1} 3 = 3 \\ J_\infty(\mathcal{T}, \gamma^*) &= 0. \end{aligned}$$

Hence, we have that

$$\lim_{n \rightarrow \infty} J_\infty(\mathcal{T}, \gamma_n^*) = 3 \neq J_\infty(\mathcal{T}, \gamma^*) = 0.$$

Notice that for this example, γ_n^* would not have been optimal for $\mathcal{T}_n$ if the initial state were between $-1/n$ and $1/n$ and, in particular, it does not satisfy the ACOE.

Theorem 8. We have that $|J_\infty(\mathcal{T}, \gamma_n^*) - J_\infty^*(\mathcal{T})| \rightarrow 0$ for any stationary optimal policy γ_n^* for $\mathcal{T}_n$

- c1. under Assumption 2 A, B, C' and D' if $\mathcal{T}_n(\cdot|x, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ in total variation for any $u_n \rightarrow u$ for every fixed x .

Proof. We write:

$$|J_\infty(\mathcal{T}, \gamma_n^*) - J_\infty^*(\mathcal{T})| \leq |J_\infty^*(\mathcal{T}_n) - J_\infty^*(\mathcal{T})| + |J_\infty(\mathcal{T}, \gamma_n^*) - J_\infty^*(\mathcal{T}_n)|$$

the first term goes to by Theorem 4. For the second term we write

$$\begin{aligned} |J_\infty(\mathcal{T}, \gamma_n^*) - J_\infty^*(\mathcal{T}_n)| &\leq |J_\infty(\mathcal{T}, \gamma_n^*) - \frac{J_t(\mathcal{T}_n, \gamma_n^*)}{t}| + |\frac{J_t(\mathcal{T}_n, \gamma_n^*)}{t} - \frac{J_t(\mathcal{T}, \gamma_n^*)}{t}| \\ &\quad + |\frac{J_t(\mathcal{T}, \gamma_n^*)}{t} - J_\infty(\mathcal{T}, \gamma_n^*)|. \end{aligned}$$

The first and the last terms above again can be made arbitrarily small by choosing t large enough uniformly over n using Lemma 1. For the second term we use [13, Section A.2] where it is shown that under the stated assumptions

$$\sup_{\gamma \in \Gamma} |J_t(\mathcal{T}_n, \gamma) - J_t(\mathcal{T}, \gamma)| \rightarrow 0.$$

Hence the proof is complete. $\square$

4.4 Comparison with the conditions reported in the literature

In the most relevant contribution (to our knowledge), [9], the following problem is considered: Suppose we have an approximating model for the true kernel $\mathcal{T}$, so that, every time step t , an estimate is updated to $\mathcal{T}_t$ and an optimal stationary policy is found with

$$J_\infty^*(\mathcal{T}_t) = \inf_{\gamma \in \Gamma} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} E^{\mathcal{T}_t} [c(X_i, \gamma(X_i))],$$

where $E^{\mathcal{T}_t}$ denotes the expectation with transition kernel $\mathcal{T}_t$. Let γ_t^* be the optimal policy for the kernel $\mathcal{T}_t$. It is shown that ([9, Theorem 5.7]) if condition f of Theorem 2 holds for $\mathcal{T}$, $\mathcal{T}_n$ and if $\sup_{x,u} \|\mathcal{T}_t(\cdot|x, u) - \mathcal{T}(\cdot|x, u)\|_{TV} \rightarrow 0$ the following hold:

- (i) $J_\infty(\mathcal{T}_t, \gamma_t^*) \rightarrow J_\infty(\mathcal{T}, \gamma^*)$ as $t \rightarrow \infty$,
- (ii) $J_\infty(\mathcal{T}, \gamma_t^*) = J_\infty(\mathcal{T}, \gamma^*)$.

Notice that the first item (i) is the continuity problem we study in this paper (Problem 1). Therefore, it can also be shown to hold true with Theorem 4 which can be proven with

$$\sup_u \|\mathcal{T}_t(\cdot|x, u) - \mathcal{T}(\cdot|x, u)\|_{TV} \rightarrow 0$$

for fixed x . Theorem 4 also states that we can even weaken the total variation convergence of kernels to weak convergence so that it suffices to have $\mathcal{T}_t(\cdot|x_t, u_t) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly for any $(x_t, u_t) \rightarrow (x, u)$.

For the second item (ii), policies γ_t^* (since are not time-invariant) are not stationary for the model $\mathcal{T}$, however it is shown in [9] that using γ_t^* at time step t , the cost

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^{N-1} E^{\mathcal{T}} [c(X_t, \gamma_t^*(X_t))]$$

is equal to the optimal cost for the true model. To prove this result [9] uses ACOE and thus is able to use an optimality equation argument to show the convergence of policies.

The following simple example shows that the total variation convergence of the transition kernels can be too much to ask for deterministic problems. Thus, the relaxation to weak convergence of transition kernels is significant.

Example 3. Assume that $x_0 = 0$ and the transition kernels are given by

$$\mathcal{T}(\cdot|x) = \delta_0(\cdot), \quad \mathcal{T}_n(\cdot|x) = \delta_{\frac{1}{n}}(\cdot).$$

The cost function is given as

$$c(x) = \begin{cases} |x| & \text{if } |x| \leq 1, \\ 1 & \text{if } |x| > 1. \end{cases}$$

Notice that $\|\mathcal{T}_n(\cdot|x) - \mathcal{T}(\cdot|x)\|_{TV} = 2$ for all n , however, $\mathcal{T}_n(\cdot|x_n) \rightarrow \mathcal{T}(\cdot|x)$ weakly for any $x_n \rightarrow x$. Furthermore, $J_\infty^*(\mathcal{T}_n) \rightarrow J_\infty^*(\mathcal{T})$.

Hence, although the transition kernels do not converge to each other in total variation, continuity still holds.

5 Continuity and Robustness under Convergence of Transition Kernels Without Uniform Ergodicity

In this section we show that if the family of optimal policies forms an equicontinuous and stationary family we can guarantee continuity and robustness without requiring a uniform ergodicity over policies and initial points. The equicontinuity assumption is admittedly an artificial assumption.

Before the main result, we present a supporting lemma from [15, Theorem 3.5] and [21, Theorem 3.5].

Lemma 3. Suppose $\{\mu_n\}_n \subset \mathcal{P}(\mathbb{X})$, where $\mathbb{X}$ is a Polish space, converges weakly to some $\mu \in \mathcal{P}(\mathbb{X})$. For a bounded real valued sequence of functions $\{f_n\}_n$ such that $\|f_n\|_\infty < C$ for all $n > 0$ with $C < \infty$, if $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$ for all $x_n \rightarrow x$, i.e. f_n continuously converges to f , then $\lim_{n \rightarrow \infty} \int_{\mathbb{X}} f_n(x) \mu_n(dx) = \int_{\mathbb{X}} f(x) \mu(dx)$.

The following result shows that if we restrict the family of policies to an equicontinuous and stationary family we can guarantee continuity and robustness. For the result we do not require a uniform ergodicity assumption as in Assumption 1.

Assumption 3. (i) For any stationary policy γ , $\mathcal{T}_n$ and $\mathcal{T}$ lead to positive Harris recurrent chains and in particular admit unique invariant measures π_n^γ and π^γ .

(ii) $\{\pi_n^\gamma\}_{\gamma,n}$ and $\{\pi^\gamma\}_\gamma$ are tight.

(iii) Family of optimal policies $\{\gamma_n^*\}_n$ for $\mathcal{T}_n$, is an equicontinuous family of functions and the optimal policy γ^* for $\mathcal{T}$ is continuous.

Remark 3. Note that Assumption 3 (iii) holds if $\mathbb{X}$ is compact.

Theorem 9. Suppose that Assumption 3 and Assumption 2 B, C, D (for $\mathcal{T}_n$ and $\mathcal{T}$) hold. Then we have that $J_\infty^*(\mathcal{T}_n) \rightarrow J^*(\mathcal{T})$ and $J_\infty(\mathcal{T}, \gamma_n^*) \rightarrow J_\infty^*(\mathcal{T})$.

Proof. We now use the following bounds. Let γ_n^* be optimal for $\mathcal{T}_n$ and γ^* be optimal for $\mathcal{T}$. Then,

$$\begin{aligned} & |J_\infty(\mathcal{T}_n, \gamma_n^*) - J_\infty(\mathcal{T}, \gamma^*)| \\ & \leq \max \left(J_\infty(\mathcal{T}_n, \gamma^*) - J_\infty(\mathcal{T}, \gamma^*), J_\infty(\mathcal{T}, \gamma_n^*) - J_\infty(\mathcal{T}_n, \gamma_n^*) \right), \\ & |J_\infty(\mathcal{T}, \gamma_n^*) - J_\infty^*(\mathcal{T})| \\ & \leq |J_\infty(\mathcal{T}, \gamma^*) - J_\infty(\mathcal{T}_n, \gamma^*)| + |J_\infty(\mathcal{T}_n, \gamma_n^*) - J_\infty(\mathcal{T}, \gamma_n^*)|. \end{aligned} \quad (14)$$

Hence, it suffices to show that

$$\begin{aligned} & |J_\infty(\mathcal{T}_n, \gamma^*) - J_\infty(\mathcal{T}, \gamma^*)| \rightarrow 0, \\ & |J_\infty(\mathcal{T}, \gamma_n^*) - J_\infty(\mathcal{T}_n, \gamma_n^*)| \rightarrow 0. \end{aligned}$$

First notice that for any policy γ , $J_\infty(\mathcal{T}_n, \gamma) = \int c(x) \pi_n^\gamma(dx)$ and $J_\infty(\mathcal{T}, \gamma) = \int c(x) \pi^\gamma(dx)$ for all initial states $x_0 \in \mathbb{X}$ as the chains are positive Harris recurrent. Thus, we only need to show that $\rho(\pi_n^{\gamma^*}, \pi^{\gamma^*}) \rightarrow 0$ and $\rho(\pi_n^{\gamma_n^*}, \pi_n^{\gamma_n^*})$ weakly since $c \in C_b(\mathbb{X})$, where ρ metrizes the topology of weak convergence.

For a fixed policy γ^* , since γ^* belongs to an equicontinuous family, we have that $\mathcal{T}_n(\cdot|x_n, \gamma^*(x_n)) \rightarrow \mathcal{T}(\cdot|x, \gamma^*(x))$ for any $x_n \rightarrow x$ and $\mathcal{T}(\cdot|x, \gamma^*(x))$ is weakly continuous in x . Since $\pi_n^{\gamma^*}$ is a tight family, there exists a subsequence $\pi_{n_k}^{\gamma^*}$ such that $\pi_{n_k}^{\gamma^*} \rightarrow \pi^*$ weakly for some $\pi^* \in \mathcal{P}(\mathbb{X})$. As $\pi_{n_k}^{\gamma^*}$ is the invariant measure for $\mathcal{T}_{n_k}$ we have that for any $f \in C_b(\mathbb{X})$

$$\int f(x_1) \mathcal{T}_{n_k}(dx_1|x_0, \gamma^*(x_0)) \pi_{n_k}^{\gamma^*}(dx_0) = \int f(x_0) \pi_{n_k}^{\gamma^*}(dx_0)$$

using the assumption that $\mathcal{T}_{n_k}(\cdot|x_{n_k}, \gamma^*(x_{n_k})) \rightarrow \mathcal{T}(\cdot|x, \gamma^*(x))$ weakly for any $x_{n_k} \rightarrow x$ and that $\pi_{n_k}^{\gamma^*} \rightarrow \pi^*$ by taking the limit $k \rightarrow \infty$, Lemma 3 gives us

$$\int f(x_1) \mathcal{T}(dx_1|x_0, \gamma^*(x_0)) \pi^*(dx_0) = \int f(x_0) \pi^*(dx_0).$$

Since $\mathcal{T}$ has a unique invariant measure we can conclude that $\pi^* = \pi^{\gamma^*}$.

So far we have proved that any converging subsequence of $\pi_n^{\gamma^*}$ converges weakly to π^{γ^*} . Now suppose that $\pi_n^{\gamma^*}$ do not converge to π^{γ^*} . Then, there exists an $\epsilon > 0$ and a further subsequence $\pi_{n_k}^{\gamma^*}$ such that $\rho(\pi_{n_k}^{\gamma^*}, \pi^{\gamma^*}) > \epsilon$ for all k . But because of the tightness assumption there exists a further subsequence $\pi_{n_{k_l}}^{\gamma^*}$ which converges and using the same arguments above it converges to π^{γ^*} which leads us to a contradiction and completes the proof.

For the sequence of policies γ_n^* , using the equicontinuity assumption with Arzelà-Ascoli theorem ([3]), there exists a subsequence $\gamma_{n_k}^*$ converging uniformly to some $\gamma \in \Gamma$. Therefore, $\mathcal{T}_{n_k}(\cdot|x_{n_k}, \gamma_{n_k}^*(x_{n_k})) \rightarrow \mathcal{T}(\cdot|x, \gamma(x))$ and $\mathcal{T}(\cdot|x_{n_k}, \gamma_{n_k}^*(x_{n_k})) \rightarrow \mathcal{T}(\cdot|x, \gamma(x))$ for any $x_{n_k} \rightarrow x$. Hence, using the same steps above, we can show that $\pi_{n_k}^{\gamma_{n_k}^*} \rightarrow \pi^\gamma$ and $\pi^{\gamma_{n_k}^*} \rightarrow \pi^\gamma$ weakly. This completes the proof. $\square$

Corollary 1. *If $\mathbb{X}$ is a finite space, $\mathbb{U}$ is compact, $\mathcal{T}(x, u)$ is weakly continuous in u : $c(x, u)$ is continuous in u and under every stationary policy state process $\{x_t\}$ is positive Harris recurrent then we have that $J_\infty^*(\mathcal{T}_n) \rightarrow J^*(\mathcal{T})$ and $J_\infty(\mathcal{T}, \gamma_n^*) \rightarrow J_\infty^*(\mathcal{T})$.*

Example 4. *Suppose a controlled model is given by the dynamics*

$$x_{t+1} = Ax_t + Bu_t + w_t$$

where w_t is a i.i.d. Gaussian noise process. Assume that A and B are unknown by the controller. However, the controller can estimate A and B in a consistent way so that $A_n \rightarrow A$ and $B_n \rightarrow B$. Then, it can be shown that for the corresponding transition kernels we have $\mathcal{T}_n(\cdot|x_n, u_n) \rightarrow \mathcal{T}(\cdot|x, u)$ weakly for any $x_n \rightarrow x$ and

$u_n \rightarrow u$. If further we have that the step-wise cost function is in the form $c(x, u) = x^T Qx + u^T Ru$, then the optimal policies are linear and also equicontinuous if the model is controllable. Furthermore, since the noise is Gaussian, the chain is Lebesgue irreducible and thus there exists a unique invariant measure and it can be reached from any initial point. Hence, the assumptions of Theorem 9 are satisfied and the continuity and robustness can be established for this example. In other words; the optimal cost for the estimates A_n and B_n , $J_\infty^*(\mathcal{T}_n)$, converges to optimal cost for A and B , $J_\infty^*(\mathcal{T})$. Also, if we apply the optimal policies γ_n^* designed for A_n and B_n , to the true model (A, B) we get $J_\infty(\mathcal{T}, \gamma_n^*) \rightarrow J_\infty^*(\mathcal{T})$.

6 Conclusion

We studied regularity properties of optimal cost for the average infinite horizon setups on the space of transition kernels, and applications to robustness of optimal control policies designed for an incorrect model applied to an actual system. We made the observation that sufficient conditions presented for infinite horizon discounted optimal cost functions are usually not enough for average cost problems and we need further conditions under weak convergence of transition kernels. We also saw that with some ergodicity properties on the controlled Markov models, total variation convergence of transition kernels guarantees continuity and robustness.

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A Proof of (8)

We first write

$$v_n^*(x_n) - v^*(x) = (v_n^*(x_n) - v_n^t(x_n)) + (v_n^t(x_n) - v^t(x)) + (v^t(x) - v^*(x))$$

where v_n^t and v^t are the results of operator (4) applied to the 0-function, t times for kernels the $\mathcal{T}_n$ and $\mathcal{T}$. Notice that v_n^t and v^t are the value functions for t -step cost problem and by the assumptions ([13, Theorem 4.4]) we have that $|v_n^t(x_n) - v^t(x)| \rightarrow 0$ for every fixed t .

For the first and the last terms, we use the fact that the operator (4) is a contraction under Assumption 2 for span semi-norm and hence both terms go

to some constants as $t \rightarrow \infty$ uniformly for all n , that is $v_n^*(x_n) - v_n^t(x_n) \rightarrow c_1$ and $v^t(x) - v^*(x) \rightarrow c_2$ for some $|c_1|, |c_2| < \infty$. Thus, we have that $\lim_{n \rightarrow \infty} (v_n^*(x_n) - v^*(x)) = c$ for some $c < \infty$.

B Proof of (10)

We first define the following operator $\hat{T}_{n,z}$ for some fixed $z \in \mathbb{X}$ by

$$\hat{T}_{n,z}v(x) := \hat{T}_nv(x) - \hat{T}_nv(z)$$

where $\hat{T}_n$ is as in (9). We use triangle inequality to write

$$\hat{v}_n^*(x_n) - v^*(x) = (\hat{v}_n^*(x_n) - \hat{T}_{n,z}^k v^*(x_n)) + (\hat{T}_{n,z}^k v^*(x_n) - v^*(x))$$

where $\hat{T}_{n,z}^k$ is the operator $\hat{T}_{n,z}$ applied k consecutive times. We note that $\hat{T}_{n,z}$ is also a contraction under the *span* semi-norm. Hence, the first term converges to some $\hat{c}_1$ as $k \rightarrow \infty$ uniformly for all n since $\hat{T}_{n,z}$ is a contraction uniformly for all n under the span seminorm and its fixed point is $\hat{v}_n^*$.

For the second term, we wish to show that $(\hat{T}_{n,z}^k v^*(x_n) - v^*(x)) \rightarrow -v^*(z)$ as $n \rightarrow \infty$ for every fixed $k < \infty$ for all $x_n \rightarrow x$.

We prove this by induction. For $k = 1$, we have

$$\begin{aligned} \hat{T}_{n,z}v^*(x_n) &= c(x_n, \gamma_n^*(x_n)) + \int v^*(y) \mathcal{T}(dy|x_n, \gamma_n^*(x_n)) \\ &\quad - \left(c(z, \gamma_n^*(z)) + \int v^*(y) \mathcal{T}(dy|z, \gamma_n^*(z)) \right). \end{aligned}$$

Now assume that $|\hat{T}_{n_k,z}v^*(x_{n_k}) - v^*(x) + v^*(z)| > \epsilon$ for some $\epsilon > 0$ for every k along some subsequence n_k . We know that there exists a further subsequence, say n_{k_l} along which $\gamma_{n_{k_l}}^*(x_{n_{k_l}}) \rightarrow u^*$ and $\gamma_{n_{k_l}}^*(z) \rightarrow u^{**}$ for some $u^*, u^{**} \in \mathbb{U}$ where u^* is an optimal action for the state x for the kernel $\mathcal{T}$ and u^{**} is optimal for z . If we take the limit along this subsequence, i.e.

$$\begin{aligned} \lim_{l \rightarrow \infty} \hat{T}_{n_{k_l},z}v^*(x_{n_{k_l}}) &= \lim_{l \rightarrow \infty} \left(c(x_{n_{k_l}}, \gamma_{n_{k_l}}^*(x_{n_{k_l}})) + \int v^*(y) \mathcal{T}(dy|x_{n_{k_l}}, \gamma_{n_{k_l}}^*(x_{n_{k_l}})) \right) \\ &\quad - \lim_{l \rightarrow \infty} \left(c(z, \gamma_{n_{k_l}}^*(z)) + \int v^*(y) \mathcal{T}(dy|z, \gamma_{n_{k_l}}^*(z)) \right) \\ &= c(x, u^*) + \int v^*(y) \mathcal{T}(dy|x, u^*) - \left(c(z, u^{**}) + \int v^*(y) \mathcal{T}(dy|z, u^{**}) \right) \\ &= v^*(x) + j^* - v^*(z) - j^* = v^*(x) - v^*(z). \end{aligned}$$

Hence we reach a contradiction and it must be that $\hat{T}_{n,z}v^*(x_n) - v^*(x) \rightarrow -v^*(z)$.
Now assume that the claim holds for k that is $(\hat{T}_{n,z}^k v^*(x_n) - v^*(x)) \rightarrow -v^*(z)$.

$$\begin{aligned}\hat{T}_{n,z}^{k+1}v^*(x_n) &= c(x_n, \gamma_n^*(x_n)) + \int \hat{T}_{n,z}^k v^*(y) \mathcal{T}(dy|x_n, \gamma_n^*(x_n)) \\ &\quad - \left(c(z, \gamma_n^*(z)) + \int \hat{T}_{n,z}^k v^*(y) \mathcal{T}(dy|z, \gamma_n^*(z)) \right).\end{aligned}$$

If we take a subsequence, indexed by say m , along which $\gamma_m^*(x_m) \rightarrow u^*$ and $\gamma_m^*(z) \rightarrow u^{**}$ where u^* is an optimal action for x and u^{**} is optimal for z , then by taking the limit along this subsequence we get

$$\begin{aligned}\lim_{m \rightarrow \infty} \hat{T}_m^{k+1}v^*(x_m) &= \lim_{m \rightarrow \infty} c(x_m, \gamma_m^*(x_m)) + \int \hat{T}_m^k v^*(y) \mathcal{T}(dy|x_m, \gamma_m^*(x_m)) \\ &\quad - \lim_{m \rightarrow \infty} c(z, \gamma_m^*(z)) + \int \hat{T}_m^k v^*(y) \mathcal{T}(dy|z_m, \gamma_m^*(z)) \\ &= c(x, u^*) + \int (v^*(y) - v^*(z)) \mathcal{T}(dy|x, u^*) \\ &\quad - \left(c(z, u^{**}) + \int (v^*(y) - v^*(z)) \mathcal{T}(dy|z, u^{**}) \right) \\ &= v^*(x) - v^*(z).\end{aligned}$$

Hence, a similar contradiction argument, we used for the case $k = 1$ yields that $(\hat{T}_{n,z}^k v^*(x_n) - v^*(x)) \rightarrow -v^*(z)$ as $n \rightarrow \infty$ for every fixed $k < \infty$ for all $x_n \rightarrow x$.

Thus, we have that

$$\hat{v}_n^*(x_n) - v^*(x) = (\hat{v}_n^*(x_n) - \hat{T}_{n,z}^k v^*(x_n)) + (\hat{T}_{n,z}^k v^*(x_n) - v^*(x)) \rightarrow \hat{c}_1 - v^*(z) := \hat{c}.$$

References

- [1] P. Billingsley. *Probability and Measure*. Wiley, 3rd ed.), New York, 1995.
- [2] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu. On the sample complexity of the linear quadratic regulator. *arXiv preprint arXiv:1710.01688v2*, 2018.
- [3] R. M. Dudley. *Real Analysis and Probability*. Cambridge University Press, Cambridge, 2nd edition, 2002.
- [4] P. Dupuis, M. R. James, and I. Petersen. Robust properties of risk-sensitive control. *Mathematics of Control, Signals and Systems*, 13(4):318–332, 2000.

- [5] E. Erdoğan and G. N. Iyengar. Ambiguous chance constrained problems and robust optimization. *Mathematical Programming*, 107(1-2):37–61, 2005.
- [6] P. M. Esfahani and D. Kuhn. Data-driven distributionally robust optimization using the Wasserstein metric: Performance guarantees and tractable reformulations. *Mathematical Programming*, pages 1–52, 2017.
- [7] J. K. Ghosh and R. V. Ramamoorthi. *Bayesian Nonparametrics*. Springer, New York, 2003.
- [8] R. M. Gray. *Entropy and Information Theory*. Springer-Verlag, New York, 1990.
- [9] O. Hernández-Lerma. *Adaptive Markov Control Processes*. Springer-Verlag, 1989.
- [10] O. Hernández-Lerma, R. Montes-De-Oca, and R. Cavazos-Cadena. Recurrence conditions for Markov decision processes with Borel state space: a survey. *Ann. Oper. Res.*, 28(1):29–46, 1991.
- [11] D. Jacobson. Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games. *IEEE Transactions on Automatic control*, 18(2):124–131, 1973.
- [12] A. D. Kara and S. Yüksel. Robustness to incorrect system models in stochastic control and application to data-driven learning. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 2753–2758, Dec 2018.
- [13] A. D. Kara and S. Yüksel. Robustness to incorrect system models in stochastic control. *SIAM Journal on Control and Optimization*, to appear (also on *arXiv:1803.06046*), 2020.
- [14] M. L. Kleptsyna and A. Yu. Veretennikov. On robustness of discrete time optimal filters. *Mathematical Methods of Statistics*, 25(3):207–218, 2016.
- [15] Hans-Joachim Langen. Convergence of dynamic programming models. *Mathematics of Operations Research*, 6(4):493–512, 1981.
- [16] K.R. Parthasarathy. *Probability Measures on Metric Spaces*. AMS Bookstore, 1967.
- [17] I. Petersen, M. R. James, and P. Dupuis. Minimax optimal control of stochastic uncertain systems with relative entropy constraints. *IEEE Transactions on Automatic Control*, 45(3):398–412, 2000.

- [18] P. Dai Pra, L. Meneghini, and W. J. Runggaldier. Connections between stochastic control and dynamic games. *Mathematics of Control, Signals and Systems*, 9(4):303–326, 1996.
- [19] H. L. Royden. *Real Analysis*. Macmillan, New York, 1968.
- [20] A. V. Savkin and I. R. Petersen. Robust control of uncertain systems with structured uncertainty. *Journal of Mathematical Systems, Estimation, and Control*, 6(3):1–14, 1996.
- [21] R. Serfozo. Convergence of lebesgue integrals with varying measures. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 380–402, 1982.
- [22] H. Sun and H. Xu. Convergence analysis for distributionally robust optimization and equilibrium problems. *Mathematics of Operations Research*, 41:377–401, 07 2015.
- [23] V. A. Ugrinovskii. Robust H-infinity control in the presence of stochastic uncertainty. *International Journal of Control*, 71(2):219–237, 1998.
- [24] C. Villani. *Optimal transport: old and new*. Springer, 2008.
- [25] L. Xie and E. de Souza Carlos. Robust H infinity control for linear systems with norm-bounded time-varying uncertainty. *IEEE Transactions on Automatic Control*, 37(8):1188–1191, Aug 1992.