arXiv:1409.0268v1 [math-ph] 31 Aug 2014

Totally Asymmetric Simple Exclusion Process by means of Probabilistic Cellular Automata

Benedetto Scoppola, Carlo Lancia, Riccardo Mariani [1]

[1] Dipartimento di Matematica, University of Roma "Tor Vergata"
Via della Ricerca Scientifica - 00133 Roma, Italy
scoppola@mat.uniroma2.it

Abstract

We define a particle system in which each particle has an independent probability p to jump at each step, with the rule of the usual Totally Asymmetric Simple Exclusion Process. We find the stationary distribution of this system on the circle, which is not uniform in this case, and we compute the current. We define then the blockage problem (see [4, 3]) for this system, and we find the exact solution of the stationary distribution for the case $p = 1$, showing that the current decreases for all the blockage probabilities. We present some numerical results for the case $p < 1$, listing some open problems.

1 Introduction

The Totally Asymmetric Simple Exclusion Process (TASEP) is one of the more popular example of discrete particle system driven by a Markov irreversible dynamics [6, 7]. The system can be defined, in finite space, on a discrete segment $\Lambda = \{1, 2, ..., 2L\}$, or on a discrete circle, imposing periodic boundary condition to the segment. A configuration $\sigma \in \{0, 1\}^{\Lambda}$ can be viewed as a set of particles living in Λ. The dynamics is defined by two steps: first, a particle is chosen uniformly at random. Second, the chosen particle is moved to its right if the site on its right is empty. If such site is occupied, the configuration does not change and a new particle is chosen. The system can also be defined, with continuous time, on the whole $\mathbb{Z}$; however, in this paper we will consider only finite space.

Despite its simplicity, this model has many interesting features. On a finite circle the stationary measure is uniform because its transition matrix is doubly Markov, while on the

finite segment the stationary state depends on the boundary probability to enter (say on the left) and to exit (on the right) from the system (see [2, 9]). An interesting quantity to measure is the *current*, defined as the probability to have a particle in a given site, with an empty site on its right. This quantity does not depend on the point where it is computed. The current can be exactly computed in the models mentioned above, and considering, for instance, the model defined on the circle, it is possible to see that the current depends only on the number of particles in the system, and it is maximum (and equal to $1/4$ in the limit $L \to \infty$) when the system is half-filled, i.e. when there are L particles in the circle $\Lambda = \{1, 2, ..., 2L\}$.

A very natural question in the study of the irreversible system is whether the effect of small perturbations in the dynamics has local effects, as in the case of reversible system far from critical points, or global effects. In the case of the TASEP this question has been investigated imposing the so-called *blockage* ([4, 3]): in a defined point (say, without loss of generality, in the point $2L$ of the circle) the probability to jump to the empty site 1 if the particle in $2L$ is selected is $1 - \varepsilon$ with $\varepsilon > 0$. The question about global effects on the system consists here in the evaluation of the current: if a blockage in a single point affects the value of the current then the effects of that blockage are obviously global. Unfortunately it is not clear if the presence of the blockage of intensity ε has global effects for all $\varepsilon > 0$. It is believed (see [1]) that the current decreases, for small ε, with a non analytical dependence on ε. Numerical evaluation of the current suggest that it is also possible that there is a critical value $\varepsilon_c > 0$ such that for $\varepsilon < \varepsilon_c$ the current does not change.

In this paper we provide a slightly different model, in which at each step each particle followed by an empty site has a finite probability p to jump. We call this parallel dynamics PCA-TASEP. We show that this model has similar features with respect to the standard TASEP; in particular, considering the blockage problem, we see numerically that for $p < 1$ the behaviour of the current is very similar to the standard TASEP case, because for small ε the current seems to be equal to the unblocked one. On the other hand, for $p = 1$, the system is exactly solvable and it can be proved that $\varepsilon_c = 0$.

The paper is organised as follows: in Section 2 we define our parallel irreversible dynamics on the circle, and we compute its stationary distribution. In Section 3 we compute explicitly the current of our model, showing how the standard TASEP can be

considered a particular case ($p \approx 1/L$) of our PCA-TASEP. In Section 4 we define the blockage problem in this parallel context, and we solve it in the particular case $p = 1$. Finally, in Section 5.2 we present some numerical results, and we list some open problems that these results seem to suggest.

2 Definitions

We define a Markov chain on a discrete circle, i.e. on the set $\Lambda = \{1, 2, ..., 2L\}$ with periodic boundary conditions. As in the case of standard TASEP, a configuration σ is a point in the space $\{0, 1\}^\Lambda$. We will denote with σ_i the local configuration of σ in the point i and we will say that in the site i there is a particle if $\sigma_i = 1$, while we will say that the site i is empty, or equivalently that in the site i there is a hole, if $\sigma_i = 0$. We will say that in i we have a particle free to move if $\sigma_i = 1$ and $\sigma_{i+1} = 0$. To move a particle means to substitute the values $\sigma_i = 1$ and $\sigma_{i+1} = 0$ with $\sigma_i = 0$ and $\sigma_{i+1} = 1$. Given a configuration σ we will define $m(\sigma) = \sum_{i=1}^{L} \sigma_i$ and we will assume from now on that $m(\sigma) \leq L$. The dynamics is such that $m(\sigma)$ is conserved. We will denote with $l(\sigma) \leq m(\sigma)$ the number of particles free to move in the configuration σ. We now define the weight of the transition from a configuration σ to a configuration τ in the following way

$$w(\sigma, \tau) = \begin{cases} w^n & \text{if } \tau \text{ can be reached by moving } n \text{ particles in } \sigma, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $w > 0$ is a positive parameter measuring the tendency to move of each particle. The transition probabilities are therefore

$$P(\sigma, \tau) = \frac{w(\sigma, \tau)}{w(\sigma)}, \tag{2}$$

where

$$w(\sigma) = \sum_{\tau} w(\sigma, \tau) = \sum_{k=1}^{l(\sigma)} \binom{l(\sigma)}{k} w^k = (1 + w)^{l(\sigma)}. \tag{3}$$

We call this Markov chain a PCA-TASEP. This dynamics is really similar to a standard TASEP when w is very small (say $w < 1/L$). For finite values of w the dynamics is a TASEP in which at each step each particle free to move actually moves with probability $w/(1 + w)$. When $w \to \infty$ all the free particles moves contemporarily at each step. This Markov chain is manifestly irreversible, but it is easy to see that for all the configuration

the *global balance principle* [5]

$$\sum_{\tau} w(\sigma,\tau) = \sum_{\tau} w(\tau,\sigma) \tag{4}$$

is verified, because for each sequence of consecutive 1 there is a right endpoint that can be moved and a left endpoint that can be reached by a move. Property (4) is also known in literature as *dynamical reversibility*. Due to this property it is easy to verify that the stationary measure of the process is $\pi(\sigma) = w(\sigma)/W$, where $W = \sum_{\sigma} w(\sigma)$. In fact,

$$\sum_{\sigma} \pi(\sigma)P(\sigma,\tau) = \sum_{\sigma} \frac{w(\sigma)}{W}\frac{w(\sigma,\tau)}{w(\sigma)} = \sum_{\sigma} \frac{w(\sigma,\tau)}{W} = \sum_{\sigma} \frac{w(\tau,\sigma)}{W} = \frac{w(\tau)}{W} = \pi(\tau)\,. \tag{5}$$

By (3), the stationary measure is

$$\pi(\sigma) = \frac{(1+w)^{l(\sigma)}}{W}\,. \tag{6}$$

Note that when $w \to 0$ the measure tends to the uniform measure on all the configurations with fixed $m(\sigma)$, while in the limit $w \to \infty$ the measure tends to be uniform on all the configurations with $l(\sigma) = m(\sigma)$, since all the other configurations have a weight smaller by a factor at least $1/(1+w)$.

3 Current for the half-filled PCA-TASEP

As discussed in the introduction, the current J in the standard TASEP is, for a certain site i, the quantity $J = \lim_{\Lambda\to\infty} \pi_{TASEP}(\sigma_i = 1, \sigma_{i+1} = 0)$, where π_{TASEP} is the stationary distribution of the model. The event $\sigma_i = 1, \sigma_{i+1} = 0$ is independent of i, but it depends only on the number of particles $m(\sigma)$.

In what follows, we consider the half-filled case $m(\sigma) = L$, i.e. the number of particles is exactly one half of the number of sites. Then, for the standard TASEP,

$$J = \pi_{TASEP}(\sigma_i = 1, \sigma_{i+1} = 0) \sim \pi_{TASEP}(\sigma_i = 1)\pi_{TASEP}(\sigma_{i+1} = 0) = 1/4$$

since the stationary probability π_{TASEP} is uniform, and therefore the probability to have the configuration $\sigma_i = 1, \sigma_{i+1} = 0$ tends, for large Λ, to the product of the probabilities to have $\sigma_i = 1$ and $\sigma_{i+1} = 0$ (and they both equal $1/2$).

We want now to find the value J^{PCA} of the current in the case of the stationary probability defined in (6). We prove the following lemma

Lemma 1

The value of the current $J^{PCA} = \lim_{\Lambda\to\infty} \pi(\sigma_i = 1, \sigma_{i+1} = 0)$ *for the irreversible Markov chain defined by the transition probabilities in (2) is given by*

$$J^{PCA} = \frac{1}{2}\ \frac{\sqrt{1+w}}{1+\sqrt{1+w}}\,. \tag{7}$$

Proof:

We rewrite J^{PCA} in the following form

$$J^{PCA} = \lim_{L\to\infty} \frac{1}{2L}\ \frac{\sum_\sigma l(\sigma)(1+w)^{l(\sigma)}}{\sum_\sigma (1+w)^{l(\sigma)}} = \lim_{L\to\infty} \frac{1}{2L}\ \frac{\sum_{l=1}^{L} l\ n(l)(1+w)^l}{\sum_{l=1}^{L} n(l)(1+w)^l}\,, \tag{8}$$

where $n(l)$ is the number of configurations σ having $l(\sigma) = l$ particles free to move. We have therefore to count such $n(l)$. This can be done as follows: we have to count the number of ways to divide L particles in l distinct groups, and the number to divide L holes in l distinct groups. Then we count the number of configurations having $\sigma_1 = 1$ simply by fixing the number l_1 of particles in the first subset of particles and multiplying the number of configurations by l_1, due to the fact that the first set of particles has l_1 ways to choose inside it the particle in the site $i = 1$. We then multiply by 2, since the number of configurations having $\sigma_1 = 0$ is the same. Since n objects can be divided in l groups in $\binom{n-1}{l-1}$ ways,

$$n(l) = 2\sum_{l_1=1}^{L-l+1} l_1 \binom{L-l_1-1}{l-2}\binom{L-1}{l-1}\,. \tag{9}$$

If we put (9) into (8),

$$J^{PCA} = \lim_{L\to\infty} \frac{1}{2L}\ \frac{\sum_{l=1}^{L}\sum_{l_1=1}^{L-l+1} l_1\binom{L-l_1-1}{l-2}\binom{L-1}{l-1}\ l\ (1+w)^l}{\sum_{l=1}^{L}\sum_{l_1=1}^{L-l+1} l_1\binom{L-l_1-1}{l-2}\binom{L-1}{l-1}(1+w)^l}\,. \tag{10}$$

In order to evaluate (10) we write $l = \alpha L$, $l_1 = \alpha_1 L$ and we use the leading order approximation

$$\binom{n}{\alpha n} \approx e^{nI(\alpha)}\,, \tag{11}$$

where $I(\alpha) = -\alpha \ln \alpha - (1-\alpha)\ln(1-\alpha)$. Then,

$$J^{PCA} = \lim_{L\to\infty} \frac{1}{2L}\, L \frac{\int_0^1 d\alpha \int_0^{1-\alpha} d\alpha_1 \; \alpha \; \alpha_1 \exp\left[L\left((1-\alpha_1) I(\frac{\alpha}{1-\alpha_1}) + I(\alpha) + \alpha \ln(1+w)\right)\right]}{\int_0^1 d\alpha \int_0^{1-\alpha} d\alpha_1 \; \alpha_1 \exp\left[L\left((1-\alpha_1) I(\frac{\alpha}{1-\alpha_1}) + I(\alpha) + \alpha \ln(1+w)\right)\right]}\,. \tag{12}$$

Calling now

$$f(\alpha, \alpha_1) = (1-\alpha_1) I\left(\frac{\alpha}{1-\alpha_1}\right) + I(\alpha) + \alpha \ln(1+w)\,,$$

it is a standard argument (saddle point method) to write $J^{PCA} = \lim_{L\to\infty} \left[\frac{1}{2}\bar{\alpha} + O\left(\frac{1}{L}\right)\right]$, where $\bar{\alpha}$ is the value of α that maximises $f(\alpha, \alpha_1)$. The latter is easy to identify because by deriving $f(\alpha, \alpha_1)$ we see that it is decreasing in α_1, and that, choosing $\alpha_1 = 0$, $\bar{\alpha} = \frac{\sqrt{1+w}}{1+\sqrt{1+w}}$. This yields

$$J^{PCA} = \frac{1}{2}\, \frac{\sqrt{1+w}}{1+\sqrt{1+w}}\,. \tag{13}$$

Q.E.D.

Note that for $w = O\left(\frac{1}{L}\right)$ we have that, as expected, $J^{PCA} = \frac{1}{4}$ as in the standard TASEP. Such current is an increasing function of w giving, for $w \to \infty$, $J^{PCA} = \frac{1}{2}$.

4 An (easy) blockage problem for the half-filled PCA-TASEP

As mentioned in the introduction, a very interesting and very difficult open problem for the standard TASEP is the following (see [1] and references therein). Assume that the dynamics is defined by the fact that at each step a particle is chosen u.a.r., and if it is free to move it is moved with probability 1, except in the case in which the particle occupy the site $2L$. In this case, if it is free to move (i.e. if $\sigma_1 = 0$), it is moved with probability $1-\varepsilon$. We do not have yet an explicit expression for the current in this model. Numerical and perturbative arguments show that, considering always the half-filled case in which the number of particles is $m(\sigma) = L$, the current seems to remain equal to its maximum value $J = \frac{1}{4}$ until a critical value of ε. However this has not been rigorously proved, and it is also possible that the expression of J as a function of ε starts from a very high order in ε, or also (as it is conjectured in [1]) that J has an essential singularity in $\varepsilon = 0$.

An analogous way to face a blockage problem in this PCA case is the following: we define

the weight of the transition from the configuration σ to the configuration τ by

$$w(\sigma,\tau)=\begin{cases} w^n(1-\varepsilon\ \chi(\sigma_{2L}=1,\tau_{2L}=0)) & \text{if } \tau \text{ can be reached by moving } n \text{ particles in } \sigma, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where $0<\varepsilon<1$ and $\chi(\sigma_{2L}=1,\tau_{2L}=0)$ is 1 if both condition $\sigma_{2L}=1$ and $\tau_{2L}=0$ are verified, and 0 otherwise. Define the PCA-TASEP with blockage by the following transition probabilities

$$P_\varepsilon(\sigma,\tau)=\frac{w(\sigma,\tau)}{w(\sigma)}\,. \tag{15}$$

In this case the global balance principle is not satisfied by the chain, and therefore we are not able to find the stationary measure of the process.

We will discuss from a numerical point of view in the section below this problem for finite values of w. It seems that the problem is similar to the classical TASEP case: the current seems to be constant until ε reaches a critical value, which is a decreasing function of w. The possibility to prove rigorously such a behaviour seems to be, however, as hard as in the standard TASEP case. There is, however, an exception, which is the case $w\to\infty$, i.e. the case in which *all* the particles free to move actually moves, except the particle in the site $2L$, that moves with probability $1-\varepsilon$. This case is easy to solve due to the circumstance that the dynamics preserves a symmetry.

Definition 1

A configuration σ such that, for all $i=1,2,...,L/2$, $\sigma_i=1-\sigma_{2L-i+1}$ is said to be particle-hole symmetric. Let us denote with PH the set of all the particle-hole symmetric configurations

The particle-hole symmetry is preserved by the dynamics when $w\to\infty$.

Lemma 2

Consider the transition probabilities

$$P_{\varepsilon,\infty}(\sigma,\tau)=\lim_{w\to\infty}P_\varepsilon(\sigma,\tau)\,. \tag{16}$$

For all $0<\varepsilon<1$ and for all $\sigma\in PH$, we have that if τ is such that $P_{\varepsilon,\infty}(\sigma,\tau)>0$ then

$\tau \in PH$.

Proof:
Since the configuration σ is symmetric, $\sigma_i = 1 - \sigma_{2L-i+1}$ and, moreover, $\sigma_{i+1} = 1 - \sigma_{L-i}$ and $\sigma_{i-1} = 1 - \sigma L - i + 2$. We can prove that after a step of our dynamics $\tau_i = 1 - \tau_{2L-i+1}$ by considering all the possible local configurations $(\sigma_{i-1}\sigma_i, \sigma i + 1)$. Suppose for instance that initially in i there is a particle, $\sigma_i = 1$. If the particle is free to move, i.e. if $\sigma_{i+1} = 0$, due to the particle-hole symmetry we will have that in $L - i$ there is a particle free to move. Hence $\tau_i = 0$ and $\tau_{L-i+1} = 1$. If the particle in i is not free to move, i.e. if $\sigma_{i+1} = i$, due to the particle-hole symmetry we will have holes both in $L - i$ and in $L - i + 1$. Hence $\tau_i = 1$ and $\tau_{L-i} = 0$ The two remaining case can be treated analogously, considering the configuration of σ in the sites $i - 1$ and $L - i + 1$. For the sites 1 and L the proof is similar.

Q.E.D.

We want now to show two lemmas in order to identify the set of recurrent states of our Markov chain $P_{\varepsilon,\infty}(\sigma, \tau)$.

Lemma 3
For the Markov chain $P_{\varepsilon,\infty}(\sigma, \tau)$ all the states $\sigma \notin PH$ are transient.

Proof:
An easy way to prove this lemma is the following observation: for all initial configurations, with a finite probability $p > \varepsilon^{2L}$ the system goes in a time $2L$ in the configuration σ_{queue} such that $\sigma_i = 0$ for $i = 1, 2, ..., L$ and $\sigma_i = 1$ for $i = L + 1, L + 2, ..., 2L$: if the particle in $2L$ do not move for a time $2L$ then all the other particles will be in queue behind it. The state σ_{queue} is of course symmetric. Hence starting from any initial state $\sigma \notin PH$ we have a finite probability to arrive in a symmetric state after $2L$ steps, and hence with finite probability we will never visit again σ.

Q.E.D.

The next easy but important observation is the following: if the system reach at some time a symmetric configuration in which all the particles in $\{1, 2, ..., L\}$ are free to move, then in that half of the circle all the particles will be free to move at any subsequent time. This

is due to the fact that all the particles in $\{1, 2, ..., L-1\}$ can never reach the preceding particle, since they all move at any step with probability one, and if in $i = L$ there is a particle, being the state symmetric, it is free to move because the related site which has to have a hole is the site $i = 2L - L + 1 = L + 1$ which is the following site. We will call Ω_∞ the set of states $\sigma \in PH$ such that all the particles in $\{1, 2, ..., L\}$ are free to move.

Lemma 4

For the Markov chain $P_{\varepsilon,\infty}(\sigma, \tau)$ all the states $\sigma \notin \Omega_\infty$ are transient.

Proof:

Again, any state $\sigma \notin \Omega_\infty$ have a finite probability to arrive in the configuration σ_{queue} in time $2L$ and $\sigma_{queue} \in \Omega_\infty$. The lemma is proved repeating the argument leading to Lemma 3.

Q.E.D.

The stationary probability $\pi_{\varepsilon,\infty}$ of the Markov chain $P_{\varepsilon,\infty}(\sigma, \tau)$ is therefore supported on Ω_∞, where the Markov chain is manifestly ergodic.

Call as before $J_{\varepsilon,\infty} = \lim_{\Lambda\to\infty} \pi_{\varepsilon,\infty}(\sigma_i = 1, \sigma_{i+1} = 0)$. Calling $r(\sigma)$ the number of particles in $\{1, 2, ..., L\}$, that are all free to move, and observing that the number of particles free to move in $\{L + 1, L + 2, ..., 2L\}$ is again $r(\sigma)$, we can compute $J_{\varepsilon,\infty}$ by computing the average value $R = \pi_{\varepsilon,\infty}(r)$ and writing

$$J_{\varepsilon,\infty} = \lim_{L\to\infty} \frac{R}{L} \,. \tag{17}$$

We are now ready to prove the basic result of this section.

Proposition 1

The current $J_{\varepsilon,\infty}$ of the Markov chain $P_{\varepsilon,\infty}(\sigma, \tau)$ is given by

$$J_{\varepsilon,\infty} = \frac{1-\varepsilon}{2-\varepsilon} \,. \tag{18}$$

Proof:

The first part of the proof is the computation of the stationary measure of the process. Let us say that at each step in which $\sigma_{2l} = 1$ the blockage is driven by a binary random variable, that can be "green", giving in the successive step $\tau_{2L} = 0$, or "red", giving in the successive step $\tau_{2L} = 1$. Due to the symmetry we can say that the probability of each state can be written in terms of a sequence of green and red lights. In particular, when the particle has passed the blockage, and therefore $\sigma_{2L} = 0, \sigma_1 = 1$, we know that in the next step we will have for sure $\tau_1 = 0, \tau_2 = 1$. By symmetry, this means that we have now a particle in the site $2L$. Hence the next generation of a particle in the set $\{1, 2, ..., L\}$ is due only to the values of the red light. It is easy to realise that the (stationary) probability to have r particles in $\{1, 2, ..., L\}$ is

$$\pi_{\varepsilon,\infty}(r) = (1-\varepsilon)^r \varepsilon^{L-2r}. \tag{19}$$

The exponent $L-2r$ is due to the fact that each green light occupies the site of the particle and the subsequent one, which is for sure empty. Using (19) we can find the value of the current by (17). We have that

$$R = \sum_{r=1}^{L/2} r \binom{L-r}{r} (1-\varepsilon)^r \varepsilon^{L-2r}. \tag{20}$$

We can again evaluate this sum simply using the approximation in (11) and the saddle point method. We call $x = \frac{r}{L}$ and we rewrite (20) as

$$R \approx L \frac{\int_0^{L/2} x \exp[L((1-2x)\ln\varepsilon + x\ln(1-\varepsilon) - x\ln\frac{x}{1-x} - (1-2x)\ln\frac{1-2x}{1-x})]}{\int_0^{L/2} \exp[L((1-2x)\ln\varepsilon + x\ln(1-\varepsilon) - x\ln\frac{x}{1-x} - (1-2x)\ln\frac{1-2x}{1-x})]}. \tag{21}$$

When $L \to \infty$ we have that $J_{\varepsilon,\infty} = \lim_{L\to\infty} \frac{R}{L} \approx \bar{x}$, being $\bar{x}$ the value of x that maximises

$$f(x) = (1-2x)\ln\varepsilon + x\ln(1-\varepsilon) - x\ln\frac{x}{1-x} - (1-2x)\ln\frac{1-2x}{1-x}.$$

It is a standard task to prove then the statement of the proposition

$$J_{\varepsilon,\infty} = \frac{1-\varepsilon}{2-\varepsilon}. \tag{22}$$

Q.E.D.

Remark

Despite its simplicity, this computation proves, in this completely parallel context, that a very small perturbation of the transition probabilities in a single site extends its effect over all the volume, without any fading: actually the uniform density of the particles in the set $\{1, 2, ..., L\}$ is $\frac{1-\varepsilon}{2-\varepsilon}$ while in the set $\{L+1, L+2, ..., 2L\}$ the uniform density is $\frac{1}{2-\varepsilon}$.

5 Numerical results

In this section we present a series of numerical results obtained in a half-filled PCA-TASEP system with blockage probability ϵ on the last site, on a ring lattice with 500 particles ($L = 250$). In particular we will compute experimentally the current $J(p, \epsilon)$ and the density $\rho(x, p, \epsilon)$ defined as follows: let $x \in \{1, 2, ..., 2L/10\}$. Then

$$\rho(x, p, \epsilon) = \frac{1}{10} \left[\sum_{i=10x}^{10x+9} \sigma_i \right] .$$

The problem of a rigorous evaluation of the mixing time of our model is not studied in this paper. However the argument leading to the evaluation of the stationary measure in the case $p = 1$ shows that, in that case, the system reach the stationary distribution after a time proportional to L. In the general case we run the dynamics for a time

$$T = 2 \, \frac{L}{p} \, log(L) \, .$$

We are not aware of an estimate of the mixing time for the standard TASEP. In the case of the symmetric exclusion process on the circle Morris proved that $T = L^2 \log L$ (see [8]). and this corresponds in our case to the choice $p = \frac{1}{2L}$.

Our numerical experiments are particularly focused on two facts:

1. We know (see [1]) that for the standard TASEP the current remains very close to the limit value without blockage, up to a certain value $\tilde{\epsilon}_c$ of the blockage probability. We know also that for the explicitly solvable blockage system discussed above, with $p = 1$, there is no critical blockage probability, in the sense that the current has a decrease that is proportional to ϵ near the value $\epsilon = 0$. We want to check numerically

if the existence of $\tilde{\epsilon}_c \neq 0$ is a particular feature of the single spin flip dynamic, or if it survives to the PCA procedure. As it is clear in the following figures, the second scenario seems to be true from a numerical point of view.

2. We want to see if the density, when the current decreases, is an increasing function of the site x or if the presence of the blockage implies simply a *queueing* of particles close to it. The latter is the case in the solvable model with $p = 1$.

5.1 Current

The measure of the current J is simply made counting the average number of the particles free to move during the evolution of the system and weighting such value with the total volume of the system $2L$.

Figure 1 shows the surface obtained interpolating 441 measures of J with every combination of the parameters $p \in [\frac{1}{n}, ..., 1]$ and $\epsilon \in [0, ..., 1]$. The figure clearly shows an excellent fit with the currents computed in Lemma 1 and in Proposition 1.

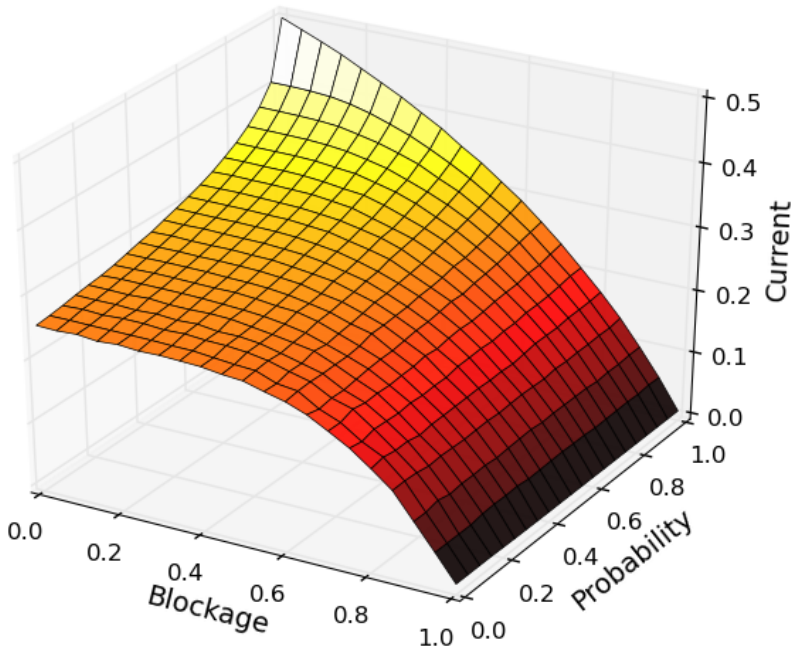


Figure 1: Current for the PCA-TASEP.

Figure 2 presents the behaviour of the current for specific values of p plotting the side projection of the 3D graph. It clearly appears that except in the case $p = 1$, where the current decrease with a finite slope for all $\varepsilon > 0$, the decrease of J starts only after a certain value of the blockage $\tilde{\epsilon}_c$. In this respect, the critical behaviour of the standard TASEP, corresponding in our model to the $p = \frac{1}{n}$ case, seems to be conserved for all the probabilities except $p = 1$.

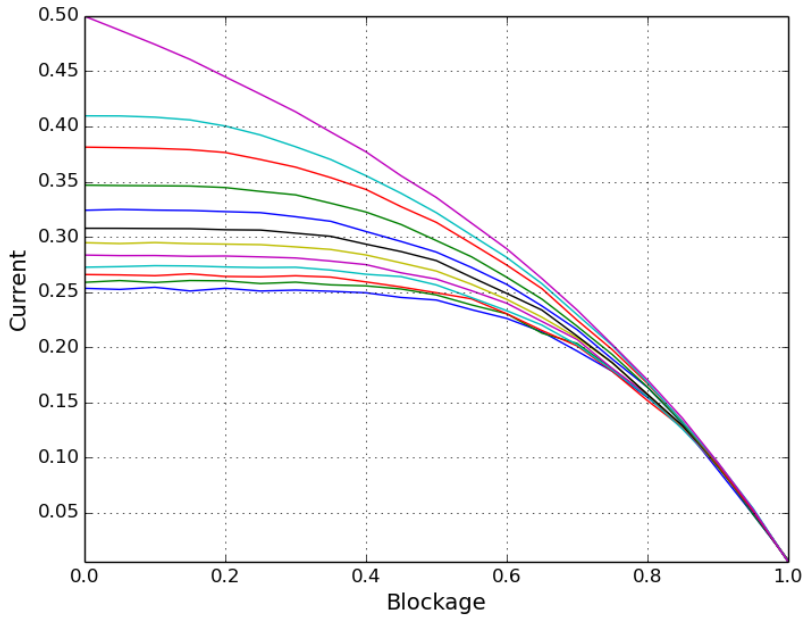


Figure 2: Profile curves of the PCA-TASEP current for different values of the probability p, from $p = 1$ (up) to $p = \frac{1}{n}$ (down).

Figure 3 plots the threshold values of ϵ for which the current deviates more than the 1% from its initial value in the absence of blockage. This gives an indication of the shape of the region in which the current remains numerically constant having however $\varepsilon > 0$.

5.2 Density

The second indication we tried to obtain from our simulation is the distribution of the particles in the whole configuration during the evolution of the system. In order to obtain this, we coarse-grained the particles on segments of length 10, in order to obtain the density ρ defined above. The density diagrams below describe the density of segments of the configuration, assigning a darker colour to the more dense segment.

Every diagram of Figure 4 is an instantaneous plot of the configuration at the time

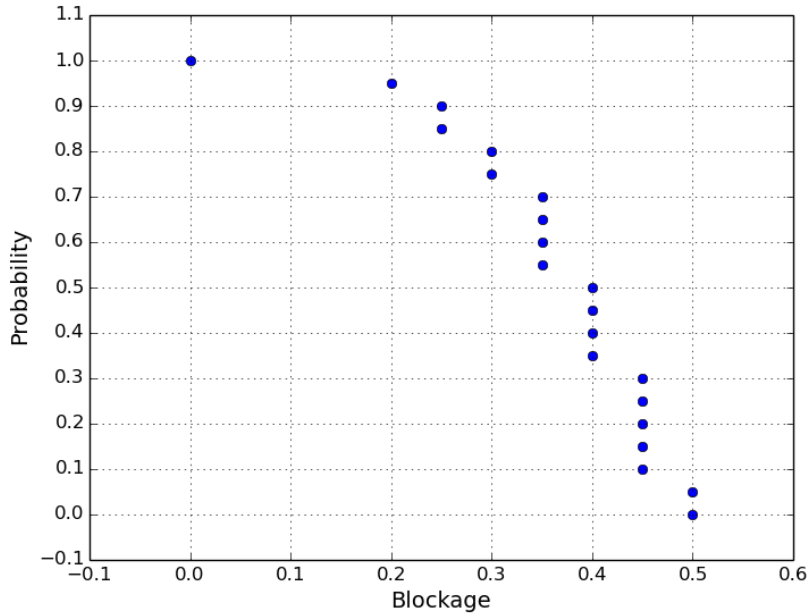


Figure 3: Threshold values of ϵ for a 1% deviation from the no-blockage current.

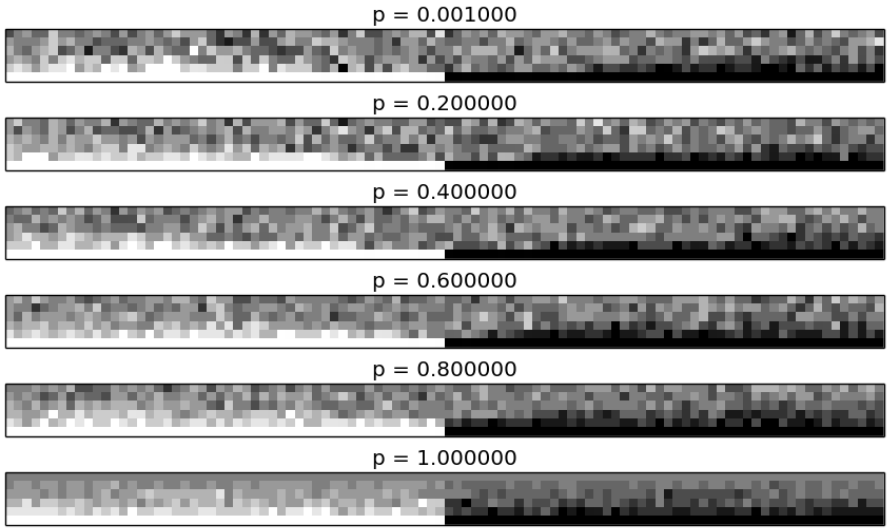


Figure 4: Density for the PCATASEP.

T defined above. Each row corresponds to a specific value of $\epsilon \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Obviously the last row of every diagram is split in half white-empty and half black-full dots, due the total congestion of the blockage at $\varepsilon = 1$.

Figure 5 plots the mean of 100 density diagrams. Exactly like the previous graph, the first row corresponds to the complete absence of blockage, and its uniform colour fairly reflects the space invariance of the system.

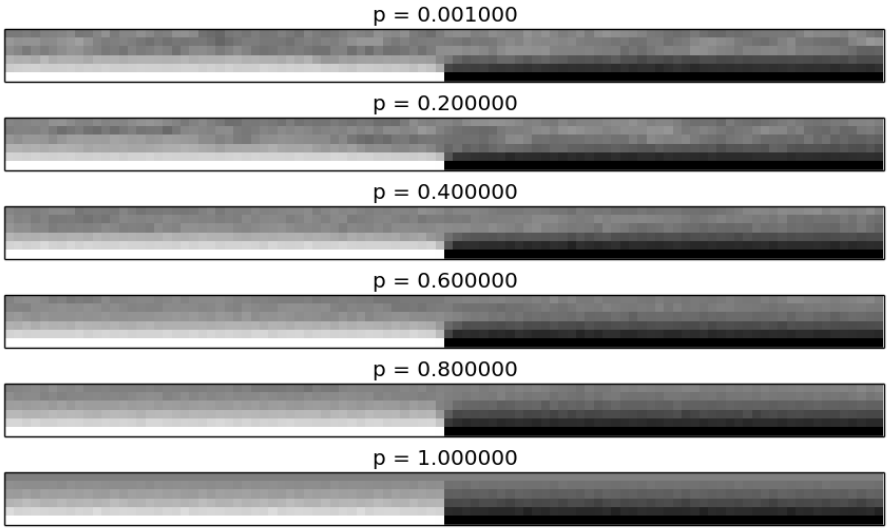


Figure 5: Mean density for 100 steps of 100 iterations each, after time T.

Finally, we focused on an intermediate case of blockage, plotting the values of the density of the segments at time T, specifically for $p = 0.4$, $\epsilon = 0.4$ (Figure 6). In green we represent a linear regression of the results indicating the predictable growth of the concentration of particles approaching the blocked site.
The numerical evidences above show that there are many open interesting questions about our model, namely,

- prove that, in absence of blockage, the mixing time τ of the process is of the order of $\frac{L}{p}\ log(L)$;
- show that except in the case $p = 1$, the probability to have a particle in a site is increasing along the circle starting from the blockage point;
- investigate, in the general case $p < 1$, the stationary measure of the system in

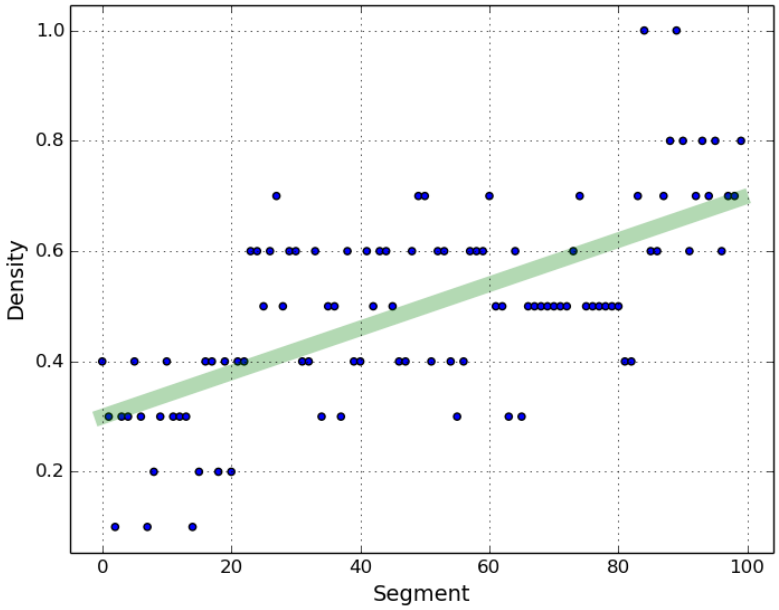


Figure 6: Density in the specific case $p = 0.4$, $\epsilon = 0.4$.

presence of blockage. This last point may be possibly tackled via some perturbative approach with respect to the two cases that are completely known ($\varepsilon = 0$ and $p = 1$).

References

[1] O Costin, JL Lebowitz, ER Speer, and A Troiani. The blockage problem. *arXiv preprint arXiv:1207.6555*, 2012.

[2] Bernard Derrida, MR Evans, Vincent Hakim, and Vincent Pasquier. Exact solution of a 1d asymmetric exclusion model using a matrix formulation. *Journal of Physics A: Mathematical and General*, 26(7):1493, 1993.

[3] SA Janowsky and JL Lebowitz. Exact results for the asymmetric simple exclusion process with a blockage. *Journal of Statistical Physics*, 77(1-2):35–51, 1994.

[4] Steven A Janowsky and Joel L Lebowitz. Finite-size effects and shock fluctuations in the asymmetric simple-exclusion process. *Physical Review A*, 45(2):618, 1992.

[5] Carlo Lancia and Benedetto Scoppola. Equilibrium and non-equilibrium ising models by means of pca. *Journal of Statistical Physics*, 153(4):641–653, 2013.

[6] Thomas M Liggett. *Interacting particle systems*. Springer-Verlag, Berlin, 1985.

[7] Thomas M Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324. Springer, 1999.

[8] Ben Morris. The mixing time for simple exclusion. *The Annals of Applied Probability*, pages 615–635, 2006.

[9] G Schütz and E Domany. Phase transitions in an exactly soluble one-dimensional exclusion process. *Journal of Statistical Physics*, 72(1-2):277–296, 1993.