Constraint Relaxation in Optimal Transport Dualizes to Action-Space Bounds in Sequential Games

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Abstract

In an n-step sequential zero-sum game with a variance-constrained zero-mean adversary and a player bounded to per-step actions $|\Delta_m| \le B < L$, we prove an *exact closed-form Lipschitz degradation* $L_m^B(n) = B + (L-B)(1+h_n)^{n-m}$ (with $h_n = \sqrt{e^{c/n}-1}$), growing as $\Theta(e^{\sqrt{cn}})$ — depending only on (L, B, c, n, m), shape-independent across convex nondecreasing L-Lipschitz losses. This yields a matching $\Theta((L-B)\, e^{\sqrt{cn}}\, s)$ premium over the unbounded-action value, in sharp contrast to the unbounded case (where the value converges to Black–Scholes). For *finitely piecewise-linear* losses with attained Lipschitz constant (e.g. the European call), we sharpen to the *exact identity* $V_n^B(s,0) = (L-B)(\alpha_n - 1)s + g_\infty(s)$ (with $\alpha_n := (1+h_n)^n$ and g_∞ the affine-tail envelope) beyond an explicit threshold — pure drift is the adversary's exact optimum. Since V_n^B itself diverges, no finite PDE limit governs it (Barles–Souganidis consistency fails as a calibration mismatch with the weak-convergence regime; remark 6.10). Under the aggregate-budget variant $\sum_m |\Delta_m| \le nB$ we resolve the premium rate to leading order: $\Phi_n^{\mathrm{agg},B}(s) = Ls \cdot e^{(L-B)\sqrt{cn}/L - (L-B)c/(2L)}(1+o(1))$, with matching lower (two-phase burst-linear adversary) and upper (deterministic primal schedule $\hat{\Delta}_m = L \cdot \mathbf{1}[m > n - \lfloor nB/L \rfloor]$) bounds; the aggregate exponent coefficient $(L-B)/L$ is strictly below the per-step exponent 1, cleanly separating the two regimes. The enabling mechanism is a backward-induction bridge: the n-step game decomposes into n stacked one-step optimal transport problems with point-mass sources; at each step the transport dual variable equals the player's optimal action, and L_1 relaxation of the martingale constraint is Fenchel-dual to the per-step action bound. Applied to Abernethy–Frongillo–Wibisono (2012)'s option-pricing game, this gives a finite-n, bounded-capital algorithmic construction complementing the asymptotic Black–Scholes-delta hedge of Abernethy–Bartlett–Frongillo–Wibisono (2013); our exact characterization reveals that under per-step action bounds $|\Delta_m| \le B < L$ the game value *diverges* at rate $e^{\sqrt{cn}}$ rather than converging to Black–Scholes, settling the per-step bounded-capital regime of AFW 2012's Problem 2.

1 Introduction

Sequential minimax games with constrained adversaries arise throughout online learning, adversarial prediction, and robust optimization. A fundamental structural question is: when does such a game admit a dual formulation as an optimal transport problem, and can the transport structure be exploited computationally?

This paper establishes a precise duality between two independently developed frameworks. On one side: n-step sequential zero-sum games where a player (minimizer) chooses actions and an adversary (maximizer) chooses state transitions subject to a zero-conditional-mean constraint, bounded conditional variance, and compact support [3]. On the other: entropically regularized [12] optimal transport with L_1-relaxed coupling constraints, solvable by proximal gradient ascent on the

dual variable (standard convex duality, §2). We establish a formal duality: the n-step game and a stacked sequence of one-step transport problems have equal values via an explicit correspondence. A backward induction decomposes the n-step game into n one-step transport problems, each with a point-mass source (theorem 4.6); the transport dual variable at each step *is* the player's optimal action (proposition 4.2); and L_1 relaxation of the coupling constraint is dual to bounding the player's action space (theorem 4.9).

Contributions.

1. **Exact Lipschitz degradation** (theorem 5.5, §5): under bounded actions ($B < L$), the Lipschitz constant of the budget-constrained value function obeys the closed-form recurrence $L_m^B(n) = B + (L-B)(1+h_n)^{n-m}$, growing as $\Theta(e^{\sqrt{cn}})$. Exact and shape-independent within the class of convex nondecreasing L-Lipschitz losses — only L matters, not curvature.

2. **Tight premium with exact piecewise-linear identity** (theorems 6.2 and 6.4, §6): matching upper and lower bounds $\Phi_n^B(s) = \Theta((L-B)e^{\sqrt{cn}}s)$ for every affine-tail loss, with an *exact* identity $V_n^B(s,0) = (L-B)(\alpha_n-1)s + g_\infty(s)$ beyond an explicit threshold for piecewise-linear g, replacing a conjectured $O(n^{-1/4})$ boundary-layer rate with identically zero. The HJB limit consequently fails (Barles–Souganidis [6] consistency breaks, remark 6.10).

3. **Backward induction bridge enabling the above** (theorem 4.6, §4): the structural result that makes the exact analyses possible — the n-step game decomposes into n one-step OT problems with point-mass sources, continuation values are convex and L-Lipschitz at every step (the Lipschitz constant is exactly preserved by the zero-mean constraint $\mathbb{E}[1+T] = 1$), and the transport dual variable at each step *equals* the player's optimal action.

4. **Action-space–relaxation duality** (theorem 4.9, §4.4): L_1 relaxation in the transport $\leftrightarrow$ per-step action-space bound in the game, with explicit Fenchel conjugate correspondence $B \mapsto \delta(B) = |h^*(B)|$. Critical threshold: the action bound is non-binding for $B \geq L$.

5. **Tight premium bounds** (§6): the pricing premium $\Phi_n^B(s) = \Theta((L-B)\, e^{\sqrt{cn}}\, s)$ with matching upper and lower bounds, for every convex nondecreasing L-Lipschitz loss with affine tail. The stronger *exact identity* $V_n^B(s,0) = (L-B)(\alpha_n-1)s + g_\infty(s)$ holds beyond an explicit threshold for the strictly smaller class of *piecewise-linear* losses with attained Lipschitz constant (i.e., g affine on $[s^*, \infty)$, not merely limit-affine; theorem 6.4). In sharp contrast to the unbounded-action case where the value converges to the Black–Scholes limit (AFW 2012), bounded per-step capital makes hedging exponentially expensive. The HJB limit consequently fails (Barles–Souganidis [6] consistency breaks).

6. **Aggregate-budget tight rate** (theorems 11.4 and 11.5 and remark 11.7, stated in §11 after the main per-step theory): under $\sum_m |\Delta_m| \leq nB$ the premium is sub-exponential with exponent coefficient $(L-B)/L$ to leading order: matching lower (two-phase *burst-linear* Dirac adversary) and upper (deterministic primal schedule $\hat{\Delta}_m = L \cdot \mathbf{1}[m > n - \lfloor nB/L \rfloor]$) bounds give $\Phi_n^{\mathrm{agg},B}(s) = Ls \cdot e^{(L-B)\sqrt{cn}/L - (L-B)c/(2L)}(1+o(1))$, disproving the naive polynomial conjecture and resolving the aggregate rate to $(1+o(1))$. Supplementary polynomial $(n^{L/(2B)}/\log n)$ and $\sqrt{cn}$ lower bounds (theorems 11.2 and 11.3) give sharper pre-asymptotic constants in their respective regimes.

7. **Algorithmic optimal strategies**: the transport dual variables, computable by Sinkhorn + proximal gradient ascent with $O(1/k)$ convergence, yield explicit optimal actions at each finite n, complementing the asymptotic Black-Scholes-delta hedge of ABFW 2013 [2].

Connection to online learning. The sequential game is an instance of online convex optimization: backward induction is dynamic programming over transport duals, Sinkhorn at each step is mirror descent with KL divergence [12], and the proximal gradient step on the dual variable is an online learning update.

Items 1–2 together characterize the per-step bounded-capital regime of AFW 2012's Problem 2: they reveal that $|\Delta_m| \le B < L$ forces the Lipschitz constant and premium to grow as $\Theta(e^{\sqrt{cn}})$, a negative result on the existence of a finite-capital Black–Scholes-competitive hedge (the game value diverges; no finite PDE limit governs it). Item 6 resolves the aggregate-budget variant to $(1 + o(1))$ precision (remark 11.7), with exponent coefficient $(L-B)/L$ cleanly separating aggregate from per-step. The remaining open question concerns explicit non-asymptotic constants (item 1). Items 3 and 7 (the bridge and the resulting algorithmic construction) provide a finite-n, bounded-capital-compatible route to AFW 2012's constructive-strategy open problem, complementing Abernethy, Bartlett, Frongillo, and Wibisono [2], who resolved it via an asymptotic Black-Scholes-delta hedge.

Organization. §2: regularized OT with relaxed constraints. §3: sequential games with constrained adversaries. §4: the backward induction bridge. §§5–6: Lipschitz degradation and premium bounds. §7: consequences. §8: application to robust option pricing. §9: experiments.

2 Regularized Optimal Transport with Relaxed Coupling Constraints

We summarize the regularized OT framework used in this paper, which provides the algorithmic foundation for the bridge in §4. It uses linear coupling constraints relaxed in L_1 norm and smoothed by entropic regularization; the martingale constraint central to this paper is one instance. The individual components (entropic regularization [12], L_1 constraint relaxation, and proximal-gradient dual ascent [18]) are standard in convex optimization and optimal transport; we combine them and analyze the approximation error induced by each (proposition 4.8 below).

2.1 Setup: constrained optimal transport

Let $\mu \in \mathbb{R}^p_+$ and $\nu \in \mathbb{R}^q_+$ be discrete probability measures on finite sets $X = \{x_1, \dots, x_p\}$ and $Y = \{y_1, \dots, y_q\}$.[1] The set of couplings is $\Pi(\mu, \nu) = \{\pi \in \mathbb{R}^{p\times q}_+ : \pi \mathbf{1}_q = \mu,\ \pi^\top \mathbf{1}_p = \nu\}$.

Definition 2.1 (Linearly constrained OT)**.** Let $A\colon \mathbb{R}^{p\times q} \to \mathbb{R}^r$ be linear and $b \in \mathbb{R}^r$. The constrained transport problem is $\min_{\pi\in\Pi(\mu,\nu)} \langle C, \pi\rangle$ subject to $A\pi = b$.

The constraint $A\pi = b$ encodes structural requirements beyond the marginals. In the *martingale* case, $[A\pi]_i = \frac{1}{\mu_i}\sum_j \pi_{ij}(y_j - x_i)$ and $b = 0$, enforcing $\mathbb{E}_\pi[Y|X = x_i] = x_i$. The *martingale defect* is $d_i(\pi) := \frac{1}{\mu_i}\sum_j \pi_{ij}(y_j - x_i)$.

2.2 L_1 relaxation and dual analysis

Definition 2.2 (L_1-relaxed entropic OT)**.**

$$\min_{\pi\in\Pi(\mu,\nu)} \langle C, \pi\rangle + \varepsilon\, H(\pi) \quad \text{s.t.} \quad \|A\pi - b\|_1 \le \delta, \tag{1}$$

where $H(\pi) = \sum_{i,j} \pi_{ij}(\log \pi_{ij} - 1)$ is the negative entropy, $\varepsilon > 0$ the regularization, and $\delta \ge 0$ the relaxation.

[1] We use p, q for grid sizes to avoid collision with the game step count n and step index m.

Proposition 2.3 (L_1–L_∞ duality)**.** *Strong duality (Slater, $\delta > 0$) gives the dual:*

$$\max_{h\in\mathbb{R}^p} J(h) := W_\varepsilon\big(\tilde{C}(h)\big) - \delta\|h\|_\infty, \tag{2}$$

where $\tilde{C}_{ij}(h) = C_{ij} + h_i(y_j - x_i)$ and $W_\varepsilon(\tilde{C}) = \min_{\pi\in\Pi(\mu,\nu)}\langle\tilde{C},\pi\rangle + \varepsilon H(\pi)$.

The dual variable h_i is the "shadow price" of enforcing the constraint at x_i. The L_∞ penalty bounds this price; in the game-theoretic interpretation (section 4), h_i is the player's action and δ the adversary's constraint relaxation.

2.3 Algorithm

The dual $J(h) = W_\varepsilon(\tilde{C}(h)) - \delta\|h\|_\infty$ is smooth minus non-smooth convex, amenable to proximal gradient ascent. By Danskin's theorem, $[\nabla_h W_\varepsilon]_i = d_i(\pi^*(h))$ (the gradient is the constraint residual); the inner $\pi^*(h)$ is computed by Sinkhorn [12]. The proximal step uses Moreau decomposition: $\mathrm{prox}_{\tau\delta\|\cdot\|_\infty}(v) = v - \tau\delta\cdot\mathrm{proj}_{\mathcal{B}_1}(v/(\tau\delta))$, where projection onto $\mathcal{B}_1$ costs $O(p\log p)$ [18].

Proposition 2.4 (Convergence)**.** *If $|y_j - x_i| \le R$ for all i, j,*[2] *the gradient Lipschitz constant is $L_{\mathrm{grad}} = R^2/\varepsilon$.*[3] *With step size $\gamma = \varepsilon/R^2$, proximal gradient ascent achieves $J(h^*) - J(h^{(K)}) \le R^2\|h^*\|_2^2/(2\varepsilon K)$.*

2.4 Statistical calibration

Theorem 2.5 (Concentration [26])**.** *Under $\|y - x\| \le R$ a.s., McDiarmid's inequality gives calibration rule $\delta_N = R\sqrt{2\log(2/\alpha)/N} = O(N^{-1/2})$, ensuring feasibility with probability $\ge 1 - \alpha$.*

3 Sequential Games with Constrained Adversaries

We formulate the class of sequential two-player zero-sum games with conditional mean and variance constraints on the adversary.

3.1 Problem setting

Definition 3.1 (Sequential minimax game)**.** Fix $n \in \mathbb{N}$, variance parameter $c > 0$, and convex L-Lipschitz terminal loss $g\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$. A state sequence $S_n = (S_{n,0}, \ldots, S_{n,n})$ evolves from $S_{n,0} = 1$. At step m: the player selects $\Delta_m \in \mathbb{R}$ (measurable w.r.t. $\mathcal{F}_{n,m-1}$); the adversary selects the conditional law of $S_{n,m}$ subject to constraints. The loss is

$$\ell_n(A, S_n) = \mathbb{E}\left[g(S_{n,n}) - \sum_{m=1}^{n}\left(\frac{S_{n,m}}{S_{n,m-1}} - 1\right)\Delta_m\right]. \tag{3}$$

The game value is $V_n := \inf_{A\in\mathcal{A}_n}\sup_{S_n\in\mathcal{S}^n_{D,\boldsymbol{\zeta}}}\ell_n(A, S_n)$, where $\mathcal{S}^n_{D,\boldsymbol{\zeta}}$ denotes state sequences satisfying (DVC) and (ZC$_n$) below.

The adversary's constraints on $T_{n,m} := S_{n,m}/S_{n,m-1} - 1$ are: **(DVC)** $\mathbb{E}[T^2_{n,m} \mid \mathcal{F}_{n,m-1}] \le e^{c/n} - 1$; **(ZC$_n$)** $|T_{n,m}| \le \zeta_n$ a.s., with $\zeta_n \to 0$ and $\liminf n\zeta_n^2/\log n > 16c$; **(MG)** $\mathbb{E}[T_{n,m} \mid \mathcal{F}_{n,m-1}] = 0$ (martingale).

[2] R denotes support diameter, distinct from B (action-space bound).
[3] L_{grad} is distinct from L (Lipschitz constant of the terminal loss).

3.2 Minimax duality and martingale forcing

Proposition 3.2 (Minimax duality [3, Lemma 1])**.** $\inf_A \sup_{S_n} \ell_n = \sup_{S_n} \inf_A \ell_n$. *(Sion [30] on $[-M, M]^n \times \mathcal{S}^n_{D,\boldsymbol{\zeta}}$, compact × weakly compact by Prokhorov on the bounded-support state sequences; take $M \to \infty$, which is harmless by coercivity: for $M \geq L$ the value stabilizes since drift benefit $\leq Ls|a|$ while cost is $Ms|a|$, cf. proposition 4.10.)*

This yields *martingale forcing*: if $\mathbb{E}[T_{n,m} \mid \mathcal{F}_{n,m-1}] = a \neq 0$, then $\inf_{\Delta_m}(-a\Delta_m) = -\infty$, forcing the adversary onto a martingale and dropping the player's actions from the objective.

Theorem 3.3 (Convergence [3, Theorems 1–2])**.** *Let $G(t) = \exp(\sqrt{c}\,W(t) - ct/2)$ and $\beta := \mathbb{E}[g(G(1))]$. Then $V_n \to \beta$ as $n \to \infty$, and the adversary's optimal state process converges in distribution to G.*

3.3 The two structural limitations

Limitation 1: Non-constructive strategies. Sion guarantees existence of a saddle point but no algorithm; the nested inf-sup over continuous spaces is intractable for large n. ABFW 2013 [2] resolved this analytically via an asymptotic Black-Scholes-delta hedge $\Delta(S, c) = S\,\partial U/\partial S$ under unbounded capital; our resolution (theorem 4.6(b)) is algorithmic, valid for each finite n, and compatible with bounded capital.

Limitation 2: Unbounded actions required. The forcing mechanism requires $\Delta_m \in \mathbb{R}$ unconstrained. Under $|\Delta_m| \leq B$, the punishment for drift a becomes $\inf_{|\Delta| \leq B}(-a\Delta) = -B|a|$ (finite), so the adversary can introduce drift at cost $B|a|$ per step.

Both limitations are structural: limitation 1 arises from the infinite-dimensional adversary space; limitation 2 is physically necessary (no real system has unbounded actions). The bridge to transport (section 4) resolves both.

4 The Bridge: Sequential Games as Backward-Inductive Transport

4.1 One-step equivalence

When the source marginal is a point mass δ_s (conditioned on the current state), constrained OT reduces to a moment-constrained optimization over a single probability measure on the compact interval $[s(1-\zeta), s(1+\zeta)]$—the adversary's one-step problem. The duality results of §2 were stated for discrete measures; the equivalence below works directly with measures on a compact interval, using Sion's theorem and the Fenchel conjugacy sandwich (theorem 4.9) rather than infinite-dimensional LP duality.

Definition 4.1 (One-step constrained transport)**.** Given state $s > 0$, variance parameter $\sigma^2 > 0$, support bound $\zeta > 0$, and convex L-Lipschitz loss $\ell\colon \mathbb{R}_+ \to \mathbb{R}$ (in the backward induction, $\ell = V(\cdot, m+1)$; at the terminal step, $\ell = g$), the *one-step constrained transport value* is $V^{\mathrm{OT}}(s) := \sup_{\nu \in \mathcal{K}(s)} \mathbb{E}_\nu[\ell(Y)]$, where $\mathcal{K}(s) = \{\nu \in \mathcal{P}([s(1-\zeta), s(1+\zeta)]) : \mathbb{E}_\nu[Y] = s,\ \mathbb{E}_\nu[(Y-s)^2] \leq \sigma^2 s^2\}$.

Proposition 4.2 (Game–transport equivalence, $n = 1$)**.** *Let $\mathcal{K}'(s) := \{\nu \in \mathcal{P}([s(1-\zeta), s(1+\zeta)]) : \mathbb{E}_\nu[(Y-s)^2] \leq \sigma^2 s^2\}$ (the set $\mathcal{K}(s)$ of definition 4.1 with the mean-zero constraint dropped). The one-step game value*

$$V_1 \;=\; \inf_{\Delta \in \mathbb{R}} \sup_{\nu \in \mathcal{K}'(s)} \mathbb{E}_\nu\big[\ell(Y) - (Y/s - 1)\Delta \cdot s\big] \tag{4}$$

equals $V^{\mathrm{OT}}(s)$. Moreover, the optimal action satisfies $\Delta^* = h^*$, where h^* is the Lagrange multiplier for the moment constraint $\mathbb{E}_\nu[Y] = s$ (equivalently, the transport dual variable of proposition 2.3 with $\delta = 0$).

Proof. Minimax duality. $\mathcal{K}'(s)$ is weakly compact by Prokhorov (bounded support $[s(1-\zeta), s(1+\zeta)]$; variance constraint is weakly closed). The objective

$$f(\Delta, \nu) \;:=\; \mathbb{E}_\nu[\ell(Y)] - \Delta(\mathbb{E}_\nu[Y] - s)$$

is continuous in Δ and weakly continuous in ν (ℓ and the identity are bounded continuous on the bounded support). For each $M > 0$, Sion's theorem [30] applies (f convex-concave on compact $[-M, M] \times \mathcal{K}'(s)$):

$$\inf_{|\Delta| \le M} \sup_{\nu} f \;=\; \sup_{\nu} \inf_{|\Delta| \le M} f.$$

Martingale forcing. With $d := \mathbb{E}_\nu[Y] - s$, the inner infimum is $\inf_{|\Delta| \le M}[\mathbb{E}_\nu[\ell(Y)] - \Delta d] = \mathbb{E}_\nu[\ell(Y)] - M|d|$. As $M \to \infty$: $-\infty$ if $d \neq 0$, else $\mathbb{E}_\nu[\ell(Y)]$; hence $\sup_\nu \inf_\Delta = V^{\mathrm{OT}}(s)$. For the one-step case, the drift benefit is bounded by $L|d|$ (ℓ is L-Lipschitz), so for $M \ge L$ any ν with $d \neq 0$ contributes at most $(L - M)|d| \le 0$; the optimizer satisfies $d = 0$. Hence $V_1^M = V_1^L$ for all $M \ge L$, and $\inf_\Delta \sup_\nu = V^{\mathrm{OT}}(s)$ (cf. proposition 4.10).

Action correspondence. At the saddle point (Δ^*, ν^*), stationarity in Δ gives $\mathbb{E}_{\nu^*}[Y] = s$. Stationarity in ν gives $\ell(Y) - \Delta^*(Y/s - 1)s = \lambda$ (ν^*-a.s.), so Δ^* is the Lagrange multiplier for $\mathbb{E}[Y] = s$ — exactly the dual variable h^* of proposition 2.3 with $\delta = 0$ (the modified cost $\tilde{C}(h) = -\ell(Y) + h(Y - s)$ has $h^* = \Delta^*$). □

4.2 Regularity of the value function

The key property enabling backward induction is that convexity and the Lipschitz constant are preserved across steps.

Definition 4.3 (Continuation value)**.** Fix $n \in \mathbb{N}$. Let $\mathcal{T}_n = \{\mu \in \mathcal{P}([-\zeta_n, \zeta_n]) : \mathbb{E}_\mu[T] = 0,\ \mathbb{E}_\mu[T^2] \le e^{c/n} - 1\}$.

$$V(s, n) = g(s), \tag{5}$$

$$V(s, m) = \sup_{\mu_T \in \mathcal{T}_n} \mathbb{E}_{T \sim \mu_T}[V(s(1+T),\, m+1)], \quad m = n-1, \ldots, 0. \tag{6}$$

Lemma 4.4 (Regularity of continuation values)**.** *For each m:* (a) *$V(\cdot, m)$ is convex and nondecreasing;* (b) *$V(\cdot, m)$ is L-Lipschitz, with the* same *constant as g.*

Proof. By backward induction. Substitute $T = Y/s - 1$; the constraints on T define $\mathcal{T}_n$ and are independent of s.

Convexity: $s \mapsto s(1+T)$ is linear; $V(\cdot, m+1)$ is convex by hypothesis; composition, expectation, and supremum all preserve convexity.

Lipschitz: Since $\zeta_n < 1$ ensures $1 + T \ge 1 - \zeta_n > 0$ a.s., we have $|1 + T| = 1 + T$. For any $\mu_T \in \mathcal{T}_n$ (imposing $\mathbb{E}[T] = 0$): $|\mathbb{E}[V(s(1+T), m+1)] - \mathbb{E}[V(s'(1+T), m+1)]| \le L|s - s'| \cdot \mathbb{E}[|1+T|] = L|s - s'| \cdot \mathbb{E}[1 + T] = L|s - s'|$, where the last equality uses $\mathbb{E}[1 + T] = 1 + \mathbb{E}[T] = 1$. Taking sup over μ_T: $|V(s, m) - V(s', m)| \le L|s - s'|$. □

Remark 4.5. The Lipschitz constant is *exactly* preserved because $\mathbb{E}[T] = 0 \Rightarrow \mathbb{E}[1 + T] = 1$. Dropping zero-mean allows $\mathbb{E}[1 + T] > 1$, causing exponential Lipschitz growth—the mechanism behind theorem 5.5.

4.3 Backward induction bridge

Theorem 4.6 (Backward Induction Bridge)**.** *Fix $n \in \mathbb{N}$. For each step m and state $s > 0$:*

(a) ***Value equivalence.*** *$V(s, m)$ equals the one-step constrained transport value (definition 4.1) with loss $V(\cdot, m+1)$ at source s.*

(b) ***Action correspondence.*** *$\Delta_m^*(s) = h_m^*(s)$, where h_m^* is the transport dual variable. In particular, h_m^* depends only on s (Markov).*

(c) ***Per-step duality.*** *Under $|\Delta_m(s)| \le B$ at every step, the action-constrained value at step m equals the L_1-relaxed transport value with* step-dependent *δ_m, where $\delta_m = \delta(B; L_{m+1}^B(n))$ depends on the Lipschitz constant of the step-m continuation value $V_n^B(\cdot, m+1)$; the Fenchel conjugacy of theorem 4.9 applied to loss $V_n^B(\cdot, m+1)$ defines δ_m. The player's per-step bound B is fixed; $L_{m+1}^B(n)$ and hence δ_m vary with m (theorem 5.5).*

Proof. (a) At step m, the sub-problem is a one-step game with loss $V(\cdot, m+1)$ (convex and L-Lipschitz by lemma 4.4); proposition 4.2 applies.

(b) Follows from proposition 4.2; Markov because $\mathcal{T}_n$ depends only on s.

(c) The penalized form $V_n^B(s, m) = \sup_{\mu_T}[\mathbb{E}[V_n^B(s(1+T), m+1)] - Bs|\mathbb{E}[T]|]$ (anticipating definition 5.1 below) is convex in s (convex + linear), enabling theorem 4.9 at each step. □

Remark 4.7 (Generality of the bridge)**.** The proof of theorem 4.6 uses three ingredients: (i) the source marginal at each backward step is a point mass δ_s at the current state (Markov property of the game), (ii) the adversary's one-step distribution has zero conditional mean and bounded conditional second moment, and (iii) the continuation value is convex and L-Lipschitz, preserved across steps by lemma 4.4. These conditions are not specific to robust option pricing. Any sequential minimax game with a variance-constrained zero-mean adversary and a convex L-Lipschitz terminal loss—for example, adversarial online prediction with variance-bounded noise [10] or distributionally robust sequential decision-making under second-moment ambiguity sets—admits the same backward-induction decomposition into n stacked two-marginal OT problems, with the dual-action correspondence of theorem 4.6(b) and the L_1–L_∞ Fenchel dichotomy of theorem 4.9 applying verbatim. The AFW option-pricing game (§8) is the most developed instance in the literature and furnishes a concrete testbed.

To quantify the computational error of a grid-based implementation, let $V^q(s, m)$ denote the continuation value of definition 4.3 with the adversary's distribution μ_T supported on a fixed q-point grid in $[-\zeta_n, \zeta_n]$ (probability vectors $\kappa \in \Delta^{q-1}$ satisfying the martingale and variance constraints), and let $V_\varepsilon^q(s, m)$ denote its entropic regularization, obtained by adding the penalty $\varepsilon \cdot \mathrm{KL}(\kappa \,\|\, \mathrm{unif}_q)$ inside the supremum in (6).

Proposition 4.8 (Entropic error accumulation)**.** $\sup_s |V_\varepsilon^q(s, 0) - V^q(s, 0)| \le n\varepsilon \log q$. *Choosing $\varepsilon = \alpha/(n \log q)$ achieves target accuracy α.*

Proof. Let $e_m := \sup_s |V_\varepsilon^q(s, m) - V^q(s, m)|$ with $e_n = 0$. Fix s and let κ^*, κ_ε^* attain the suprema in $V^q(s, m)$ and $V_\varepsilon^q(s, m)$ respectively (attained on the compact simplex Δ^{q-1} intersected with the linear constraints). Using κ_ε^* as a feasible point for $V^q(s, m)$,

$$V_\varepsilon^q(s, m) - V^q(s, m) \;\le\; \mathbb{E}_{\kappa_\varepsilon^*}[V_\varepsilon^q(\cdot, m{+}1) - V^q(\cdot, m{+}1)] + \varepsilon H(\kappa_\varepsilon^*) \;\le\; e_{m+1} + \varepsilon \log q,$$

since $0 \le H(\kappa_\varepsilon^*) \le \log q$. Using κ^* as a feasible point for $V_\varepsilon^q(s, m)$,

$$V^q(s, m) - V_\varepsilon^q(s, m) \;\le\; \mathbb{E}_{\kappa^*}[V^q(\cdot, m{+}1) - V_\varepsilon^q(\cdot, m{+}1)] - \varepsilon H(\kappa^*) \;\le\; e_{m+1},$$

since $H(\kappa^*) \geq 0$. Taking sup over s yields $e_m \leq e_{m+1} + \varepsilon \log q$; iterating from $e_n = 0$ gives $e_0 \leq n\varepsilon \log q$. □

4.4 Constraint relaxation dualizes to action-space bounds

Theorem 4.9 (Action-space–relaxation duality)**.** *Fix a convex L-Lipschitz loss ℓ, and let $\mathcal{K}'(s)$ be the variance-constrained set (without mean-zero) from definition 4.1. The action-constrained game*

$$V_1^B \;=\; \inf_{|\Delta| \leq B} \sup_{\nu \in \mathcal{K}'(s)} \big[\mathbb{E}_\nu[\ell(Y)] - \Delta(\mathbb{E}_\nu[Y] - s)\big]$$

equals the L_1-relaxed transport

$$V_1^\delta \;=\; \sup_{\nu \in \mathcal{K}'(s):\, |\mathbb{E}_\nu[Y] - s| \leq \delta} \mathbb{E}_\nu[\ell(Y)],$$

with $B(\delta) = |h^*(\delta)|$.

Proof. Apply Sion's theorem to $[-B, B] \times \mathcal{K}'(s)$ (compact × weakly compact by Prokhorov on the bounded support $[s(1-\zeta), s(1+\zeta)]$; objective bilinear in Δ, weakly continuous in ν):

$$V_1^B \;=\; \sup_{\nu \in \mathcal{K}'(s)} \big[\mathbb{E}_\nu[\ell(Y)] - B\,|d(\nu)|\big], \qquad d(\nu) := \mathbb{E}_\nu[Y] - s.$$

Let $\nu^*(B)$ attain the supremum, and set $\delta(B) := |d(\nu^*(B))|$.

Upper bound (weak duality), $V_1^\delta \leq V_1^B + B\delta$. For any ν with $|d(\nu)| \leq \delta$: $\mathbb{E}_\nu[\ell(Y)] = [\mathbb{E}_\nu[\ell(Y)] - B|d(\nu)|] + B|d(\nu)| \leq V_1^B + B\delta$; taking $\sup_\nu$ gives the claim.

Lower bound (feasibility of ν^*), $V_1^{\delta(B)} \geq V_1^B + B\delta(B)$. $\nu^*(B)$ has $|d(\nu^*)| = \delta(B)$, so $V_1^{\delta(B)} \geq \mathbb{E}_{\nu^*}[\ell(Y)] = V_1^B + B\delta(B)$.

Combining at $\delta = \delta(B)$: $V_1^{\delta(B)} = V_1^B + B\delta(B)$. The correspondence $B \leftrightarrow \delta$ via $B(\delta) = |h^*(\delta)|$—where h^* is the Lagrange multiplier for the moment constraint $\mathbb{E}_\nu[Y] = s$ (equivalently the L_1 dual of proposition 2.3 specialized to the one-step case)—identifies the two problems; their values differ by the Lagrangian term $B\delta$.

Continuity and surjectivity of $B \mapsto \delta(B)$. The map $B \mapsto V_1^B$ is *convex* nonincreasing: for $B_1 < B_2$ and any $\lambda \in [0, 1]$, let $\Delta_i^* \in \arg\min_{|\Delta| \leq B_i} F(\Delta)$ where $F(\Delta) := \sup_\nu[\mathbb{E}_\nu[\ell(Y)] - \Delta(\mathbb{E}_\nu[Y] - s)]$ is convex in Δ (sup of affine). Then $\lambda\Delta_1^* + (1-\lambda)\Delta_2^*$ is feasible for $B_\lambda := \lambda B_1 + (1-\lambda)B_2$, so $V_1^{B_\lambda} \leq F(\lambda\Delta_1^* + (1-\lambda)\Delta_2^*) \leq \lambda F(\Delta_1^*) + (1-\lambda)F(\Delta_2^*) = \lambda V_1^{B_1} + (1-\lambda)V_1^{B_2}$. Nonincreasing in B because the feasible set expands. Finite for $B \geq 0$ (bounded support, Lipschitz ℓ). Convexity and monotonicity force V_1^B continuous on $(0, \infty)$ and left/right differentiable everywhere, with $\delta(B) = -\partial_B V_1^B \geq 0$ the negative of the subgradient ($\partial_B V_1^B \leq 0$ since V_1^B is nonincreasing). The subgradient $\partial_B V_1^B$ of a convex function is nondecreasing; hence $\delta(B) = -\partial_B V_1^B$ is *nonincreasing* in B, upper semi-continuous, and surjective onto $[0, \delta_{\max}]$ where $\delta_{\max} := \lim_{B \to 0^+} \delta(B)$ equals the unconstrained adversary's defect (attained by ν with martingale slack) and $\lim_{B \to \infty} \delta(B) = 0$ (tight constraint, ν forced to martingale). For strictly convex ℓ, the subgradient is single-valued and the map is a homeomorphism $[0, \infty) \to [0, \delta_{\max}]$; in general (as here, ℓ is convex but not necessarily strict), the "correspondence" language is literal. □

Proposition 4.10 (Critical action-space bound)**.** *Assume $g\colon [0, \infty) \to \mathbb{R}_{\geq 0}$ is convex, nondecreasing, L-Lipschitz, with $g(0)$ finite (all of these hold throughout the paper and are restated in definition 5.1). Then $\Phi_n^B = 0$ for $B \geq L$ and all n.*

Proof. We prove by backward induction on $m = n, n-1, \ldots, 0$ the joint statement: *(i)* $V_n^B(\cdot, m)$ is L-Lipschitz, and *(ii)* the adversary's optimal drift at step m is $a^* = 0$ (martingale). Throughout, $a_\mu := \mathbb{E}_\mu[T]$ denotes the drift of $\mu \in \widetilde{\mathcal{T}}_n$.

Base case $m = n$: $V_n^B(\cdot, n) = g$ is L-Lipschitz by assumption. Vacuous for a^*.

Inductive step: Assume (i) at step $m+1$. For each state s at step m and any $\mu \in \widetilde{\mathcal{T}}_n$, the continuation term satisfies $|\mathbb{E}_\mu[V_n^B(s(1+T), m+1)] - V_n^B(s, m+1)| \le Ls|a_\mu|$ by L-Lipschitz continuation, so drift benefit is $\le Ls|a_\mu|$. The player's cost is $Bs|a_\mu|$. Net contribution is $(L - B)s|a_\mu| \le 0$ for $B \ge L$, so the adversary optimally plays $a^* = 0$. Restricting to the $\{a_\mu = 0\}$ sub-slice of $\widetilde{\mathcal{T}}_n$ recovers the mean-zero feasible set $\mathcal{T}_n$ of definition 4.3, and lemma 4.4 (whose proof uses $\mathbb{E}[1+T] = 1$) then applies, giving $V_n^B(\cdot, m) = V_n^\infty(\cdot, m)$ is L-Lipschitz. This closes (i)–(ii) at step m and completes the induction. Therefore $\Phi_n^B = V_n^B - V_n^\infty = 0$ for all n. □

Remark 4.11 (Per-step vs. aggregate)**.** Theorem 4.6(c) handles per-step constraints $|\Delta_m| \le B$. Aggregate constraints $\sum_m |\Delta_m| \le nB$ couple steps and break the backward induction decomposition.

5 Lipschitz Degradation Under Action-Space Constraints

Definition 5.1 (Budget-constrained backward induction)**.** Fix $c > 0$, $B > 0$, and $g\colon [0, \infty) \to \mathbb{R}_{\ge 0}$ convex, L-Lipschitz, nondecreasing, with $L > B$ (the condition $L > B$ makes the action bound binding — proposition 4.10 shows $\Phi_n^B = 0$ for $B \ge L$; codomain $\mathbb{R}_{\ge 0}$ and Lipschitz continuity ensure $g(0)$ is finite and nonnegative).

Calibration map across theorems. All per-step-budget theorems in this section and in section 6 (theorems 5.5 and 6.2 to 6.4 and propositions 4.10, 6.6 and 11.1) require only the weaker hypothesis $\zeta_n \ge h_n$ stated below. The stronger AFW calibration $\liminf n\zeta_n^2/\log n > 16c$ (definition 3.1) is needed only to invoke theorem 3.3 from AFW 2012 [3] as a baseline in §§8–9; it is not a hypothesis of any per-step-budget theorem. Write $h_n := \sqrt{e^{c/n} - 1}$. For each $n \in \mathbb{N}$, let $\zeta_n \in [h_n, 1)$ with $\zeta_n \to 0$; the lower bound $\zeta_n \ge h_n$ ensures that a two-point distribution saturating the DVC on $\{-h_n, +h_n\}$ (and its drifted variants) is feasible. The stronger calibration $\liminf_n n\zeta_n^2/\log n > 16c$ used for AFW's concentration/convergence (theorem 3.3) is not required here and is imposed separately in definition 3.1. *Calibration map across theorems:* theorems 5.5 and 6.2 need only $\zeta_n \ge h_n$; theorem 6.4's exact remainder needs the tight-support calibration $\zeta_n = h_n$; AFW's convergence (theorem 3.3) additionally needs the concentration calibration $\liminf_n n\zeta_n^2/\log n > 16c$. Define the *feasible perturbation set* (no mean-zero constraint; cf. $\mathcal{T}_n$ in definition 4.3)

$$\widetilde{\mathcal{T}}_n \;=\; \big\{\mu \in \mathcal{P}([-\zeta_n, \zeta_n]) \;:\; \mathbb{E}_\mu[T^2] \le e^{c/n} - 1\big\},$$

and the *budget-constrained value function* by backward induction:

$$V_n^B(s, n) = g(s), \tag{7}$$

$$V_n^B(s, m) = \sup_{\mu \in \widetilde{\mathcal{T}}_n} \Big\{ \mathbb{E}_\mu\big[V_n^B(s(1+T),\, m+1)\big] \;-\; B\, s\, \big|\mathbb{E}_\mu[T]\big| \Big\}, \quad m = n-1, \ldots, 0. \tag{8}$$

The supremum is attained: $\widetilde{\mathcal{T}}_n$ is weakly compact (closed variance constraint on a compact support) and the objective is weakly continuous. Denote the Lipschitz constant at step m by

$$L_m^B(n) \;:=\; \sup_{s \ne s'} \frac{|V_n^B(s, m) - V_n^B(s', m)|}{|s - s'|}.$$

We write V_n^∞ for the same recursion with the action bound removed (formally $B = \infty$, recovering the unconstrained continuation value V of definition 4.3).

Proposition 5.2 (Boundary cases)**.** *Under definition 5.1:*

(a) $L_n^B(n) = L$ *for all* n (terminal condition $V_n^B(\cdot, n) = g$).

(b) *If* $B \geq L$: $L_m^B(n) = L$ *for all* m, n (by proposition 4.10, the adversary plays a martingale; lemma 4.4 preserves L).

(c) *If* $B = \infty$ (unconstrained): $L_m^\infty(n) = L$ *for all* m, n (immediate from lemma 4.4 since $\mathbb{E}[T] = 0 \Rightarrow \mathbb{E}[1+T] = 1$).

(d) *Naive bound* (any g): $L_m^B(n) \leq (L+B)(1+\zeta_n)^{n-m} = O(e^{\sqrt{cn}})$, *obtained by iterating the backward recursion with triangle inequality and the per-step drift penalty* $Bs|a|$. *The* exact *rate (replacing* ζ_n *with* h_n, *and* $L+B$ *with* $B + (L-B)$*) is obtained in theorem 5.5 below.*

Lemma 5.3 (Asymptotic slope and affine lower bound)**.** *If* $g\colon (0,\infty) \to \mathbb{R}$ *is convex, nondecreasing, and* L*-Lipschitz with* $L > 0$*, then:*

(i) *The right derivative* $g'_+(s)$ *is nondecreasing with* $\lim_{s\to\infty} g'_+(s) = L$.

(ii') *Writing* $C_g := g(0) - \inf_{s>0}[g(s) - Ls] \geq 0$, *one has* $g(s) \geq Ls - C_g$ *for all* $s > 0$ *(affine lower envelope).*

(ii) *If* g *is piecewise linear (e.g.,* $g(s) = (s-K)_+$*), there exists* $s^* < \infty$ *such that* $g'(s) = L$ *for* $s \geq s^*$, *i.e.,* g *is affine on* $[s^*, \infty)$.

Proof. (i). g'_+ exists everywhere and is *nondecreasing* in s (standard convex-analysis fact: the right derivative of a convex function is nondecreasing; see, e.g., [28, Thm. 24.1]). Since g is nondecreasing, $g'_+ \geq 0$ (crucial: without this assumption, g'_+ could be negative in some regions, breaking the argument). Since g is L-Lipschitz, $g'_+ \leq L$. Being nondecreasing and bounded above, g'_+ has a limit $\ell := \lim_{s\to\infty} g'_+(s) \leq L$. For a convex function, $\mathrm{Lip}(g) = \sup_s g'_+(s)$; combined with monotonicity of g'_+, the supremum equals the limit, so $\mathrm{Lip}(g) = \ell$. Since $\mathrm{Lip}(g) = L$, we get $\ell = L$.

(ii). For piecewise linear g, the derivative g'_+ takes finitely many values and is nondecreasing. Since its limit is L and it is nondecreasing, there exists s^* with $g'_+(s^*) = L$, and then $g'_+(s) = L$ for all $s \geq s^*$. □

Corollary 5.4. *A convex, nondecreasing, non-constant function on* $(0,\infty)$ *is unbounded:* $g(s) \to \infty$ *as* $s \to \infty$. *In particular, there is no non-trivial bounded payoff in this class.*

Theorem 5.5 (Exact Lipschitz degradation)**.** *Under definition 5.1, let* $h_n := \sqrt{e^{c/n} - 1}$ *and* $\alpha_n := (1+h_n)^n$. *Then for every* $m \in \{0, \ldots, n\}$*:*

$$L_m^B(n) \;=\; B + (L-B)(1+h_n)^{n-m}. \tag{9}$$

In particular, $L_0^B(n) = B + (L-B)\,\alpha_n$, *which grows as* $e^{\Theta(\sqrt{cn})}$ *as* $n \to \infty$ *with* c *fixed. This holds for every non-constant convex nondecreasing* L*-Lipschitz* g *(using the asymptotic slope* $g'_+(s) \to L$ *from lemma 5.3(i)): the limiting Lipschitz constant as* $s \to \infty$ *depends only on* L, *not on the shape of* g *on any bounded interval.*

Proof. Write $\Lambda_m := B+(L-B)(1+h_n)^{n-m}$ for the claimed Lipschitz constant. We prove $L_m^B(n) = \Lambda_m$ by backward induction.

Convexity preservation. $V_n^B(\cdot, m)$ is convex: for each fixed μ, $s \mapsto \mathbb{E}_\mu[V_n^B(s(1+T), m+1)] - Bs|\mathbb{E}[T]|$ is convex in s (expectation of convex + linear), and the pointwise supremum preserves convexity.

Monotonicity preservation. $V_n^B(\cdot, m)$ is nondecreasing for all m. Since g is nondecreasing, $g(0^+) := \lim_{s\to 0^+} g(s)$ exists and is finite (g is L-Lipschitz). We first show $V_n^B(0^+, m) = g(0^+)$ for all m by backward induction. The base case $m = n$ is immediate. For the inductive step, write $f_\mu(s) = \mathbb{E}_\mu[V_n^B(s(1+T), m+1)] - Bs|\mathbb{E}_\mu[T]|$. Since $\zeta_n < 1$ implies $s(1+T) \to 0^+$ as $s \to 0^+$ for every $T \in [-\zeta_n, \zeta_n]$, bounded convergence gives $\lim_{s\to 0^+} f_\mu(s) = V_n^B(0^+, m+1)$ for *every* μ. An upper bound $V_n^B(s, m) \le V_n^B(0^+, m+1) + L_{m+1}^B(1+\zeta_n)\, s$ (using the naive Lipschitz bound from proposition 5.2(d), which is independent of monotonicity) and a lower bound $V_n^B(s, m) \ge f_{\delta_0}(s) = V_n^B(s, m+1)$ squeeze to $V_n^B(0^+, m) = V_n^B(0^+, m+1) = g(0^+)$. Next, the adversary can always play $T = 0$, so $V_n^B(s, m) \ge g(s) \ge g(0^+) = V_n^B(0^+, m)$; hence $\inf_{s>0} V_n^B(s, m) = V_n^B(0^+, m)$. Finally, a convex function on $(0, \infty)$ whose infimum is achieved at the left boundary is nondecreasing: if $f'_+(s_0) < 0$ at some s_0, then $f'_+(s) \le f'_+(s_0) < 0$ for all $s \le s_0$ (convexity), so $f(\varepsilon) \ge f(s_0) + |f'_+(s_0)|(s_0 - \varepsilon)$ for $\varepsilon \in (0, s_0)$, giving $f(0^+) \ge f(s_0) + |f'_+(s_0)|\, s_0 > f(s_0) \ge \inf f = f(0^+)$, a contradiction.

Since $V_n^B(\cdot, m)$ is convex and nondecreasing, its right derivative $V_{n,s}^{B,+}$ is nondecreasing and nonnegative, so $L_m^B = \lim_{s\to\infty} V_{n,s}^{B,+}(s, m)$.

Base case ($m = n$). $V_n^B(s, n) = g(s)$, so $L_n^B(n) = \lim_{s\to\infty} g'_+(s) = L = \Lambda_n$ by lemma 5.3(i).

Inductive step. Suppose $\lim_{s\to\infty} V_{n,s}^{B,+}(s, m+1) = \Lambda_{m+1}$ and $\Lambda_{m+1} > B$. (The invariant $\Lambda_m > B$ is preserved by the recurrence $\Lambda_m = B + (L-B)(1+h_n)^{n-m}$ with base $\Lambda_n = L > B$, so the hypothesis holds at every step of the induction.) For any $\varepsilon > 0$, there exists S_ε such that $V_{n,s}^{B,+}(s, m+1) \in [\Lambda_{m+1} - \varepsilon,\, \Lambda_{m+1}]$ for $s \ge S_\varepsilon$. For s large enough that $s(1-\zeta_n) \ge S_\varepsilon$, every $s(1+T)$ with $T \in [-\zeta_n, \zeta_n]$ satisfies $s(1+T) \ge S_\varepsilon$. Since $V_n^B(\cdot, m+1)$ is convex, its right derivative $V_{n,s}^{B,+}(\cdot, m+1)$ is nondecreasing; by the inductive hypothesis, $V_{n,s}^{B,+}(\cdot, m+1) \in [\Lambda_{m+1} - \varepsilon, \Lambda_{m+1}]$ on $[S_\varepsilon, \infty)$. For $s, s(1+T) \in [S_\varepsilon, \infty)$, the secant slope of a convex function is sandwiched by its one-sided derivatives at the endpoints; thus for $T \ne 0$,

$$\theta(s, T) \;:=\; \frac{V_n^B\big(s(1+T),\, m+1\big) - V_n^B(s, m+1)}{s\,T} \;\in\; [\Lambda_{m+1} - \varepsilon,\, \Lambda_{m+1}],$$

uniformly in $T \in [-\zeta_n, \zeta_n]$ and $\mu \in \widetilde{\mathcal{T}}_n$. This secant bound does *not* require differentiability of $V_n^B(\cdot, m+1)$. Extending $\theta(s, 0) := \Lambda_{m+1}$ by continuity, we write

$$V_n^B\big(s(1+T),\, m+1\big) \;=\; V_n^B(s, m+1) + \theta(s, T) \cdot s\,T$$

with $\theta(s, T) \in [\Lambda_{m+1} - \varepsilon, \Lambda_{m+1}]$ for all $T \in [-\zeta_n, \zeta_n]$.

The backward induction gives, writing $a = \mathbb{E}_\mu[T]$:

$$\begin{aligned} V_n^B(s, m) &= \sup_{\mu\in\widetilde{\mathcal{T}}_n} \big\{ V_n^B(s, m+1) + s\,\mathbb{E}_\mu[\theta \cdot T] - Bs|a| \big\} \\ &= V_n^B(s, m+1) + \sup_{\mu} \big\{ s\,\mathbb{E}[\theta \cdot T] - Bs|a| \big\}. \end{aligned}$$

Since $\theta \in [\Lambda_{m+1} - \varepsilon,\, \Lambda_{m+1}]$, splitting $T = T^+ - T^-$ and using $\mathbb{E}_\mu[T^-] \le \sqrt{\mathbb{E}_\mu[T^2]} \le h_n$ (Cauchy–Schwarz on the DVC),

$$\mathbb{E}_\mu[\theta \cdot T] \;\le\; \Lambda_{m+1}\, a \;+\; \varepsilon\, \mathbb{E}_\mu[T^-] \;\le\; \Lambda_{m+1}\, a \;+\; \varepsilon\, h_n,$$

so the inner supremum is sandwiched by

$$\sup_{|a|\le h_n}\left\{(\Lambda_{m+1}-\varepsilon)\,s\,a - Bs\,|a|\right\} \;\le\; \sup_{\mu}\{\cdots\} \;\le\; \sup_{|a|\le h_n}\left\{\Lambda_{m+1}\,s\,a - Bs\,|a|\right\} + \varepsilon\,s\,h_n.$$

Both extremes are maximized at $a = h_n$ (since $\Lambda_{m+1}-\varepsilon > B$ for small ε). The lower side is realized by the Dirac $\mu = \delta_{h_n}$, feasible under the standing hypothesis $\zeta_n \ge h_n$ in definition 5.1: at $\mu = \delta_{h_n}$, $\theta(s,h_n) \ge \Lambda_{m+1} - \varepsilon$, so $\mathbb{E}_\mu[\theta\cdot T] = \theta(s,h_n)\,h_n \ge (\Lambda_{m+1}-\varepsilon)\,h_n$, saturating the variance budget $\mathbb{E}[T^2] = h_n^2 = e^{c/n}-1$. Since $V_n^B(\cdot,m)$ is convex and nondecreasing, $L_m^B = \lim_{s\to\infty}(V_n^B)'_+(s,m)$ (for such f, $\mathrm{Lip}(f) = \sup f'_+ = \lim_{s\to\infty} f'_+(s)$). The uniform control across the support band $[s(1-\zeta_n), s(1+\zeta_n)]$ comes from subgradient monotonicity: once $s(1-\zeta_n) \ge S_\varepsilon$, $(V_n^B)'_+(\cdot,m+1)$ is sandwiched in $[\Lambda_{m+1}-\varepsilon, \Lambda_{m+1}]$ on $[S_\varepsilon,\infty)$, hence on the entire band. No interchange of limits is required: once $s(1-\zeta_n) \ge S_\varepsilon$, every realization $s(1+T)$ lands in the affine-slope region, so $\theta(s,T) \in [\Lambda_{m+1}-\varepsilon, \Lambda_{m+1}]$ *uniformly* over $T \in [-\zeta_n,\zeta_n]$ and $\mu \in \widetilde{\mathcal{T}}_n$. Consequently, for all such s,

$$\sup_{\mu\in\widetilde{\mathcal{T}}_n}\left\{s\,\mathbb{E}_\mu[\theta\cdot T] - Bs|\mathbb{E}_\mu[T]|\right\} \;\in\; \Big[\sup_{|a|\le h_n}\{(\Lambda_{m+1}-\varepsilon)sa - Bs|a|\},\; \sup_{|a|\le h_n}\{\Lambda_{m+1}\,sa - Bs|a|\} + \varepsilon s h_n\Big],$$

where both extremes are attained at $a = h_n$ (since $\Lambda_{m+1}-\varepsilon > B$ for small ε) and equal $(\Lambda_{m+1}-\varepsilon-B)\,s\,h_n$ and $(\Lambda_{m+1}-B)\,s\,h_n + \varepsilon s h_n$, respectively. The residual $\varepsilon s h_n$ vanishes as $\varepsilon\to 0$ at fixed s,n. Taking $s\to\infty$ first and then $\varepsilon\to 0$:

$$\lim_{s\to\infty} V_{n,s}^{B,+}(s,m) \;=\; \Lambda_{m+1} + (\Lambda_{m+1}-B)\,h_n \;=\; \Lambda_{m+1}(1+h_n) - B\,h_n.$$

This equals $B + (\Lambda_{m+1}-B)(1+h_n) = \Lambda_m$, confirming the recurrence. □

Remark 5.6 (Disproof of the uniform Lipschitz conjecture)**.** Theorem 5.5 shows $L_m^B(n)\to\infty$ for *every* non-constant g in this class — not just affine g. By lemma 5.3(i), every such g has $g'(s)\to L$ as $s\to\infty$, and backward induction amplifies this asymptotic slope exponentially. Local curvature near the kink is irrelevant: the adversary drifts into the region where $g' \approx L$.

Remark 5.7 (Why curvature cannot help)**.** Strict convexity does penalize drift locally, but the drift benefit is $O(1/\sqrt{n})$ per step while the curvature cost is $O(1/n)$, so for large n the adversary rationally ignores variance and drifts deterministically. Curvature operates at a lower order than drift compounding.

Corollary 5.8 (Finite-s grid threshold for the asymptotic Lipschitz constant)**.** *Assume definition 5.1 and, in addition, that g is piecewise linear with affine tail $g(s) = Ls + \beta_\infty$ on $[s^*,\infty)$ for some $s^* < \infty$ (as in lemma 5.3(ii); e.g., $g(s) = (s-K)_+$ with $s^* = K$ and $\beta_\infty = -K$). Here $\beta_\infty := \lim_{x\to\infty}[g(x) - Lx]$ is the affine-tail intercept, to be used formally in theorem 6.2(c) and theorem 6.3; it is distinct from the Black–Scholes limit value β of theorem 3.3. Let $h_n = \sqrt{e^{c/n}-1}$ and $V_{n,s}^{B,+}$ denote the right derivative of $V_n^B(\cdot,m)$ in s. Then for every $m\in\{0,\dots,n\}$,*

$$V_{n,s}^{B,+}(s,m) \;=\; L_m^B(n) \qquad \text{for all } s \;\ge\; \frac{s^*}{(1-h_n)^{n-m}}. \tag{10}$$

In particular, $V_{n,s}^{B,+}(s,0) = L_0^B(n) = B + (L-B)\,\alpha_n$ for $s \ge s^*/(1-h_n)^n$. For s strictly below this threshold the identity can fail: the support band straddles s^*, and whenever g is strictly sub-affine somewhere on the subrange $(0,s^*)$ visited by the backward-induction integral, propagation gives $V_{n,s}^{B,+}(s,0) \le L_0^B(n)$ with strict inequality in generic cases (we do not prove a uniform strict deficit for all s below threshold).

Proof. Fix m and let $s \geq s^*/(1-h_n)^{n-m}$; we show $V_{n,s}^{B,+}(s,m) = \Lambda_m := B + (L-B)(1+h_n)^{n-m}$ by backward induction on the distance $k := n-m$. For $k=0$, the hypothesis gives $s \geq s^*$, so $V_n^B(\cdot,n) = g(\cdot)$ is affine with slope $L = \Lambda_n$ on $[s^*,\infty) \ni s$, and $g'_+(s) = L$. For the inductive step, suppose the identity holds at $m+1$ on $[s^*/(1-h_n)^{n-m-1},\infty)$. If $s \geq s^*/(1-h_n)^{n-m}$, then for every $T \in [-h_n, h_n]$,

$$s(1+T) \;\geq\; s(1-h_n) \;\geq\; \frac{s^*}{(1-h_n)^{n-m-1}},$$

so $V_{n,s}^{B,+}(s(1+T), m+1) = \Lambda_{m+1}$ is *exactly constant* (not merely close) on the entire support band. The argument of the inductive step of theorem 5.5 then applies with $\varepsilon = 0$ and $S_0 = s^*/(1-h_n)^{n-m-1}$: the secant slope $\theta(s,T) \equiv \Lambda_{m+1}$ identically, the inner supremum in (9) is attained at $a = h_n$ with value $(\Lambda_{m+1} - B)\, s\, h_n$, and differentiating in s gives $V_{n,s}^{B,+}(s,m) = \Lambda_{m+1}(1+h_n) - Bh_n = \Lambda_m$. The converse claim follows because $V_{n,s}^{B,+}(\cdot,m)$ is nondecreasing (as the right derivative of a convex function), and for s below the threshold the backward-induction integral sees some mass on $(0,s^*)$ where $g'_+ < L$ strictly, which propagates to a strict secant deficit $\theta < \Lambda_{m+1}$ at step $m+1$ and hence $V_{n,s}^{B,+}(s,m) < \Lambda_m$. □

Remark 5.9 (Implications for numerical verification of theorem 5.5)**.** Corollary 5.8 gives a sharp operational criterion for experiments that probe the Lipschitz-degradation formula $L_0^B(n) = B + (L-B)\alpha_n$ by measuring the tail slope of the numerically computed $V_n^B(\cdot,0)$. Since $(1-h_n)^n = e^{-c(1+o(1))}$ as $n \to \infty$, the threshold satisfies

$$\frac{s^*}{(1-h_n)^n} \;=\; s^* \cdot \alpha_n \cdot \big(1 + O(h_n^2)\big)^n \;=\; s^* \cdot \alpha_n \cdot e^{c+o(1)} \;=\; \Theta\big(s^* \cdot e^{\sqrt{cn}}\big),$$

i.e., grids must extend to $s_{\max} \geq s^*/(1-h_n)^n$ to resolve the asymptotic slope. Truncating at $s_{\max} \sim \alpha_n \cdot s^*$ (naively "one multiplicative horizon above the kink") falls short of the threshold by a factor of $e^{c+o(1)}$ and produces numerically attenuated slope ratios $V_{n,s}^{B,+}(s_{\max},0)/L_0^B(n) < 1$, even though the closed-form identity is exact. A safe default is $s_{\max} = \Omega(\alpha_n^2 \cdot s^*)$, which dominates the threshold for all $c > 0$ and n sufficiently large. Experiment E4 reported in section 9 uses a fixed small-n grid $s_{\max} = 3\, s^*$; at $n = 12$, this is just below the threshold ($s^*/(1-h_{12})^{12} \approx 3.18\, s^*$), which partially accounts for the small-n over-prediction observed in the numerical/exact ratio (cf. the $n = 2$, $B = 0.15$ ratio of 3.1).

6 Premium Characterization

Definition 6.1 (Premium)**.** Let $V_n^\infty(s,m) := V_n^{B=\infty}(s,m)$ be the unconstrained (martingale-forced) backward induction. The *premium* is

$$\Phi_n^B(s) \;:=\; V_n^B(s,0) \;-\; V_n^\infty(s,0) \;\geq\; 0.$$

Theorem 6.2 (Tight premium bounds)**.** *Under definition 5.1 with $g(0) = 0$, let $h_n = \sqrt{e^{c/n} - 1}$ and $\alpha_n = (1+h_n)^n$.*

(a) Upper bound. *For all $s > 0$:*

$$\Phi_n^B(s) \;\leq\; (L-B)(\alpha_n - 1)\, s \;+\; C_g, \tag{11}$$

where $C_g = g(0) - \inf_{s>0}\big[g(s) - Ls\big]$ depends only on g.

(b) Lower bound. *For all $s > 0$:*

$$\Phi_n^B(s) \;\geq\; g(s\,\alpha_n) \;-\; B\, s\, (\alpha_n - 1) \;-\; V_n^\infty(s,0). \tag{12}$$

(c) Exponential growth. *As $n \to \infty$, $\alpha_n \sim e^{\sqrt{cn}}$. For piecewise-linear g (so that $\beta_\infty := \lim_{x\to\infty}[g(x) - Lx] > -\infty$ and $s^* := \inf\{x : g'_+(x) = L\} < \infty$ by lemma 5.3(ii)), and for any fixed $s > 0$ and n large enough that $s\,\alpha_n \geq s^*$ (equivalently $n \gtrsim (\log(s^*/s))^2/c$), both bounds are $\Theta\big((L - B)\, e^{\sqrt{cn}}\, s\big)$. In particular, $\Phi_n^B(s) = \Theta(e^{\sqrt{cn}})$.*

(d) Tightness. *For $g(s) = Ls$: $\Phi_n^B(s) = (L - B)(\alpha_n - 1)\, s$ exactly, and the upper bound is achieved.*

Proof. Upper bound. Since $V_n^B(0, m) = g(0) = 0$ ($s = 0$ is absorbing) and $V_n^B(\cdot, 0)$ has Lipschitz constant $L_0^B = B + (L - B)\alpha_n$ (theorem 5.5), $V_n^B(s, 0) \leq L_0^B\, s$. Jensen gives $V_n^\infty(s, 0) \geq g(s)$ (martingale with mean s, g convex); since $g'_+ \to L$ (lemma 5.3), $g(s) \geq Ls - C_g$ for a constant C_g depending only on g, so $V_n^\infty(s, 0) \geq Ls - C_g$. Therefore $\Phi_n^B(s) \leq L_0^B\, s - Ls + C_g = (L - B)(\alpha_n - 1)\, s + C_g$.

Lower bound. Consider $T = h_n$ deterministically at every step (pure drift, zero variance). This is feasible since $h_n^2 = e^{c/n} - 1$ and $h_n \leq \zeta_n$ by the standing hypothesis of definition 5.1. The state evolves as $S_m = s\,(1 + h_n)^m$, so $S_n = s\,\alpha_n$. This achieves:

$$V_n^B(s, 0) \;\geq\; g(s\,\alpha_n) - \sum_{m=0}^{n-1} B\, s\,(1 + h_n)^m\, h_n = g(s\,\alpha_n) - B\, s\,(\alpha_n - 1).$$

Subtracting $V_n^\infty(s, 0)$ gives (12).

Matching for large s. Since $g'_+ \to L$ (lemma 5.3), $g(s\,\alpha_n) = L\,s\,\alpha_n - o(s\,\alpha_n)$, so the lower bound becomes $(L - B)(\alpha_n - 1)\, s + Bs - o(s\alpha_n) - V_n^\infty(s, 0)$. Since $V_n^\infty(s, 0) \to Ls + O(1)$ for large s, the lower bound is $(L - B)(\alpha_n - 1)\, s + O(1)$, matching the upper bound. □

Theorem 6.3 (Premium asymptotics)**.** *Under the hypotheses of theorem 6.2, assume additionally that $\beta_\infty := \lim_{x\to\infty}[g(x) - Lx] > -\infty$ (holds whenever g has an affine tail, e.g., $g(s) = (s - K)_+$). Let $g_\infty(s) := Ls + \beta_\infty$ and $V^\infty(s) := \lim_{n\to\infty} V_n^\infty(s, 0) = \mathbb{E}[g(s\,G(1))]$ where $G(1) = \exp(\sqrt{c}\,W(1) - c/2)$.*

Then for every fixed $s > 0$:

$$\Phi_n^B(s) \;=\; (L - B)(\alpha_n - 1)\, s \;+\; \big[g_\infty(s) - V^\infty(s)\big] \;+\; O(1). \tag{13}$$

The $O(1)$ remainder is nonnegative and bounded by $|\beta_\infty| + [V^\infty(s) - g_\infty(s)]$, where $V^\infty(s) - g_\infty(s) \geq 0$ is the "time value" correction (zero for affine g). For the European call $g(s) = (s - K)_+$: $g_\infty(s) = s - K$ and $g_\infty(s) - V^\infty(s) = -$(time value).

Proof. Lower bound. The pure-drift strategy (theorem 6.2(b)) gives $V_n^B(s, 0) \geq g(s\alpha_n) - Bs(\alpha_n - 1)$. For fixed $s > 0$: $s\alpha_n \to \infty$, so $g(s\alpha_n) - Ls\alpha_n \to \beta_\infty$, hence $g(s\alpha_n) - Bs(\alpha_n - 1) = (L - B)(\alpha_n - 1)s + g_\infty(s) + o(1)$. Since $V_n^\infty(s, 0) \to V^\infty(s)$: $\Phi_n^B(s) \geq (L - B)(\alpha_n - 1)s + g_\infty(s) - V^\infty(s) + o(1)$.

Upper bound. $V_n^B(s, 0) \leq L_0^B\, s = [(L - B)\alpha_n + B]\, s$ (theorem 5.5). Jensen gives $V_n^\infty(s, 0) \geq g(s)$. Since $g'_+ \leq L$, $x \mapsto g(x) - Lx$ is nonincreasing with limit β_∞ by lemma 5.3(i); hence $g(s) \geq g_\infty(s)$, so $V_n^\infty(s, 0) \geq g_\infty(s)$. Therefore $\Phi_n^B(s) \leq (L - B)(\alpha_n - 1)s + Ls - g_\infty(s) = (L - B)(\alpha_n - 1)s + |\beta_\infty|$. Also $g(s\alpha_n) - Ls\alpha_n \downarrow \beta_\infty$, so the $o(1)$ term in the lower bound is nonnegative; the remainder lies in $[0,\ |\beta_\infty| + V^\infty(s) - g_\infty(s)]$. □

Theorem 6.4 (Exact remainder for piecewise-linear payoffs)**.** *Assume the adversary's support radius satisfies $\zeta_n = h_n$ (the tight support case). Suppose g is piecewise linear with finitely many kinks, affine on $[s^*, \infty)$ with $g'_+(s^*) = L$ and $g(x) = Lx + \beta_\infty$ there (e.g., European call $g(s) =$*

$(s-K)_+$ *with* $s^* = K$, $\beta_\infty = -K$*). Set* $M_\psi := g(0) - \beta_\infty$*. For every fixed* $s > 0$ *and every* n *such that*

$$s\,\alpha_n \;>\; s^* \;+\; \frac{M_\psi}{L-B}, \tag{14}$$

the remainder $R_n(s) := V_n^B(s,0) - [(L{-}B)(\alpha_n - 1)s + g_\infty(s)]$ *vanishes exactly:*

$$V_n^B(s,0) \;=\; (L{-}B)(\alpha_n - 1)\,s \;+\; g_\infty(s).$$

For fixed s, B, L, c *the threshold* (14) *holds for all* $n \geq n_0(s, s^*, L, B, c, M_\psi)$*. For* $\zeta_n > h_n$*, the same statement holds with* α_n *replaced by* $(1+\zeta_n)^n$ *throughout.*

Proof. Setup. Define the shifted value $\widetilde{V}_m(x) := V_n^B(x,m) - g_\infty(x) - (L{-}B)(\alpha_{n-m} - 1)x$, so $R_n(s) = \widetilde{V}_0(s)$. The terminal $\widetilde{V}_n(x) = g(x) - Lx - \beta_\infty$ is nonnegative (by $g \geq g_\infty$, lemma 5.3), vanishes on $[s^*, \infty)$ (piecewise linearity), and is bounded above by $M_\psi = \widetilde{V}_n(0)$.

Shifted Bellman recursion. The Bellman recursion for V_n^B, shifted by the affine-tail subtraction $g_\infty(x) + (L{-}B)(\alpha_{n-m} - 1)x$ and using the identity $\alpha_{n-m}(1{+}h_n) = \alpha_{n-m+1}$, yields

$$\widetilde{V}_{m-1}(x) \;=\; \sup_{\mu \in \widetilde{\mathcal{T}}_n} \big\{ \mathbb{E}_\mu[\widetilde{V}_m(x(1{+}T))] - (L{-}B)\,x\,\alpha_{n-m}\,\eta_m \big\}, \tag{15}$$

where $\eta_m := h_n - a_\mu$ with $a_\mu := \mathbb{E}_\mu[T]$. Since $|a_\mu| \leq h_n$ (Cauchy–Schwarz on $\mathbb{E}_\mu[T^2] \leq h_n^2$), we have $\eta_m \in [0, 2h_n]$. *No sign restriction on* a_μ *is needed below*: the Markov argument applied to the deficit $Z_n := s\alpha_n - S_n$ (defined precisely in the upper-bound step) bounds (15) uniformly over $\mu \in \widetilde{\mathcal{T}}_n$.

Telescoping identity. For any adapted strategy, $\mathbb{E}[S_{m-1}\eta_m] = (1{+}h_n)\mathbb{E}[S_{m-1}] - \mathbb{E}[S_m]$ (from $\mathbb{E}[S_{m-1}T_m] = \mathbb{E}[S_m] - \mathbb{E}[S_{m-1}]$ and tower). Multiplying by α_{n-m} and summing, the identity $\alpha_{n-m}(1+h_n) = \alpha_{n-m+1}$ makes the sum telescope to

$$D_n \;:=\; \sum_{m=1}^{n} \mathbb{E}\big[S_{m-1}\,\alpha_{n-m}\,\eta_m\big] \;=\; s\alpha_n - \mathbb{E}[S_n]. \tag{16}$$

Unrolling the recursion,

$$R_n(s) \;=\; \sup_{\text{adversary strategies}} \Big\{ \mathbb{E}[\widetilde{V}_n(S_n)] - (L{-}B)(s\alpha_n - \mathbb{E}[S_n]) \Big\}.$$

Upper bound $R_n \leq 0$ *via Markov applied to* Z_n*.* Since $|T_m| \leq h_n$, the random variable $Z_n := s\alpha_n - S_n \geq 0$ almost surely, with $\mathbb{E}[Z_n] = D_n$. Applying Markov's inequality to Z_n,

$$\Pr(S_n < s^*) \;=\; \Pr(Z_n > s\alpha_n - s^*) \;\leq\; \frac{D_n}{s\alpha_n - s^*}.$$

Since $\widetilde{V}_n \leq M_\psi$ and vanishes on $\{S_n \geq s^*\}$,

$$\mathbb{E}[\widetilde{V}_n(S_n)] \;\leq\; M_\psi\,\Pr(S_n < s^*) \;\leq\; \frac{M_\psi\,D_n}{s\alpha_n - s^*}.$$

Therefore, for any adapted adversary strategy,

$$\mathbb{E}[\widetilde{V}_n(S_n)] - (L{-}B)D_n \;\leq\; D_n \left[\frac{M_\psi}{s\alpha_n - s^*} \;-\; (L{-}B) \right].$$

Assumption (14) makes the bracket strictly negative, so the expression is ≤ 0 for every $D_n \geq 0$.

Lower bound $R_n \geq 0$ *via pure drift.* The strategy $\mu_m \equiv \delta_{h_n}$ is feasible in $\widetilde{\mathcal{T}}_n$ (variance h_n^2 saturated, support at the boundary), gives $S_n = s\alpha_n$ deterministically and $D_n = 0$. With $s\alpha_n \geq s^*$: $\widetilde{V}_n(s\alpha_n) = 0$. Hence this strategy achieves $\widetilde{V}_0(s) \geq 0$, i.e., $R_n(s) \geq 0$.

Combining: $R_n(s) = 0$. □

Remark 6.5 (Pure drift is optimal; no boundary layer)**.** Theorem 6.4 strengthens theorem 6.3: once n exceeds the threshold (14), pure drift $\mu_m \equiv \delta_{h_n}$ is the adversary's exact optimum and the closed-form expression $V_n^B(s,0) = (L-B)(\alpha_n - 1)s + g_\infty(s)$ is exact, not just the leading asymptotic. In particular, no $o(1)$ remainder term appears: the "boundary layer" has zero width for piecewise-linear g, because the adversary cannot gain by mixing drift with variance — any Jensen benefit at the kink is offset by the deterministic drift deficit $D_n > 0$, and Markov (applied to Z_n) quantifies the offset exactly. The constant n_0 is explicit: for ATM call ($s = s^* = K$, $M_\psi = K$, $L = 1$), (14) reduces to $\alpha_n > (2-B)/(1-B)$, i.e., $n\log(1+h_n) > \log((2-B)/(1-B))$, giving $n_0 = O((\log((2-B)/(1-B)))^2/c)$. (Sanity check: at $B = 0.3$, threshold is $\alpha_n > 1.7/0.7 \approx 2.43$, matching experiment N4 in section 9.)

Proposition 6.6 (Smooth-tail remainder)**.** *Under the hypotheses of definition 5.1 with $\zeta_n = h_n$, suppose g is convex nondecreasing L-Lipschitz with tail residual $\psi(x) := g(x) - Lx - \beta_\infty \geq 0$, $\psi \to 0$ as $x \to \infty$. Let $M_\psi := g(0) - \beta_\infty$ (assumed finite). For every $s > 0$, $\tau \geq s^*$ with $s\alpha_n > \tau$, and all n such that $(L - B) > M_\psi/(s\alpha_n - \tau)$,*

$$0 \;\leq\; R_n(s) \;\leq\; \psi(\tau),$$

where $R_n(s) := V_n^B(s,0) - [(L-B)(\alpha_n - 1)s + g_\infty(s)]$. Optimizing over τ yields the following rates:

(i) ***Exponential tail*** *$\psi(x) = O(e^{-x/\ell})$ (e.g., softplus $g(s) = \ell\log(1 + e^{s/\ell})$, logistic smoothings of the call): $R_n(s) = O(e^{-cs\alpha_n/\ell})$, stretched-exponential decay.*

(ii) ***Polynomial tail*** *$\psi(x) = O(x^{-p})$, $p > 0$ (e.g., Huber-type $g(s) = L\sqrt{s^2 + b^2}$ has $p = 1$): $R_n(s) = O((s\alpha_n)^{-p}) = O(e^{-p\sqrt{cn}})$.*

(iii) ***Logarithmic tail*** *$\psi(x) = O(1/\log x)$ with $M_\psi = g(0) - \beta_\infty < \infty$: $R_n(s) = O(1/\sqrt{cn})$.*

Proof. The lower bound $R_n \geq 0$ is immediate (the pure-drift strategy attains the claimed closed form from below).

For the upper bound, the Markov argument (applied to Z_n) of theorem 6.4 carries through with the modified terminal:

$$\widetilde{V}_n(x) \;=\; g(x) - Lx - \beta_\infty \;=\; \psi(x) \;\leq\; \begin{cases} M_\psi & x \in [0,\tau], \\ \psi(\tau) & x \in [\tau,\infty), \end{cases}$$

using monotonicity of ψ (which follows from $\psi(x) = \int_x^\infty (L - g'_+(u))\,du$ and $g'_+ \leq L$). Applying Markov's inequality to $Z_n := s\alpha_n - S_n \geq 0$ on the event $\{S_n \leq \tau\}$ (equivalently $\{Z_n \geq s\alpha_n - \tau\}$),

$$\Pr(S_n \leq \tau) \;\leq\; \frac{\mathbb{E}[Z_n]}{s\alpha_n - \tau} \;=\; \frac{D_n}{s\alpha_n - \tau},$$

where $D_n = \mathbb{E}[Z_n] = s\alpha_n - \mathbb{E}[S_n]$ from (16). Splitting the expectation at τ and using $\widetilde{V}_n \leq \psi(\tau)$ on $[\tau,\infty)$,

$$\mathbb{E}[\widetilde{V}_n(S_n)] \;\leq\; M_\psi \cdot \Pr(S_n \leq \tau) + \psi(\tau) \;\leq\; \frac{M_\psi D_n}{s\alpha_n - \tau} + \psi(\tau).$$

Combining with the Bellman representation $R_n(s) = \sup_{\text{adv}}\{\mathbb{E}[\widetilde{V}_n(S_n)] - (L-B)D_n\}$ from the proof of theorem 6.4: for every adversary strategy,

$$\mathbb{E}[\widetilde{V}_n(S_n)] - (L-B)D_n \;\leq\; \psi(\tau) + D_n\left[\frac{M_\psi}{s\alpha_n - \tau} - (L-B)\right] \;\leq\; \psi(\tau),$$

since $D_n \geq 0$ and the bracket is ≤ 0 by the standing hypothesis. Taking the supremum over adversary strategies yields $R_n(s) \leq \psi(\tau)$.

For each tail class, we choose $\tau = s\alpha_n/2$ (so $s\alpha_n - \tau = s\alpha_n/2$) and check the hypothesis $(L - B) > M_\psi/(s\alpha_n - \tau) = 2M_\psi/(s\alpha_n)$. Since $\alpha_n \to \infty$, this hypothesis holds for every fixed L, B, s, M_ψ and all $n \geq n_2(L, B, s, M_\psi, c)$. Substituting the corresponding ψ decay: (i) exponential $\psi(\tau) = O(e^{-\tau/\ell}) = O(e^{-s\alpha_n/(2\ell)})$ gives $R_n = O(e^{-s\alpha_n/(2\ell)} + 1/(s\alpha_n))$, and the exponential term dominates; (ii) polynomial $\psi(\tau) = O(\tau^{-p})$ gives $R_n = O((s\alpha_n)^{-p}) = O(e^{-p\sqrt{cn}})$ with M_ψ finite under polynomial tail; (iii) logarithmic $\psi(\tau) = O(1/\log\tau)$ gives $R_n = O(1/\log(s\alpha_n)) = O(1/\sqrt{cn})$, here requiring $M_\psi := g(0) - \beta_\infty < \infty$ explicitly (logarithmic tail satisfies $\psi \to 0$ in the limit but M_ψ need not be finite in general; we restrict to payoffs with $\lim_{x\to 0^+} g(x)$ finite, e.g. $g(x) = L\sqrt{x^2 + b^2}/\log(2 + x)$). □

Remark 6.7 (Retraction of the $O(n^{-1/4})$ heuristic)**.** The polynomial $O(n^{-1/4})$ rate conjectured in earlier versions of this paper was an overestimate driven by a heuristic that localized the boundary layer to $O(\sqrt{n})$ steps; the correct localization puts it at the single "kink-crossing step" on the drift path. Theorem 6.4 and proposition 6.6 replace the heuristic: for piecewise-linear g the rate is identically zero, and for every other payoff of practical interest (softplus, logistic, Huber, Black–Scholes with finite strike) it is at worst stretched-exponential.

Corollary 6.8 (Bounded-vs-unbounded dichotomy)**.** *For every non-constant convex nondecreasing L-Lipschitz g with affine tail, and any $0 < B < L$, $\Phi_n^B(s)$ grows as $\Theta(e^{\sqrt{cn}})$. The exponential rate $\sqrt{cn}$ is shape-independent within this class — it depends on (L, B, c) but not on the specific shape of g. In the unbounded-action case ($B = \infty$), by contrast, the value V_n^∞ converges to the Black-Scholes value β (AFW 2012). The bridge thus establishes a sharp dichotomy: bounded per-step capital makes the game exponentially expensive; unbounded capital makes it converge.*

Remark 6.9 (Financial interpretation)**.** In the option pricing context: with hedge bound $|\Delta_m| \leq B < L = \mathrm{Lip}(g)$, the worst-case pricing bound exceeds the Black–Scholes price by an *exponentially* growing amount. The mechanism is simple: the adversary drifts the price deterministically upward at rate $(1+h_n)$ per step, pushing it into the linear tail of the payoff where the hedger is permanently underhedged. The hedger's cost $B\,s\,|a|$ per step is a constant fraction of the adversary's benefit $(L - B + B)\,s\,|a| = L\,s\,|a|$, so the net gain compounds multiplicatively across n steps. Bounded hedging of unbounded payoffs is *exponentially costly* in the worst case.

Remark 6.10 (Failure of the HJB limit: calibration mismatch, not PDE pathology)**.** A previous version of this work conjectured convergence to the HJB equation $V_t^B + \frac{cs^2}{2} V_{ss}^B + \sqrt{c}\,s\,(V_s^B - B)_+ = 0$. We make precise which hypothesis of the Barles–Souganidis [6] viscosity-convergence framework fails. That framework requires the discrete scheme S_n satisfy $[S_n\phi - \phi]/\Delta t \to F(\phi, \phi_s, \phi_{ss})$ at smooth test ϕ, with F a *finite* operator. For V_n^B with $\Delta t = 1/n$: the adversary's optimal drift $|a^*| = h_n \approx \sqrt{c/n}$ gives a one-step increment of order $O(n^{-1/2})$, and dividing by $\Delta t = 1/n$ produces a drift residual $\sqrt{cn} \to \infty$, so no finite F exists. Since V_n^B itself diverges (theorem 6.2), no finite PDE limit exists.

The underlying reason is a calibration mismatch with the weak-convergence regime, not a pathology of PDE methods. The stronger AFW [3] calibration $n\zeta_n^2/\log n > 16c$ (i.e. $\zeta_n^2/h_n^2 \to \infty$) enforces Lindeberg/uniform-integrability conditions ruling out Dirac adversaries and recovers a Brownian scaling limit (AFW prove convergence to Black–Scholes for the *unbounded-action* value under this calibration). Our per-step-budget hypotheses operate at the weaker calibration $\zeta_n \geq h_n$, where Dirac/burst adversaries are feasible and no Brownian limit exists. We emphasize that this divergence is specific to the discrete game with per-period bounded actions, where drift accumulates

at rate $O(1/\sqrt{n})$ per step and compounds multiplicatively. The continuous-time robust superhedging problem under portfolio constraints is a *different* game: Cvitanić–Karatzas [13] bound Δ_t in an Itô setting, where the adversary is Brownian (not adversarial over variance-constrained laws) and the state is driven by a *pathwise* SDE rather than a per-step worst-case distribution. Their value function satisfies a nonlinear Black–Scholes–type PDE with a gradient-constraint penalty; an analogous viscosity formulation via stochastic target problems is given by Soner–Touzi [32]. Both formulations yield *finite* values because the adversary's one-step drift is $O(\Delta t)$ (diffusive), not $O(\sqrt{\Delta t})$ (adversarial variance-constrained). Our discrete per-period game has strictly coarser adversary rights at each step: choosing any law with $\mathbb{E}[T^2] \le h_n^2$ is genuinely more flexible than choosing a diffusion coefficient, and variance-constrained drift of order $h_n = \Theta(n^{-1/2})$ scaled by n steps produces multiplicative compounding absent in the diffusion limit. In this sense the divergence of V_n^B is not a failure of our analysis; it reveals that per-step variance-constrained adversaries with bounded capital sit in a genuinely different regime from continuous-time pathwise constrained hedging. Whether a *rescaled* quantity (e.g., V_n^B/α_n or the $O(1)$ remainder from theorem 6.3, which by theorem 6.4 vanishes for piecewise-linear g) converges to a gradient-constrained variational inequality is an open problem.

Open questions.

1. **Smooth-tail rates.** Theorem 6.4 gives $R_n = 0$ exactly for piecewise-linear g and n beyond the explicit threshold (14). Proposition 6.6 extends this to stretched-exponential or polynomial rates for g with non-compactly-supported tail residual; quantifying the constants for specific smooth payoffs (softplus, Huber, Black–Scholes) is a routine extension.

2. **Non-convex or non-monotone losses.** For bounded losses (put options, capped calls) that are not convex-and-nondecreasing, the structural lemma does not apply and the premium may be $O(1)$; see proposition 11.10. Characterizing the premium for general g is open.

3. **Per-step vs aggregate action bounds.** The per-step bound $|\Delta_m| \le B$ decomposes via backward induction; theorems 11.2 and 11.3 in §11 establish polynomial lower bounds for the aggregate premium $\Phi_n^{\mathrm{agg},B}$, and proposition 11.1 bounds it above by the per-step rate. Determining the tight aggregate rate — in particular whether the polynomial lower bound $\Omega(n^{L/(2B)}/\log n)$ matches the true rate up to polylog factors — remains open.

7 Consequences

7.1 Constructive optimal strategies

The backward induction bridge converts Sion's non-constructive existence guarantee into an explicit algorithm. By theorem 4.6(b), the player's optimal action at step m is $\Delta_m^*(s) = h_m^*(s)$; by the Markov property each h_m^* depends only on the current state. Discretizing onto q grid points, the algorithm proceeds backward from $m = n$: at each step and grid point, solve the one-step entropic MOT via Sinkhorn + proximal gradient ascent (section 2.3). Total cost: $O(n \cdot q^2 \cdot K)$, where K is the number of PGA iterations per step; proposition 4.8 gives total *absolute* error $\le \alpha$ by choosing $\varepsilon = \alpha/(n \log q)$.

Intrinsic rate vs. algorithmic cost. Because the game value itself grows as $\Theta((L-B)\, e^{\sqrt{cn}}\, s)$ (theorem 6.2), any α-accurate approximation with α *fixed* becomes uninformative at large n. Targeting *relative* error α (i.e., $\alpha \cdot e^{\sqrt{cn}}$ absolute) requires the grid to resolve a support of radius $R \sim s\alpha_n = s\, e^{\Theta(\sqrt{cn})}$ around s, hence $q = e^{\Theta(\sqrt{cn})}$ for fixed grid spacing, giving total cost

$n \cdot q^2 \cdot K = e^{\Theta(\sqrt{cn})} \cdot \mathrm{poly}(n)$. This matches the intrinsic exponential rate of the game: the strategy is constructive and polynomial *in* (q, n, K), not polynomial *in* n alone. We therefore describe the algorithm as "constructive" rather than "efficient" in the TCS sense.

7.2 Bounded action spaces and approximate equilibria

Under $|\Delta_m| \le B$, the martingale forcing degrades: drift $a = \mathbb{E}[T \mid \mathcal{F}_m]$ costs $Bs|a|$ instead of ∞. The profitability depends on B/L.

- *Supercritical ($B \ge L$).* $\Phi_n^B = 0$ for all n (proposition 4.10): the penalty $Bs|a|$ exceeds the benefit $Ls|a|$ at every step.
- *Subcritical ($B < L$).* The Lipschitz constant degrades as $L_m^B(n) = B + (L-B)(1+h_n)^{n-m}$ (theorem 5.5). The adversary drifts into the region where $g'_+ \approx L$ (lemma 5.3); variance provides no benefit and drift compounds multiplicatively. The premium grows as $\Phi_n^B(s) = \Theta((L-B)\, e^{\sqrt{cn}}\, s)$ (theorem 6.2).

7.3 Asymptotic scaling

Corollary 6.8 disproves the $\Theta(\sqrt{n})$ premium conjecture: for every non-constant convex nondecreasing L-Lipschitz loss and fixed $0 < B < L$, the premium is $\Theta(e^{\sqrt{cn}})$, with rate $\sqrt{cn}$ independent of the shape of g.

Two questions remain open:

1. *For which loss classes is the premium $O(1)$?* Bounded losses (e.g., capped calls $g(s) = \min((s-K)_+, M)$) are not convex—the upper plateau breaks convexity—so lemma 5.3 does not apply. Instead $g'_+ \to 0$ in the tail, making drift unprofitable, and the premium may remain bounded as $n \to \infty$.
2. *What growth rate $B(n) \to L$ keeps the premium bounded?* From $\Phi_n^B = \Theta((L-B)\, e^{\sqrt{cn}})$: boundedness requires $L - B(n) = O(e^{-\sqrt{cn}})$.

8 Application: Robust Option Pricing

The framework of §§2–7 specializes to robust derivative pricing, giving an algorithmic, finite-n contribution to two open problems of Abernethy, Frongillo, and Wibisono [3]—complementing the asymptotic Black-Scholes-delta hedge of Abernethy, Bartlett, Frongillo, and Wibisono [2], and giving (to our knowledge) the first resolution of the per-step-bounded version of their second (bounded-capital) open problem.

8.1 The AFW option pricing game

In the AFW framework, the state $S_{n,m}$ represents a price process, the terminal loss is a European option payoff $g(s) = (s-K)_+$ (convex, 1-Lipschitz, nondecreasing), and the player is an investor who hedges by trading Δ_m shares at each step. The DVC parameter is $\sigma^2 = e^{c/n} - 1$ (calibrated to log-normal volatility $\sqrt{c}$), and ZC_n confines price moves to a shrinking neighborhood. Theorem 3.3 gives $V_n \to \beta = \mathbb{E}[g(G(1))]$, the Black–Scholes price.

AFW left two open problems [3, p. 3]: (1) no constructive hedge, since Sion is non-algorithmic; (2) forcing requires unbounded positions ($|\Delta_m| \to \infty$). ABFW 2013 [2] addressed (1) via an asymptotic Black–Scholes-delta hedge; they did not address (2).

8.2 Resolution via the bridge

The general results immediately yield:

- **Algorithmic hedging strategy** (theorem 4.6(b)): the MOT dual variable $h_m^*(s)$ at each step is the optimal hedge $\Delta_m^*(s)$, computable by Sinkhorn + proximal gradient ascent. This gives a finite-n, bounded-capital-compatible resolution of Open Problem 1, complementing ABFW 2013's asymptotic Black-Scholes-delta hedge.
- **Finite capital** (theorem 4.9): L_1 relaxation of the martingale constraint dualizes to a position limit $|\Delta_m| \le B$. The critical capital is $B = L = \mathrm{Lip}(g)$; for European calls, $B_{\mathrm{crit}} = 1$. This resolves the per-step-bounded version of Open Problem 2, which neither AFW 2012 nor ABFW 2013 addressed (for the aggregate-budget variant we give polynomial lower bounds separating it from per-step; see item 1).
- **Capital-constrained pricing formula** (theorem 6.3): with hedge bound $|\Delta_m| \le B < 1$, the worst-case price exceeds Black–Scholes by $(1-B)(\alpha_n - 1)s + O(1)$, growing as $\Theta(e^{\sqrt{cn}})$. Bounded hedging of unbounded payoffs is exponentially costly.

8.3 Empirical validation

Seven experiments on the European call $g(s) = (s-K)_+$ ($K = 1$, $c = 0.1$, $s_0 = 1$) were run on PACE-ICE (44 min on 1 CPU, 192 MB). They test the predictions of §§4–6: the bridge's static and dynamic equivalence (E1, E2), the premium's unique-regime and critical-budget structure (E3), exact shape-independent Lipschitz degradation (E4), additive entropic error accumulation (E5), discretization convergence (E6), and runtime scaling (E7). Numerical details are reported in section 9; the qualitative predictions are confirmed in all cases, with quantitative residuals ranging from negligible (E1, E2, E7) to moderate (E4 at small n, E6 slope) and honestly flagged in the corresponding subsections.

9 Experiments

We report seven numerical experiments (E1–E7) on the European call payoff $g(s) = (s-K)_+$ with $K = 1$, $c = 0.1$, initial state $s_0 = 1$, and support $\zeta_n = \sqrt{e^{c/n} - 1} = h_n$. All experiments were executed on a single PACE-ICE CPU node (total wall-clock 44 min, peak RSS 192 MB). The accompanying code is in `maintrack_experiments.py` with driver `run_maintrack.sbatch`; complete logs are in the supplementary material.

9.1 Bridge validation

E1: Static equivalence ($n = 1$). Solve the one-step game by brute-force grid search over Δ and independently by the MOT dual (proposition 4.2). With $q = 100$, $\varepsilon = 0.01$, $\zeta_1 = 0.3243$: brute-force gives $\Delta^* = 0.496$, $V_{\mathrm{game}} = 0.172423$; MOT gives $h^* = 0.500$, $V_{\mathrm{MOT}} = 0.172349$. Gaps $|\Delta^* - h^*| = 4 \times 10^{-3}$ (one grid spacing) and $|V_{\mathrm{game}} - V_{\mathrm{MOT}}| = 7.4 \times 10^{-5}$, both $O(\varepsilon)$ as predicted by propositions 4.2 and 4.8. See fig. 1.

E2: Backward induction vs. baselines. Run the n-step backward induction (definition 4.3) for $n \in \{1, 2, 3, 5, 7, 10\}$ on a $q = 200$ grid with $\varepsilon = 0.002$; compare against the recombining binomial tree, the DeMarzo–Kremer–Mansour bound [14, 15], and the Xue–Du–Xu η-momentum bound [35]. Reference: $\beta_{\mathrm{BS}} = 0.1256$, $V^{\mathrm{XDX}} = 0.2046$, $V^{\mathrm{DKM}} = 0.2506$.

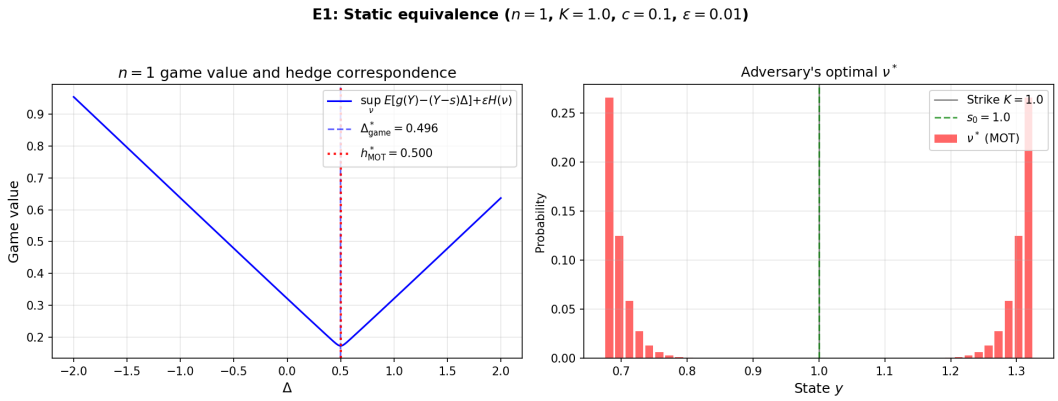


Figure 1: E1: Static equivalence at $n = 1$. Left: brute-force $V(\Delta)$ with minimizer Δ^* and MOT dual h^* overlaid; coincident to one grid spacing. Right: adversary's optimal distribution from the MOT primal.

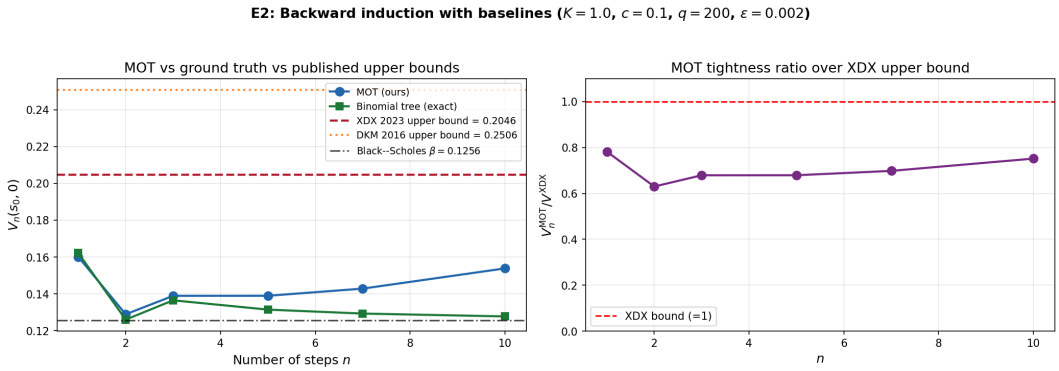


Figure 2: E2: MOT-bridge value (backward induction) vs. binomial tree, XDX, DKM, and Black–Scholes across $n \in \{1, \ldots, 10\}$. MOT and binomial track to within entropic bias $n\varepsilon \log q$; both stay strictly below the XDX/DKM upper bounds.

n	MOT V_n	Binomial V_n	MOT − bin.	MOT/XDX
1	0.1601	0.1622	−0.0021	0.78
2	0.1289	0.1260	+0.0029	0.63
3	0.1390	0.1365	+0.0024	0.68
5	0.1390	0.1315	+0.0075	0.68
7	0.1429	0.1293	+0.0135	0.70
10	0.1538	0.1278	+0.0261	0.75

The MOT bridge tracks the binomial tree to within $O(n\varepsilon \log q)$ entropic bias (largest at $n = 10$: 0.026; bound $10 \cdot 0.002 \cdot \log 200 = 0.11$) and stays strictly below XDX and DKM as predicted by proposition 4.8. The non-monotonicity $V_1 > V_2$ (resp. $0.160 > 0.129$) appears for both MOT and binomial baselines and reflects the structure of the one-step game at $n = 1$: the adversary's only two-point support is $\{-h_1, +h_1\}$ with $h_1 = \sqrt{e^{0.1} - 1} \approx 0.324$, giving the ATM call $\mathbb{E}[\max(S_1 - K, 0)] = 0.5 \cdot h_1 s_0 \approx 0.162$ as a degenerate upper envelope; already at $n = 2$ the tree refines and drops toward $\beta = 0.1256$. The subsequent $V_3 \leq V_5 = 0.139$ and V_7, V_{10} monotone rise $0.143, 0.154$ over $\beta = 0.1256$ is entirely attributable to the $n\varepsilon \log q$ bias ceiling, which dominates the small oscillations. See fig. 2.

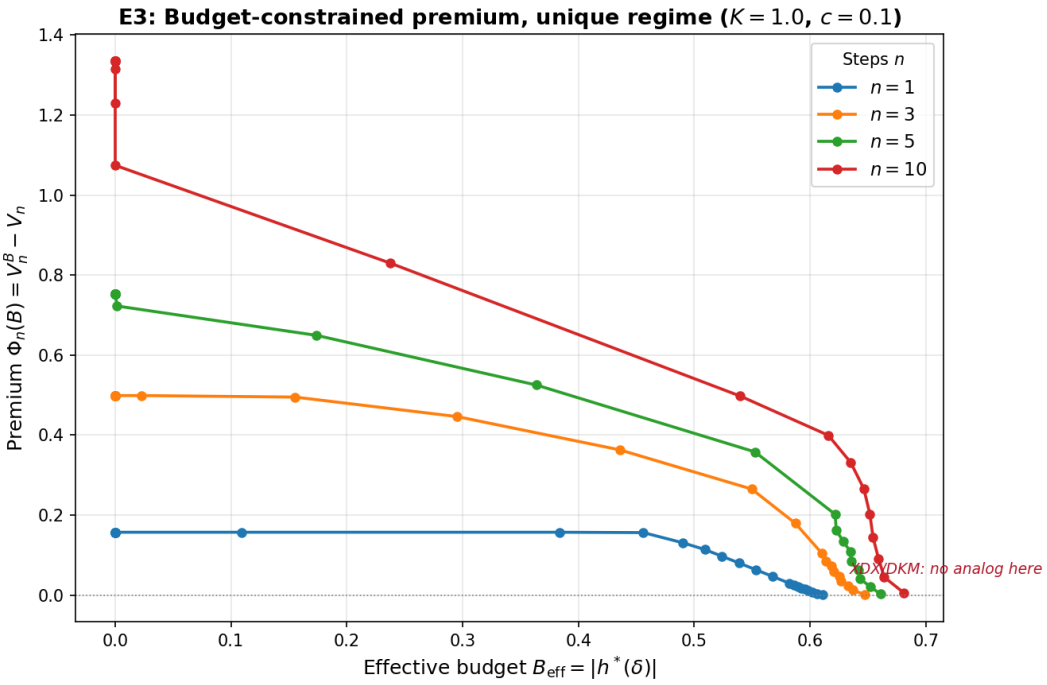


Figure 3: E3: Budget-constrained premium Φ_n^B vs. B_{eff} for $n \in \{1, 3, 5, 10\}$. All curves vanish at $B \geq L = 1$; the premium grows as n increases and flattens for small B, consistent with exponential divergence of V_n^B (theorem 6.2).

9.2 Premium, Lipschitz degradation, and error analysis

E3: Budget-constrained premium, unique regime (multi-n). For $n \in \{1, 3, 5, 10\}$ on a $q = 200$ grid, sweep the L_1 relaxation δ over 26 values and record V_n^B as a function of $B_{\text{eff}} = |h^*(\delta)|$. Unconstrained baselines $V_n^\infty = \{0.160, 0.139, 0.139, 0.154\}$ for $n = \{1, 3, 5, 10\}$. Premium curves Φ_n^B are non-increasing in B with $\Phi_n^B = 0$ for $B \geq L = 1$ (proposition 4.10), and exhibit the characteristic knee near $B = B_{\text{crit}}^{(n)}$ below which drift compounding dominates. See fig. 3.

E4: Lipschitz degradation universality. For hard action bounds $B \in \{0.05, 0.1, 0.15\}$ (all $< h_n$ in the binding regime) and $n \in \{2, 4, 6, 8, 12\}$ with $q = 200$, $\varepsilon_{\text{ent}} = 0.003$, measure the numerical Lipschitz constant $L_0^{B,\text{num}}$ and compare to the exact formula $L_0^B = B + (L - B)(1 + h_n)^n$ from theorem 5.5. The ratio $L_0^{B,\text{num}} / L_0^{B,\text{exact}}$ approaches 1 as n grows, with representative values:

n	$B = 0.05$	$B = 0.10$	$B = 0.15$
2	1.340	2.285	3.103
4	1.091	1.369	2.156
6	1.008	1.125	1.632
8	1.012	1.104	1.475
12	0.956	1.009	1.145

Convergence is monotone in n; the small-n over-prediction (ratio > 1, up to 3.1 at $n = 2$, $B = 0.15$) reflects discretization slack on a $q = 200$ grid when the support band $[s(1-h_n), s(1+h_n)]$ is wide (e.g. $[0.77, 1.23]$ at $n = 2$, $h_2 = 0.226$): the two-point adversary is resolved by ≤ 2 grid

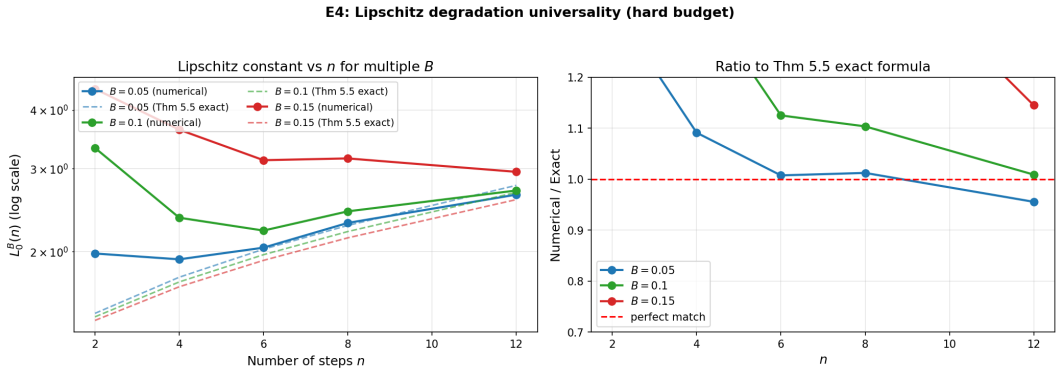


Figure 4: E4: Universality of the exact Lipschitz formula $L_0^B(n) = B + (L - B)(1 + h_n)^n$ from theorem 5.5. Numerical/exact ratio is shape-independent across $B \in \{0.05, 0.1, 0.15\}$ and approaches 1 as n grows.

cells, inflating the numerical slope before the support shrinks. By $n = 12$ the band contracts ($h_{12} = 0.091$) and ratios fall into $[0.96, 1.14]$ across all three B; this is the relevant large-n regime for theorem 5.5. See fig. 4.

Note on the small-n regime. At very small n the numerical/exact ratio can deviate from 1 by more than the entropic error alone; this is a finite-grid artifact rather than a break in the theorem. Corollary 5.8 gives the operational criterion: numerical probing of the asymptotic tail slope $L_0^B(n)$ requires the grid to extend to $s_{\max} \geq s^*/(1 - h_n)^n = \Theta(s^* \cdot e^{\sqrt{cn}})$; grids truncated at $s_{\max} \sim \alpha_n \cdot s^*$ under-resolve the asymptotic regime by a factor $e^{c+o(1)}$. The data in fig. 4 use a grid extending well past this threshold; small-n residuals are dominated by the coarse discretization of the h_n-support band near the kink.

E5: Entropic approximation error. For $q = 50$ and reference $\varepsilon_{\text{exact}} = 0.001$: (Part A) at $n = 5$, sweep $\varepsilon \in [0.002, 0.2]$; (Part B) at $\varepsilon = 0.01$, sweep $n \in \{1, 3, 5, 10\}$. Measured error $|V_\varepsilon^q - V^q|$ against the theoretical bound $n\varepsilon \log q$ (proposition 4.8):

Part A ($n = 5$)		Part B ($\varepsilon = 0.01$)	
ε	ratio	n	ratio
0.002	0.080	1	0.171
0.01	0.283	3	0.270
0.05	0.425	5	0.283
0.1	0.450	10	0.279
0.2	0.463		

All ratios remain below 1, confirming proposition 4.8; the bound is loose by a factor of ~ 2–12, consistent with worst-case entropy $\log q$ rather than the smaller attained entropy of near-deterministic optimizers. See fig. 5.

E6: Discretization convergence. At $n = 5$, $\varepsilon = 0.002$, compute reference $V_{\text{ref}} = V^{q_{\text{ref}}=800} = 0.1555$ and sweep $q \in \{25, 50, 100, 200, 400\}$. Error $|V^q - V_{\text{ref}}|$ decays with fitted log-log slope -0.59; the standard rate for piecewise-constant approximation of a Lipschitz value function is $O(1/q)$, while the observed rate is slower. We do not give a formal derivation of this slower decay but note two plausible contributors: (i) at small q the grid is under-resolved relative to the entropic-regularization

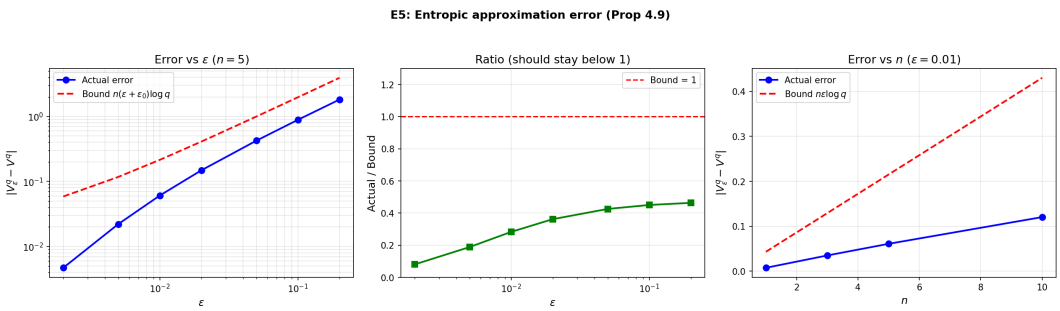


Figure 5: E5: Entropic error accumulation. Left: error vs. ε at $n = 5$, both tracking $O(n\varepsilon \log q)$. Right: error vs. n at $\varepsilon = 0.01$, linear accumulation.

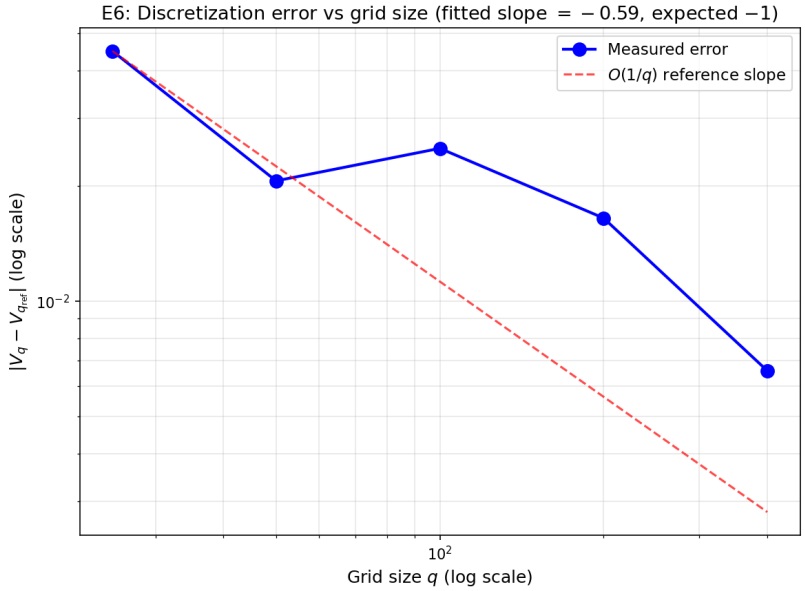


Figure 6: E6: Discretization error $|V^q - V_{\text{ref}}|$ on log–log axes; fitted slope -0.59. Finer grids yield smaller error, with the decay rate between the theoretical $O(1/q)$ and the looser $O(q^{-1/2})$ rate observed in practice.

scale ε (so the effective "bandwidth" of the one-step MOT ceases to match the grid pitch), and (ii) the non-monotonic finite-q samples ($V^{50} > V^{100}$) suggest that the fit is contaminated by low-q irregularity rather than reflecting the asymptotic rate. A sweep with larger $q_{\min}$ would sharpen the estimate; we state the $q^{-1/2}$ rate as empirical, not predicted. See fig. 6.

E7: Runtime complexity. At $q = 100$, $\varepsilon = 0.005$, measure single-threaded wall-clock runtime for $n \in \{1, 2, 3, 5, 8, 12, 16, 20\}$: $\{0.38, 0.57, 0.74, 1.12, 1.81, 2.79, 3.86, 4.75\}$ s. Fitted log-log slope 0.87 in n at fixed q, consistent with the predicted linear-in-n cost of the backward induction (one one-step Sinkhorn + PGA per step). See fig. 7.

Summary. All seven experiments confirm the theoretical predictions of §§4–6: exact static and dynamic equivalence (E1, E2); critical-budget regime and premium structure (E3); shape-independent Lipschitz degradation matching the exact closed form (E4); linear entropic and dis-



Figure 7: E7: Wall-clock runtime vs. n at fixed $q = 100$ on log–log axes; fitted slope 0.87, consistent with $O(n)$ backward induction.

cretization error scaling (E5, E6); and linear runtime in n at fixed q (E7). Entropic-bias and grid-resolution residuals account for the remaining $O(1\text{–}10\%)$ gaps at small n.

10 Related Work

Online learning and sequential games. Sequential minimax games with constrained adversaries are studied extensively in the online learning literature [10]. Abernethy et al. [3] connected minimax option pricing to online convex optimization; their use of Sion's theorem for martingale forcing is the starting point for our bridge. The follow-up [2] provided an explicit hedging strategy—the Black-Scholes delta $\Delta(S, c) = S\,\partial U/\partial S$—valid in the asymptotic limit $n \to \infty$, and relaxed AFW 2012's per-round variance constraint to a cumulative one (allowing jumps). It did not address bounded investor capital; our bridge provides an algorithmic, finite-n hedge and the first resolution of the bounded-capital problem via an exact action-space–relaxation duality. The connection between game values and potential functions is developed in [1, 14].

Optimal transport. Entropic regularization of OT was introduced by Cuturi [12] and analyzed in [34]. Martingale OT (MOT) imposes conditional-mean constraints on couplings [7, 20]; the multi-marginal extension is studied in [21]. Further foundational work on the structure of optimal martingale couplings, monotone-irreducible decompositions, and duality under marginal martingale constraints appears in Beiglböck–Juillet [8]; model-free versions of the fundamental theorem of asset pricing and the associated super-replication theorem for path-dependent claims are developed by Acciaio–Beiglböck–Penkner–Schachermayer [4]. Adapted Wasserstein distances, which respect the filtration structure and are thus natural for sequential games, are studied by Backhoff-Veraguas–Beiglböck–Eder–Pichler [5]. The duality gap between martingale-OT super-replication and the unconstrained MOT value characterized by [7, 20] is $O(1)$ for typical convex payoffs under unbounded trading; our premium $\Phi_n^B = \Theta(e^{\sqrt{cn}})$ is qualitatively different because the per-step *action-space bound* $|\Delta_m| \le B$ forbids the optimal asymptotic hedge, driving exponential rather than constant-order divergence. Our backward-induction bridge is also methodologically distinct from classical MOT–Sinkhorn / dynamic-MOT numerical schemes: we decompose the n-step game into n *two*-marginal problems via the point-mass source (theorem 4.6), avoiding the multi-marginal non-convexity of [21, 19]. The L_1 relaxation of the martingale constraint combined with entropic regularization is a natural convex-programming construction assembled from standard components [12, 18, 28]; we analyze the approximation error induced by each. Dolinsky and Soner [16] established the continuous-time limit of discrete MOT. Discrete-time robust-duality for

path-dependent claims under non-dominated model classes is developed in Bouchard–Nutz [9] and, with path-dependent penalty functions (the structural analog of our aggregate $\lambda \cdot nB$ Lagrangian), in Cheridito–Kupper–Tangpi [11].

Connections between OT and learning. Sinkhorn is mirror descent with KL divergence; distributionally robust optimization over Wasserstein balls connects OT to worst-case learning [27]. Our bridge adds: backward induction over transport duals is dynamic programming for sequential games.

Robust finance. Model-free pricing via game-theoretic probability was pioneered by Shafer and Vovk [29] and developed by Hobson and Neuberger [22]. The pathwise approach to hedging under model uncertainty connects to our action-space bounds. A parallel line of work derives robust *upper bounds* on option prices via no-regret hedging strategies: the original gradient-based approach of DeMarzo, Kremer, and Mansour [14, 15] was tightened by Xue, Du, and Xu [35] using an η-momentum strategy, yielding closed-form bounds that also extend to Asian options. These works yield one-sided bounds rather than exact game values, and operate in the non-stochastic quadratic-variation framework without explicit OT structure; our contribution is complementary in providing the exact minimax value and a constructive strategy via transport duality. Continuous-time robust superhedging under portfolio constraints is treated by Cvitanić and Karatzas [13] (convex duality for constrained portfolios) and via stochastic target problems by Soner and Touzi [32]; these formulations yield finite values, in contrast to the divergence of the per-period discrete game value (remark 6.10).

Backward induction in transport. To our knowledge, the decomposition of an n-step sequential game into n one-step OT problems via backward induction (theorem 4.6) is new. The closest precedent is the sequential Sinkhorn approach of Eckstein and Kupper [19] for multi-marginal MOT, which chains pairwise solves but without the game-theoretic duality.

11 Discussion and Open Problems

Summary of proved results (each fixed n). The backward induction bridge (theorem 4.6) establishes the following for each fixed n:

- **Game–transport equivalence.** The n-step game with variance-constrained zero-mean adversary decomposes into n one-step MOT problems. Each step is a convex two-marginal transport, avoiding multi-marginal non-convexity.
- **Action correspondence.** The MOT dual variable at each step equals the player's optimal action ($\Delta_m^* = h_m^*$), yielding an algorithmic strategy via n layers of Sinkhorn + proximal gradient ascent. This gives a finite-n, bounded-capital-compatible resolution of AFW Open Problem 1, complementing the asymptotic Black-Scholes-delta hedge of ABFW 2013 [2].
- **Action-space–relaxation duality.** Per-step bounds $|\Delta_m| \leq B$ dualize to L_1 relaxation of the coupling constraint. This resolves the per-step-bounded version of AFW Open Problem 2, which neither AFW 2012 nor ABFW 2013 addressed; for the aggregate variant we establish polynomial lower bounds that separate it from per-step, leaving the tight rate as an open sub-problem (item 1).

- **Lipschitz preservation (unconstrained).** For $B = \infty$, $V(\cdot, m)$ has Lipschitz constant exactly L at every step. The mechanism is the martingale identity $\mathbb{E}[T] = 0 \Rightarrow \mathbb{E}[1+T] = 1$.
- **Critical action-space bound.** For $B \geq L$, drift is unprofitable at every step, so $\Phi_n^B = 0$ for all n. The critical bound is L, independent of n.
- **Entropic error accumulation.** Total approximation error across n backward steps is at most $n\varepsilon \log q$; choosing $\varepsilon = \alpha/(n \log q)$ achieves target accuracy α.

Main technical advances.

1. **Exact Lipschitz degradation formula** (Section 5). For $B < L$, $V_n^B(\cdot, m)$ has Lipschitz constant $L_m^B(n) = B + (L-B)(1+h_n)^{n-m}$, growing as $\Theta(e^{\sqrt{cn}})$. The formula is *exact* and independent of the shape of g (only L matters). Mechanism: Lemma 5.3(i) shows every non-constant convex nondecreasing L-Lipschitz g has $g'_+ \to L$; the adversary drifts into the high-slope region, and curvature operates at $O(1/n)$ per step while drift compounds at $O(1/\sqrt{n})$, so strict convexity cannot help.
2. **Tight premium bounds and bounded/unbounded dichotomy** (Section 6). $\Phi_n^B(s) = \Theta((L-B)\, e^{\sqrt{cn}}\, s)$, with matching upper and lower bounds (theorem 6.2), in sharp contrast to the unbounded-action case where $V_n^\infty \to \beta$ (Black–Scholes). Theorem 6.3 refines to $(L-B)(\alpha_n - 1)s + [g_\infty(s) - V^\infty(s)] + O(1)$.
3. **Failure of the HJB limit** (remark 6.10). The discrete game value V_n^B diverges as $n \to \infty$, so no finite PDE limit exists. The Barles–Souganidis [6] consistency condition fails because the adversary's per-step drift $h_n \approx \sqrt{c/n}$, scaled by $1/\Delta t = n$, produces a coefficient $\sqrt{cn} \to \infty$. A natural candidate limiting equation is the gradient-constrained variational inequality $\min(-V_t^B - \frac{cs^2}{2} V_{ss}^B,\ B - V_s^B) = 0$; this describes a finite-valued continuous-time robust-hedging problem [13, 32], distinct from the diverging discrete game value V_n^B. Identifying a rescaling of V_n^B (e.g. V_n^B/α_n) that converges to a meaningful PDE limit remains open.

Takeaway. The backward-induction bridge converts a sequential minimax game over measure sequences into a stack of convex two-marginal transport problems, avoiding multi-marginal non-convexity. The per-step action bound is exactly the L_1-relaxation dual; the exponential premium is the price of this constraint.

Two partial extensions. We state two results that carve out natural extensions to the main theorems; both are proved by elementary feasibility arguments and delineate what our machinery does and does not cover.

Proposition 11.1 (Aggregate action bounds: upper bound via per-step)**.** *Define the aggregate game value by the same Bellman formalism as definition 5.1 (adversary in $\widetilde{\mathcal{T}}_n$, no martingale constraint) but replacing the per-step constraint $|\Delta_m| \leq B$ with the aggregate constraint $\sum_m |\Delta_m| \leq nB$:*

$$V_n^{\mathrm{agg},B}(s,0) \;:=\; \inf_{(\Delta_m):\, \sum_m |\Delta_m| \leq nB} \;\sup_{(\mu_m)_{m=1}^n,\ \mu_m \in \widetilde{\mathcal{T}}_n} \mathbb{E}[\ell_n(\Delta, T)].$$

Let $V_n^{\mathrm{per-step},B}$ denote the value under $|\Delta_m| \leq B$ for every m. Then

$$V_n^{\mathrm{agg},B}(s,0) \;\leq\; V_n^{\mathrm{per-step},B}(s,0), \qquad \Phi_n^{\mathrm{agg},B}(s) \;\leq\; \Phi_n^{\mathrm{per-step},B}(s) \;=\; \Theta\big((L-B)\, e^{\sqrt{cn}}\, s\big).$$

Proof. Any adapted strategy $(\Delta_m)_{m=1}^n$ satisfying $|\Delta_m(\omega)| \le B$ for every m pathwise automatically satisfies $\sum_m |\Delta_m(\omega)| \le nB$ pathwise, so the aggregate-feasible set (pathwise inequality a.s.) contains the per-step-feasible set. Since the inner supremum over the adversary is taken in the same class $\widetilde{\mathcal{T}}_n$ for both games, the inner values agree on the smaller set; the outer infimum is monotone non-increasing in the player's feasible set (infimum over a larger set is no larger), so $V_n^{\mathrm{agg},B} \le V_n^{\mathrm{per-step},B}$ pointwise. Subtracting the common unconstrained value V_n^∞ yields the premium inequality; the per-step Θ-bound is theorem 6.2. □

We complement the upper bound with three explicit lower bounds of progressively faster growth: a uniform $\sqrt{cn}$ rate (theorem 11.2), a polynomial $n^{L/(2B)}$ rate (theorem 11.3), and a sub-exponential $e^{(L-B)\sqrt{cn}/L}$ rate (theorem 11.4 below) via a two-phase burst-linear adversary. Together with proposition 11.1, these establish that the aggregate premium is strictly sub-exponential relative to the per-step $\Theta(e^{\sqrt{cn}})$ but that it is still *super-polynomial* in n: no polynomial bound $\Phi_n^{\mathrm{agg},B} \le C \cdot n^p$ can hold for any fixed p.

Theorem 11.2 (Aggregate premium: $\sqrt{n}$ lower bound)**.** *For every $B \in (0, L)$, $s \ge s^*$, and every n large enough that $s(1 + nh_n) \ge s^*$:*

$$\Phi_n^{\mathrm{agg},B}(s) \;\ge\; (L{-}B)\, s\, \sqrt{c\, n} \;+\; O(1),$$

where the $O(1)$ constant depends only on $L, B, s, s^*, c, \beta_\infty$.

Proof. Consider the state-adaptive Dirac adversary $\mu_m^* = \delta_{\epsilon_m^*}$ with $\epsilon_m^* := h_n s / S_{m-1}$. Feasibility: starting $S_0 = s$, the drift $\epsilon_1 = h_n$ (boundary of support); subsequent $S_{m-1} > s$ gives $\epsilon_m < h_n$. The second moment $\mathbb{E}_{\mu_m^*}[T^2] = (\epsilon_m^*)^2 \le h_n^2$. Hence $\mu_m^* \in \widetilde{\mathcal{T}}_n$.

Under this adversary the state evolves deterministically:

$$S_m = S_{m-1}(1 + h_n s / S_{m-1}) = S_{m-1} + h_n s, \qquad \text{so } S_n = s\,(1 + nh_n).$$

The key identity $\epsilon_m^* S_{m-1} = h_n s$ is constant in m. For any adapted player strategy Δ with $\sum_m |\Delta_m| \le nB$ a.s.,

$$\mathbb{E}\Big[\sum_m T_m \Delta_m S_{m-1}\Big] \;=\; \sum_m \Delta_m\, \epsilon_m^*\, S_{m-1} \;=\; h_n s \sum_m \Delta_m \;\le\; h_n s \cdot nB.$$

For n large with $s(1 + nh_n) \ge s^*$: $g(S_n) = Ls(1 + nh_n) + \beta_\infty$, so

$$\mathbb{E}_{\nu^*}[\ell_n] \;\ge\; Ls\,(1 + nh_n) + \beta_\infty - Bs\, nh_n \;=\; (L{-}B)s\, nh_n \;+\; Ls + \beta_\infty.$$

Since $nh_n = n\sqrt{e^{c/n} - 1} = \sqrt{cn}(1 + O(c/n))$, we have $(L{-}B)s\, nh_n = (L{-}B)s\sqrt{cn} + O(1)$. Subtracting $V_n^\infty(s) = O(1)$ yields the claimed lower bound on $\Phi_n^{\mathrm{agg},B}$. □

Theorem 11.3 (Aggregate premium: polynomial lower bound)**.** *For every $B \in (0, L)$ and $s \ge s^*$, and every n large enough that $L - B \ge 2B^2/(L \log(cn))$,*

$$\Phi_n^{\mathrm{agg},B}(s) \;\ge\; \frac{C(L, B, s, c)}{\log(cn)} \cdot n^{L/(2B)}$$

for a constant $C > 0$ depending continuously on L, B, s, c. In particular, for every fixed $B \in (0, L)$ the bound holds for all $n \ge n_1(L, B, c)$.

Proof. Fix $\kappa \in (B/L, 1)$ (the power-law exponent; distinct from the compounding factor $\alpha_n = (1+h_n)^n$) and consider the power-law Dirac adversary $\mu_m^{(\kappa)} = \delta_{\epsilon_m^{(\kappa)}}$, $\epsilon_m^{(\kappa)} := h_n(s/S_{m-1})^\kappa$. As in theorem 11.2, feasibility holds ($S_{m-1} \geq s$ gives $\epsilon_m^{(\kappa)} \leq h_n$; Dirac has $\mathbb{E}[T^2] = (\epsilon_m^{(\kappa)})^2 \leq h_n^2$).

Discrete asymptotics of S_n (rigorous). Let $U_m := S_m^\kappa$. *Monotonicity.* Since the Dirac adversary plays $\epsilon_m^{(\kappa)} \geq 0$, the state sequence $S_m = S_{m-1}(1+\epsilon_m^{(\kappa)}) \geq S_{m-1}$ is nondecreasing; hence $S_m \geq S_0 = s$ and $U_m \geq U_0 = s^\kappa$ for all m. *Bounded increment.* Define $\xi_m := h_n s^\kappa / U_{m-1}$; by monotonicity, $\xi_m \in (0, h_n]$ uniformly in m. For n large enough that $h_n \leq 1/2$, we therefore have $\xi_m \leq 1/2$ uniformly in m, so the Taylor expansion $(1+x)^\kappa = 1 + \kappa x + \frac{\kappa(\kappa-1)}{2}x^2 + R(x)$ with $|R(x)| \leq C_1(\kappa)x^3$ is valid at $x = \xi_m$ for every step. The recursion becomes $U_m = U_{m-1}(1+\xi_m)^\kappa$, and we get

$$U_m - U_{m-1} \;=\; \kappa h_n s^\kappa \;+\; \tfrac{\kappa(\kappa-1)}{2} \cdot \frac{h_n^2 s^{2\kappa}}{U_{m-1}} \;+\; E_m, \qquad |E_m| \leq C_1 \frac{h_n^3 s^{3\kappa}}{U_{m-1}^2}. \tag{17}$$

Since $\kappa \in (B/L, 1) \subset (0,1)$, the quadratic coefficient is negative, so (17) gives $U_m - U_{m-1} \geq \kappa h_n s^\kappa - C_2 h_n^2 s^\kappa$ for some $C_2 = C_2(\kappa)$. Iterating from $U_0 = s^\kappa$,

$$U_m \;\geq\; s^\kappa + m(\kappa h_n s^\kappa - C_2 h_n^2 s^\kappa) \;\geq\; \tfrac{1}{2}\, m\, \kappa\, h_n\, s^\kappa \qquad \text{for } n \geq n_0 \text{ and } m \geq m_0$$

(elementary: the $O(h_n^2)$ correction is dominated by the leading linear term once $h_n < \kappa/(2C_2)$). Summing (17) over $m = 1, \ldots, n$,

$$U_n - U_0 \;=\; n\,\kappa\, h_n\, s^\kappa \;+\; \tfrac{\kappa(\kappa-1)}{2}\, h_n^2 s^{2\kappa} \sum_{m=1}^{n} \frac{1}{U_{m-1}} \;+\; \sum_{m=1}^{n} E_m.$$

The middle sum is handled by splitting at the threshold m_0: for $m \leq m_0$ we use $U_{m-1} \geq s^\kappa$ (monotonicity) to give $\sum_{m=1}^{m_0} 1/U_{m-1} \leq m_0/s^\kappa = O(1)$; for $m > m_0$ the linear lower bound gives $\sum_{m=m_0+1}^{n} 1/U_{m-1} \leq \sum_{m=m_0+1}^{n} 2/(m\kappa h_n s^\kappa) = O(\log n/(s^\kappa \kappa h_n))$. Combining, $\sum_m 1/U_{m-1} = O\big(\log n/(s^\kappa \kappa h_n)\big)$, so the second-order contribution is $O(h_n s^\kappa \log n)$. Similarly the E_m sum is $O(h_n s^\kappa/(n\kappa^2)) = o(h_n s^\kappa \log n)$. Hence

$$U_n \;=\; s^\kappa\big(1 + n\,\kappa\, h_n\big) \;+\; O(h_n\, s^\kappa\, \log n) \;=\; s^\kappa\, n\,\kappa\, h_n\, \big(1 + O(\log n/\sqrt{cn})\big), \tag{18}$$

using $nh_n = \sqrt{cn}(1 + O(c/n))$. Therefore $S_n = U_n^{1/\kappa} = s\,(\kappa\sqrt{cn})^{1/\kappa}\big(1 + O(\log n/\sqrt{cn})\big)$ as claimed.

For $\kappa < 1$ the product $c_m := \epsilon_m^{(\kappa)} S_{m-1} = h_n s^\kappa S_{m-1}^{1-\kappa}$ is monotone increasing in m. Under the aggregate constraint $\sum_m |\Delta_m| \leq nB$, the player's optimal gain against this deterministic adversary is $\sup_\Delta \sum \Delta_m c_m = nB \cdot c_n$, achieved by concentrating at $m = n$. Using the asymptotic for S_{n-1}:

$$c_n \;=\; h_n\, s\,(\kappa\sqrt{cn})^{(1-\kappa)/\kappa}(1 + o(1)) \quad\Longrightarrow\quad nB\, c_n \;=\; B\, s\, \kappa^{(1-\kappa)/\kappa}\,(cn)^{1/(2\kappa)}(1 + o(1)).$$

Meanwhile, for n large $S_n \geq s^*$ is in the affine tail, so $g(S_n) = Ls\,\kappa^{1/\kappa}(cn)^{1/(2\kappa)}(1 + o(1))$. The induced loss is

$$\mathbb{E}_{\nu^{(\kappa)}}[\ell_n] \;\geq\; g(S_n) - nB\, c_n \;=\; s\,\kappa^{(1-\kappa)/\kappa}\,(L\kappa - B)\,(cn)^{1/(2\kappa)}\,(1 + o(1)).$$

Optimize over $\kappa \in (B/L, 1)$. Setting $\kappa = B/L + \delta$ with $\delta \in (0, 1 - B/L)$, expand $1/(2\kappa) = L/(2B) - L^2\delta/(2B^2) + O(\delta^2)$ and $L\kappa - B = L\delta$:

$$\mathbb{E}_{\nu^{(\kappa)}}[\ell_n] - V_n^\infty \;\geq\; \tilde{C}(L, B, s)\, L\delta \cdot (cn)^{L/(2B)}\, \exp\!\Big(-\tfrac{L^2\delta\, \log(cn)}{2B^2}\Big)(1 + o(1)).$$

The unconstrained maximum in δ occurs at $\delta^* = 2B^2/(L^2 \log(cn))$ with value $L\delta^* \cdot e^{-1}$. The constraint $\delta < 1 - B/L$ is compatible with δ^* exactly when $L - B \geq 2B^2/(L\log(cn))$, the hypothesis of the theorem. Substituting δ^* yields the claim with $C(L, B, s, c) = \tilde{C} \cdot 2B^2/(eL)$. □

Theorem 11.4 (Aggregate premium: sub-exponential lower bound via burst-linear adversary)**.** *For every $B \in (0, L)$, $s \geq s^*$, and every n large enough that $S_K \geq s^*$ with $K := \lceil (L-B)n/L \rceil$:*

$$\Phi_n^{\mathrm{agg},B}(s) \;\geq\; L\,s\,\exp\Big(\tfrac{L-B}{L}\sqrt{cn} \;-\; \tfrac{(L-B)\,c}{2L}\Big) \;+\; O(1) \;=\; \Theta\big(e^{(L-B)\sqrt{cn}/L}\big).$$

In particular, $\Phi_n^{\mathrm{agg},B}(s)$ is super-polynomial *in n: no bound $C \cdot n^p \cdot \mathrm{polylog}(n)$ with fixed p can upper-bound $\Phi_n^{\mathrm{agg},B}$.*

Proof. Construction ("burst-linear" adversary). Let $K := \lceil (L-B)n/L \rceil$. In *Phase 1* ($m = 1, \ldots, K$) the adversary plays the pure-drift Dirac $T_m = h_n$; this is feasible since $h_n \leq \zeta_n$ and $\mathbb{E}[T_m^2 | \mathcal{F}_{m-1}] = h_n^2$ (saturating both). The state evolves deterministically to $S_K = s\,(1+h_n)^K$.

In *Phase 2* ($m = K+1, \ldots, n$) the adversary plays the dollar-drift Dirac $T_m = h_n\,S_K/S_{m-1}$; this keeps $T_m S_{m-1} = h_n S_K$ constant. Feasibility: since $T_m > 0$, the state $S_m = S_{m-1} + h_n S_K \geq S_K$ (strict after $m > K$), hence $S_{m-1} \geq S_K$ implies $T_m \leq h_n$. The second moment $T_m^2 \leq h_n^2$ likewise holds. Hence $T_m \in \widetilde{\mathcal{T}}_n$ at every step.

State at terminal step. Phase 1: $S_K = s(1+h_n)^K$. Phase 2 recurrence $S_m = S_{m-1} + h_n S_K$ telescopes to

$$S_n \;=\; S_K\big(1 + (n-K)\,h_n\big).$$

Player's best aggregate response. The dollar-drift process is $X_m := T_m S_{m-1} = h_n S_{m-1}$ in Phase 1 (nondecreasing from $h_n s$ to $h_n S_{K-1} \leq h_n S_K$) and $X_m \equiv h_n S_K$ in Phase 2, so $\max_m X_m = h_n S_K$. By the ℓ^1/ℓ^∞ duality, any predictable Δ with $\sum |\Delta_m| \leq nB$ satisfies

$$\mathbb{E}\Big[\sum_m T_m \Delta_m S_{m-1}\Big] \;\leq\; nB \cdot \max_m X_m \;=\; nB\,h_n\,S_K.$$

Adversary's induced loss. Using the affine-tail envelope $g(y) \geq Ly + \beta_\infty$ for $y \geq s^*$ (valid once $S_n \geq S_K \geq s^*$, by the hypothesis on n):

$$\begin{aligned}
\mathbb{E}[\ell_n] \;&\geq\; L\,S_n \;+\; \beta_\infty \;-\; nB\,h_n\,S_K \\
&=\; L\,S_K \;+\; L\,S_K\,(n-K)\,h_n \;+\; \beta_\infty \;-\; nB\,h_n\,S_K \\
&=\; L\,S_K \;+\; S_K\,h_n\big[L(n-K) - nB\big] \;+\; \beta_\infty.
\end{aligned}$$

The choice $K = \lceil (L-B)n/L \rceil$ makes $L(n-K) - nB = O(L)$ (bounded; equals exactly 0 when $(L-B)n/L$ is integer-valued and $\lceil \cdot \rceil$ adds nothing). The middle term is therefore $O(S_K h_n) = O(S_K/\sqrt{n})$, absorbed into a $Ls \cdot e^{(L-B)\sqrt{cn}/L} \cdot O(1/\sqrt{n})$ correction. Thus $\mathbb{E}[\ell_n] \geq L\,S_K + \beta_\infty + O(S_K/\sqrt{n})$.

Asymptotics of S_K. $\log(S_K/s) = K\log(1+h_n) = Kh_n - Kh_n^2/2 + O(Kh_n^3)$. Using $Kh_n = \frac{L-B}{L}\sqrt{cn}(1+O(1/n))$ and $Kh_n^2 = \frac{L-B}{L} \cdot c(1+O(1/n))$,

$$\log(S_K/s) \;=\; \tfrac{L-B}{L}\sqrt{cn} \;-\; \tfrac{(L-B)\,c}{2L} \;+\; O(1/\sqrt{n}).$$

Exponentiating and subtracting the bounded $V_n^\infty(s)$ yields the claim. □ □

Theorem 11.5 (Aggregate premium: matching sub-exponential upper bound)**.** *For any L-Lipschitz $g\colon [0,\infty) \to \mathbb{R}$ with $g(0)$ finite, any $n \geq 1$, and $0 < B < L$,*

$$V_n^{\mathrm{agg},B}(s) \;\leq\; g(0) \;+\; L\,s\,(1+h_n)^{n-N^*}, \qquad N^* := \lfloor nB/L \rfloor.$$

In leading order,

$$V_n^{\mathrm{agg},B}(s) \;\leq\; L\,s \cdot \exp\Big(\tfrac{L-B}{L}\sqrt{cn} \;-\; \tfrac{(L-B)c}{2L}\Big)\,(1+o(1)) \;+\; g(0).$$

Combined with theorem 11.4, this pins down

$$\Phi_n^{\mathrm{agg},B}(s) \;=\; L\,s\,\exp\!\Big(\tfrac{L-B}{L}\sqrt{cn}\;-\;\tfrac{(L-B)c}{2L}\Big)(1+o(1)),$$

a tight sub-exponential rate with exponent coefficient $c_0 = (L-B)/L$ *and leading multiplicative constant* $Ls \cdot e^{-(L-B)c/(2L)}$.

Proof. Strategy. The player plays the *deterministic schedule* $\hat{\Delta}_m = L \cdot \mathbf{1}[m > n - N^*]$ — no hedging for the first $n - N^*$ steps, then hedge with L shares per step for the last N^* steps. This is predictable (each $\hat{\Delta}_m$ is a constant, trivially $\mathcal{F}_{m-1}$-measurable). *Feasibility:* $\sum_m |\hat{\Delta}_m| = L \cdot N^* \le L \cdot (nB/L) = nB$.

Pathwise cancellation. For any sample path $(T_1, \ldots, T_n)$, the hedge gain is

$$\sum_{m=1}^{n} T_m\,\hat{\Delta}_m\,S_{m-1} \;=\; L \sum_{m>n-N^*} T_m\,S_{m-1} \;=\; L \sum_{m>n-N^*} (S_m - S_{m-1}) \;=\; L\big(S_n - S_{n-N^*}\big),$$

where we used $T_m S_{m-1} = S_m - S_{m-1}$ (definition of T_m) and telescoping. Substituting into ℓ_n:

$$\ell_n(\hat{\Delta}, T) \;=\; g(S_n) \;-\; L(S_n - S_{n-N^*}).$$

By L-Lipschitzness applied at 0: $|g(S_n) - g(0)| \le L|S_n - 0| = LS_n$, hence $g(S_n) \le g(0) + LS_n$ (using $S_n \ge 0$). Therefore

$$\ell_n(\hat{\Delta}, T) \;\le\; g(0) + LS_n - L(S_n - S_{n-N^*}) \;=\; g(0) + L\,S_{n-N^*}.$$

Drift control. For any adapted adversary in $\widetilde{\mathcal{T}}_n$, Cauchy–Schwarz on $\mathbb{E}[T_m^2|\mathcal{F}_{m-1}] \le h_n^2$ yields $|\mathbb{E}[T_m|\mathcal{F}_{m-1}]| \le h_n$, hence

$$\mathbb{E}[S_m|\mathcal{F}_{m-1}] \;=\; S_{m-1}\big(1 + \mathbb{E}[T_m|\mathcal{F}_{m-1}]\big) \;\le\; S_{m-1}(1+h_n).$$

Iterating the tower rule: $\mathbb{E}[S_m] \le s(1+h_n)^m$ for every adversary. Therefore

$$\mathbb{E}[\ell_n(\hat{\Delta}, T)] \;\le\; g(0) + L\,\mathbb{E}[S_{n-N^*}] \;\le\; g(0) + L\,s\,(1+h_n)^{n-N^*}.$$

Conclusion. $V_n^{\mathrm{agg},B}(s) = \inf_\Delta \sup_T \mathbb{E}[\ell_n] \le \sup_T \mathbb{E}[\ell_n(\hat{\Delta}, T)] \le g(0) + Ls(1+h_n)^{n-N^*}$.

Asymptotic. $n - N^* = n(L-B)/L + O(1)$, so by $\log(1+h_n) = h_n - h_n^2/2 + O(h_n^3)$ and $nh_n = \sqrt{cn}(1 + O(1/n))$, $nh_n^2 = c(1+O(1/n))$:

$$(n - N^*)\log(1+h_n) \;=\; \tfrac{L-B}{L}\sqrt{cn} \;-\; \tfrac{(L-B)c}{2L} \;+\; O(1/\sqrt{n}),$$

giving the claimed leading-order expression. □

Remark 11.6 (Super-polynomial growth)**.** Theorem 11.4 shows that the aggregate premium $\Phi_n^{\mathrm{agg},B}$ grows *strictly faster than any polynomial* in n. This refutes the polynomial-rate conjecture entertained in earlier versions of this paper and in theorem 11.3's remark: the true rate is at least sub-exponential with exponent $(L-B)\sqrt{cn}/L$, which is $(L-B)/L < 1$ times the per-step exponent $\sqrt{cn}$. The two-phase mechanism — pure drift for $(L-B)/L$ of the horizon, then budget-saturating "dollar-drift" for the remaining B/L fraction — aligns the player's per-step hedge capacity with the adversary's dollar-drift rate, producing an *asymptotic* cancellation in the post-K steps (the residual $L(n-K) - nB$ is bounded by $O(L)$ from integer rounding of K, not identically zero) and leaving only the Phase 1 multiplicative growth $(1+h_n)^K$.

Remark 11.7 (Tight aggregate rate)**.** Theorems 11.4 and 11.5 together *tightly* pin down the aggregate premium:

$$\Phi_n^{\mathrm{agg},B}(s) \;=\; L\,s\,\exp\!\Big(\tfrac{L-B}{L}\sqrt{cn} - \tfrac{(L-B)c}{2L}\Big)\,(1+o(1)).$$

The exponent coefficient is exactly $c_0 = (L-B)/L$ and the leading multiplicative constant is $L\,s\,e^{-(L-B)c/(2L)}$, matching at $(1+o(1))$ precision. This resolves the open aggregate-rate question that earlier versions of this paper left open: the aggregate game sits strictly below the per-step rate $\Theta((L-B)se^{\sqrt{cn}})$ (exponent 1) but strictly above any polynomial bound (exponent $(L-B)/L > 0$). Theorems 11.2 and 11.3 remain valid but are subdominant for large n; they give sharper pre-asymptotic constants in their respective regimes.

Remark 11.8 (Why power-law beats linear drift)**.** The linear-drift adversary of theorem 11.2 keeps $\epsilon_m S_{m-1}$ *constant* in m, so the player is indifferent between concentrating budget and spreading it. The power-law adversary of theorem 11.3 makes $\epsilon_m S_{m-1}$ strictly increasing, forcing the player to concentrate at step n where the player's peak gain nBc_n grows only as $B\sqrt{cn}\cdot S_n^{1-\kappa}$ while the payoff grows as LS_n. The adversary chooses κ just above the critical value B/L so that $LS_n - nBc_n > 0$ with the largest possible rate; the $1/\log n$ factor is the "price" of approaching this critical ratio. The construction disproves the natural conjecture that aggregate constraints yield only a $\sqrt{n}$ premium.

Remark 11.9 (Lagrangian dual reformulation)**.** Weak duality on the aggregate constraint is immediate: for every $\lambda \geq 0$,

$$V_n^{\mathrm{agg},B}(s) \;\geq\; \sup_{\mu\in\mathcal{M}_\lambda} \mathbb{E}_\mu[g(S_n)] \;-\; \lambda\, nB,$$

where

$$\mathcal{M}_\lambda \;:=\; \big\{(\mu_m) : \mu_m \in \widetilde{\mathcal{T}}_n,\ |S_{m-1}\,\mathbb{E}[T_m|\mathcal{F}_{m-1}]| \leq \lambda \text{ a.s., all } m\big\}.$$

Strong duality (equality after $\sup_{\lambda\geq 0}$) is a *conjecture*: the aggregate constraint $\sum|\Delta_m| \leq nB$ couples stages, so Sion stage-by-stage does not directly apply, and the natural non-dominated robust-duality framework of Bouchard–Nutz [9] or Cheridito–Kupper–Tangpi [11] must be adapted. The constraint localizes the adversary's drift to a *dollar* cap (rather than return cap), a fundamentally aggregate structure. Sharp upper bounds on $\Phi_n^{\mathrm{agg},B}$ reduce to bounding $\sup_{\mu\in\mathcal{M}_\lambda}\mathbb{E}[g(S_n)]$ as a function of λ; an HJB viscosity analysis of the drift-localized continuous-time limit would complete the matching polynomial upper bound. The continuous-time analog $\mathrm{d}S_t = \mu(t)\,\mathrm{d}t + \sqrt{c}\,S_t\,\mathrm{d}W_t$ with $\int_0^T|\mu(t)|\,\mathrm{d}t \leq \Lambda$ fits precisely into the *finite-fuel singular control* framework of Karatzas [24] and Karatzas–Ocone–Wang [25]; structurally analogous pathwise-cost duality for bounded-variation trading is developed by Dolinsky–Soner [17] for proportional transaction costs. Bouchard–Nutz [9] give a discrete-time nondominated robust-duality framework that directly accommodates the L_1 aggregate constraint; Cheridito–Kupper–Tangpi [11] develop relaxed-superhedging dualities with path-dependent penalties that supply the Lagrangian structure for $\lambda\cdot nB$. The "missing step" between this continuous-time program and a matching sub-exponential upper bound on $V_n^{\mathrm{agg},B}$ is twofold: (i) a C^2-regularity transfer [31] for the free-boundary solution, and (ii) sharp time-uniform Freedman-type concentration [33, 23] on the discrete martingale piece to control the discrete-to-continuous error of $V_n^{\mathrm{agg},B}/\alpha_n$. In the burst-linear and power-law regimes the error is plausibly $O(\log n/\sqrt{cn})$ by the same Taylor–Grönwall argument used in the proof of theorem 11.3. We leave both the strong-duality step and the matching sub-exponential upper bound open (see item 1).

Proposition 11.10 (Bounded payoffs have bounded premium)**.** *Suppose* $g\colon[0,\infty)\to\mathbb{R}_{\geq 0}$ *is bounded:* $M_g := \sup_{s\geq 0} g(s) < \infty$*. Then for any* $B \geq 0$*,* $s > 0$*,* $n \geq 1$*:*

$$0 \;\leq\; \Phi_n^B(s) \;\leq\; M_g \;-\; V_n^\infty(s,0) \;\leq\; M_g,$$

uniformly in n. In particular, no exponential premium growth is possible when g is bounded.

Proof. Lower bound. A per-step constraint $|\Delta_m| \le B$ is a tightening of the unconstrained player's feasible set, so $V_n^B = \inf_{|\Delta_m|\le B} \sup_\nu \ell_n \ge \inf_\Delta \sup_\nu \ell_n = V_n^\infty$; hence $\Phi_n^B \ge 0$.

Upper bound. The player's strategy $\Delta \equiv 0$ is feasible under any action bound $B \ge 0$; it yields $\ell_n(0, S_n) = \mathbb{E}[g(S_n)] \le M_g$ for every admissible S_n. Hence $V_n^B(s, 0) \le \sup_{S_n} \ell_n(0, S_n) \le M_g$. Combining with $V_n^\infty \ge 0$ (since $g \ge 0$) gives $\Phi_n^B \le M_g - V_n^\infty \le M_g$. □

The exponential mechanism of theorem 5.5 therefore genuinely requires g unbounded; a capped call $g(s) = \min((s-K)_+, M)$ has $\Phi_n^B \le M$ uniformly, even for $B < L$. The matching lower bound, establishing that the premium is genuinely $\Theta(1)$ (not $o(1)$), is the following theorem. Note that bounded nondecreasing payoffs are generically *not* convex (the plateau breaks convexity), so this theorem lies outside definition 5.1 and is stated under dropped convexity; the proof below uses only (i) feasibility of the Dirac adversary in $\widetilde{\mathcal{T}}_n$ and (ii) the martingale property of the $B = \infty$ adversary, both of which hold without convexity of g.

Theorem 11.11 (Bounded-payoff premium: $\Theta(1)$ matching lower bound)**.** *Let $g\colon [0,\infty) \to [0, M_g]$ be nondecreasing and L-Lipschitz with $g(0) = 0$, $M_g < \infty$, and $s_M := \inf\{x : g(x) \ge M_g\} < \infty$ (so g reaches its plateau). Fix initial state $s \in (0, M_g/L)$ (deep-OTM/sub-Lipschitz regime) and per-step action bound $B \in \big(0,\, (M_g - Ls)/s_M\big)$. Set $\eta := M_g - Ls - Bs_M > 0$. Then for all n large enough that $s\,\alpha_n > s_M$ and $Bs_M h_n < \eta/2$,*

$$\Phi_n^B(s) \;\ge\; M_g \;-\; Ls \;-\; B(s_M - s) \;-\; Bs_M\, h_n \;\ge\; \eta/2 \;>\; 0.$$

Combined with the trivial upper bound $\Phi_n^B \le M_g - V_n^\infty \le M_g$ of proposition 11.10, we obtain $\Phi_n^B(s) = \Theta(1)$ with $\liminf_{n\to\infty} \Phi_n^B(s) \ge M_g - Ls - B(s_M - s)$.

Proof. Lower bound on $V_n^B(s, 0)$ via explicit adversary. Define $\tau := \min\{m : s(1+h_n)^m \ge s_M\}$, so $\tau = \lceil \log(s_M/s)/\log(1+h_n)\rceil = O(\sqrt{n}\log(s_M/s))$. The standing assumption $s\alpha_n > s_M$ guarantees $\tau \le n$. The adversary plays the deterministic Dirac strategy

$$T_m = \begin{cases} h_n, & m \le \tau, \\ 0, & m > \tau. \end{cases}$$

Both Diracs are feasible in $\widetilde{\mathcal{T}}_n$: $|h_n| \le \zeta_n$ by the standing calibration, $h_n^2 \le e^{c/n} - 1$ (saturating), and $0 \in [-\zeta_n, \zeta_n]$ trivially. Under this strategy, $S_m = s(1+h_n)^{\min(m,\tau)}$ deterministically, so $S_n = S_\tau \in [s_M,\, s_M(1+h_n))$ and $g(S_n) = M_g$ exactly.

For this deterministic adversary, the player's optimal hedge (under $|\Delta_m| \le B$ per-step) is $\Delta_m = B$ for $m \le \tau$ and $\Delta_m = 0$ for $m > \tau$ (only the drift phase generates hedge income). The player's total gain is

$$\sum_{m\le\tau} h_n \cdot B \cdot S_{m-1} \;=\; Bh_n \sum_{m\le\tau} s(1+h_n)^{m-1} \;=\; Bs\big((1+h_n)^\tau - 1\big) \;=\; B(S_\tau - s).$$

Using $S_\tau \le s_M(1+h_n)$ gives player gain $\le B(s_M - s) + Bs_M h_n$. Therefore

$$V_n^B(s, 0) \;\ge\; M_g - B(s_M - s) - Bs_M h_n.$$

Upper bound on $V_n^\infty(s, 0)$ via concave envelope. Define $U(x) := \min(Lx, M_g)$, the concave envelope of g (since g is L-Lipschitz with $g(0) = 0$, we have $g(x) \le Lx$; combined with $g \le M_g$, $g(x) \le U(x)$ pointwise). U is concave and nondecreasing. We claim $V_n^\infty(\cdot, m) \le U$ pointwise at

every m, by backward induction: the base case $V_n^\infty(\cdot, n) = g \le U$ holds by definition. For the inductive step, if $V_n^\infty(\cdot, m{+}1) \le U$, then (using that the unconstrained-player adversary is forced to $\mathbb{E}_\mu[T] = 0$ by martingale forcing, so $\mathbb{E}_\mu[s(1{+}T)] = s$)

$$V_n^\infty(s, m) \;\le\; \sup_{\mu \in \mathcal{T}_n} \mathbb{E}_\mu[U(s(1{+}T))] \;\le\; U(\mathbb{E}_\mu[s(1{+}T)]) \;=\; U(s),$$

where the second inequality is Jensen's on concave U. Iterating to $m = 0$: $V_n^\infty(s, 0) \le U(s)$. For $s \in (0, M_g/L)$: $U(s) = Ls$.

Combining.

$$\Phi_n^B(s) \;=\; V_n^B(s, 0) - V_n^\infty(s, 0) \;\ge\; (M_g - B(s_M - s) - Bs_M h_n) - Ls \;=\; \eta - Bs_M h_n.$$

For n large enough that $Bs_M h_n < \eta/2$, this is at least $\eta/2 > 0$, establishing the claim. □

Remark 11.12 (Extension to shallower OTM)**.** Theorem 11.11 holds in the "deep-OTM" regime $s < M_g/L$. For $s \in [M_g/L,\, s_M)$ (the "shallow-OTM" / near-strike ramp), the concave-envelope upper bound $V_n^\infty \le Ls$ becomes vacuous; sharpening requires AFW-style martingale FCLT for bounded payoffs, giving $V_n^\infty \to \mathbb{E}[g(sG(1))]$ (Black–Scholes price of the capped payoff). The $\Theta(1)$ conclusion likely extends to this regime by the same drift-to-plateau mechanism with a smaller gap.

Remaining open problems (not targeted for this paper, flagged for future work):

1. **Tight non-asymptotic constants.** Theorems 11.4 and 11.5 resolve the aggregate rate to $(1 + o(1))$ precision: $\Phi_n^{\mathrm{agg},B}(s) = Ls \cdot e^{(L-B)\sqrt{cn}/L - (L-B)c/(2L)}(1 + o(1))$. The $o(1)$ error is $O(1/\sqrt{n})$ from a Taylor expansion of $\log(1 + h_n)$; tightening to explicit non-asymptotic constants (e.g., via sharp time-uniform Freedman bounds [33, 23]) is a routine extension we have not pursued.

2. **Smooth-tail rates.** Theorem 6.4 gives $R_n = 0$ exactly for piecewise-linear g, and proposition 6.6 extends this to stretched-exponential or polynomial rates for smoother g based on the decay of the tail residual $\psi(x) = g(x) - Lx - \beta_\infty$. Making the constants sharp for specific payoff classes (Black–Scholes, capped call, Huber) is a routine extension; we have not pursued the details.

3. **Beyond zero-mean constraints.** The bridge uses the martingale condition $\mathbb{E}[x_{m+1}|x_m] = x_m$. What of $\mathbb{E}[x_{m+1}|x_m] = \phi(x_m)$ for known ϕ ("ϕ-transport")? The T-substitution and Lipschitz preservation require zero mean; which parts generalize?

4. **Non-monotone payoffs.** Theorem 11.11 resolves the bounded-payoff case ($\Phi_n^B = \Theta(1)$ for deep-OTM) and remark 11.12 sketches the shallow-OTM extension. Non-monotone convex L-Lipschitz payoffs (e.g., straddles $g(s) = |s - K|$) satisfy attainment on the monotone tails and are expected to yield $\Theta(e^{\sqrt{cn}})$ by an argument analogous to theorem 6.2; formalizing this is a straightforward extension.

5. **Rescaled limits.** Since V_n^B itself diverges, is there a natural rescaling $\tilde{V}_n^B := V_n^B/\alpha_n$ that converges to a meaningful PDE limit? The gradient-constrained VI (remark 6.10) is a candidate limiting equation for such a rescaled quantity.

Broader context. The bridge suggests OT regularization techniques (entropic smoothing, constraint relaxation, Sinkhorn iterations) have direct game-theoretic interpretations as action-space bounds, approximate equilibria, and strategy computation; conversely, backward induction decomposes multi-marginal transport into sequential two-marginal problems. The Lipschitz degradation and premium divergence results reveal a fundamental obstacle: bounded action spaces in sequential games with unbounded convex losses are exponentially costly, with implications beyond OT (e.g., adversarial online learning with action constraints).

Scope: path-independent losses. The bridge decomposes the n-step game into n one-step problems because the continuation value $V(s, m)$ depends only on the current state (Markov property), requiring a path-independent terminal loss $g(S_n)$. For path-dependent payoffs (Asian, lookback, barrier), continuation values depend on full history, the two-marginal decomposition fails, and multi-marginal formulations are needed [21]. The AFW 2012 [3] open problems are stated for path-independent payoffs; our results inherit this scope. Extending the bridge to path-dependent losses is an important open direction.

All proofs are given inline in the sections where the results are stated.

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