
A HIGHER ORDER LOCAL MESH METHOD FOR APPROXIMATING 1-LAPLACIANS ON UNKNOWN MANIFOLDS

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ABSTRACT

We introduce a numerical method for approximating arbitrary differential operators on vector fields in the weak form given point cloud data sampled randomly from a d dimensional manifold embedded in $\mathbb{R}^n$. This method generalizes the local linear mesh method to the local curved mesh method, thus, allowing for the estimation of differential operators with nontrivial Christoffel symbols, such as the Bochner or Hodge Laplacians. In particular, we leverage the potentially small intrinsic dimension of the manifold ($d \ll n$) to construct local parameterizations that incorporate both local meshes and higher-order curvature information. The former is constructed using low dimensional meshes obtained from local data projected to the tangent spaces, while the latter is obtained by fitting local polynomials with the generalized moving least squares. Theoretically, we prove the spectral convergence for the proposed method for the estimation of the Bochner Laplacian. We provide numerical results supporting the theoretical convergence rates for the Bochner and Hodge Laplacians on simple manifolds.

Keywords Bochner Laplacian · Hodge Laplacian · vector Laplacians · manifold learning · local meshes · eigenvalue convergence

1 Introduction

The graph Laplacian is a widely used tool in unsupervised learning tasks such as dimensionality reduction [3, 10], clustering [23, 30], and semi-supervised learning [2, 8]. Given a finite sample set $X = \{x_1, \dots, x_n\}$, the graph Laplacian is constructed by quantifying pairwise affinities between data points. Under the manifold assumption $X \subset M$, where M is a d -dimensional Riemannian submanifold of $\mathbb{R}^m$, a standard approach is to build the graph using a kernel that measures the closeness of data points using the ambient Euclidean metric. The key motivation is that for sufficiently local kernels [10, 5], including compactly supported kernels [28, 7], Euclidean distances in $\mathbb{R}^m$ provide accurate approximations of the geodesic distances on M . This observation firmly establishes the graph Laplacian as a fundamental manifold learning tool: it enables access to the intrinsic geometry of the data using only ambient-space information. In particular, when the graph Laplacian is shown to converge, in the large-sample limit, to its continuous analogue, the Laplace–Beltrami operator, it becomes a consistent estimator of the underlying geometric structure. Since Laplace–Beltrami operator acts on functions, this class of approach allows one to represent functions on the manifold that the data lie on.

Recent extensions for representing vector fields on manifolds (or, dually, differential 1-forms) have also been proposed. One prominent approach is vector diffusion maps [25], which generalizes diffusion maps by incorporating local parallel transport into the kernel construction and is shown to be consistent for approximating the connection Laplacian. Related frameworks, including Discrete Exterior Calculus (DEC) [16] and Finite Element Exterior Calculus (FEEC) [1], have also been developed. However, these methods are often ill-suited for point-cloud data, as they require additional geometric structure, global meshes or simplicial complexes, that may not be available in many data-driven settings. Particularly, when the point cloud data are randomly sampled from a low dimensional manifold embedded in high dimensional Euclidean space, the construction of global meshes in the high dimensional ambient space is computationally not feasible. Moreover, the spectral consistency of DEC for estimating Laplace–de Rham operators remains unclear.

These limitations are addressed by the Spectral Exterior Calculus (SEC) [4], a Galerkin-based framework that represents differential forms and vector fields using eigenfunctions of the Laplace–Beltrami operator. The accuracy of SEC, however, depends critically on the quality of the estimated Laplace–Beltrami eigenpairs, since all differential operators are constructed through spectral expansions of the zero-form Laplacian. In [4], Diffusion Maps are used to estimate these eigenfunctions. We argue that the performance of SEC can be further improved by the approach developed in this paper, which enables accurate estimation of eigensolutions of Laplace operators, including the Laplace–Beltrami, Bochner and Hodge Laplacians.

In our previous work [15], we developed a mesh-free framework for estimating arbitrary differential operators acting on tensor fields. The original formulation was based on radial basis function (RBF) interpolation, and its computational efficiency was later improved through the use of generalized moving least squares (GMLS) [21, 17]. This approach employs moving least squares to estimate local coordinate charts and subsequently approximates differential operators using standard pointwise formulas from differential geometry. However, because the method is inherently pointwise, it yields a nonsymmetric discretization of the Laplacian operators, which significantly complicates both computation and theoretical analysis. Although a variational formulation of these differential operators was also introduced in [15], the reliance on Monte Carlo integration substantially degrades the accuracy of the resulting operator estimates.

In this paper, we build on the local mesh approach of [18], replacing Monte Carlo integration over point cloud data with a higher-order local mesh construction. The original method [21] constructs linear local meshes by projecting data points onto estimated tangent spaces inferred from the point cloud. While this framework enables the estimation of the Laplace–Beltrami operator, it fails to accurately approximate operators whose local expressions depend on higher-order curvature quantities, such as Christoffel symbols. This limitation arises because the induced Riemannian metric is constant on each triangle of the local meshes.

To overcome this issue, we introduce a higher-order local curved mesh based on Generalized Moving Least Squares (GMLS), generalizing the approach in [18]. Although this framework can approximate arbitrary differential operators acting on tensor fields, we focus here on estimating 1-Laplacians on vector fields. In particular, we establish the spectral consistency with the Bochner Laplacian in the large-data limit. Extensive numerical experiments further verify the convergence of the proposed Hodge and Bochner Laplacian estimators using randomly sampled ambient point cloud data. Practically, the estimated tensor fields can directly evaluate on any out-of-sample data points on the manifold without requiring an additional interpolation scheme that is needed in graph-Laplacian based methods, including Vector Diffusion Maps and SEC.

The remainder of this paper is organized as follows. In Section 2, we review the local frameworks of [18] and [21]. We also demonstrate how the parameterization of [21] can be used on the local mesh of [18] to obtain local curved meshes. In Section 3, we formulate how the local charts of Section 2 can be used to obtain weak form estimations of arbitrary differential operators on tensor fields. In Section 4, we prove the spectral convergence for the estimation of Bochner Laplacian. Finally, in Section 5, we demonstrate the numerical convergence of the local curved mesh method in estimating the Hodge and Bochner Laplacians on some example manifolds. We supplement this paper with two appendices: In Appendix A, we report the detailed calculation for the components of the mass and stiffness matrices for both the Bochner and Hodge Laplacians. In Appendix B, we report the proofs of some technical lemmas.

2 Local Metric Learning

This section provides an overview of two existing frameworks which are key to the novel method introduced in this paper. We begin by giving a description of the Local Mesh Method [18], which leverages local tangent space approximations to construct triangular meshes near each element of a given point cloud. We also present a brief overview of the Generalized Moving Least Squares technique [21], in which local parameterizations of the form $\Phi : T_x M \rightarrow M$ are approximated by fitting polynomials on the estimated tangent spaces using a weighted least

squares. Subsequently, we introduce the Local Curved Mesh Method, which serves as the central idea of this paper, amalgamating the aforementioned methodologies. To allow for a concise description of each method, we focus on learning the Riemannian metric of an unknown manifold on a local chart throughout this section. The ideas of this section, however, can be extended to approximate more complicated objects on manifold (e.g., operators on tensor bundles, as shown in Section 3).

2.1 Local tangent space approximation

Given a point cloud $X = \{x_1, \dots, x_N\} \subseteq \mathbb{R}^n$ sampled from an unknown manifold M of dimension $d \ll n$, we wish to obtain local estimation of $T_{x_i}M$, the tangent space at each point, as each method of locally approximating a Riemannian metric described in this section hinges on such an estimation. This problem is well studied in the literature, and is frequently addressed using local SVD [12, 29, 32] or local PCA, as in [18, 21]. More recently, a 2nd order local SVD method introduced in [15] improves on local SVD by keeping track of higher order terms. For simplicity, we provide a brief overview of local PCA.

Given a point x_i , denote its k nearest neighbors by $K(i) = \{x_{i_1}, \dots, x_{i_k}\}$. In this notation, $x_i = x_{i_1}$. Consider the local $n \times n$ covariance matrix

$$P_i = \sum_{x \in K(i)} (x - \mu_i)(x - \mu_i)^\top,$$

where $\mu_i = \frac{1}{k} \sum_{x \in K(i)} x$. In practice, we found insignificant accuracy differences using either the average μ_i or the base point x_i in placing μ_i away from the boundary. After diagonalizing this matrix, one obtains n eigenpairs $(\lambda_1^{(i)}, \mathbf{t}_1^{(i)}), \dots, (\lambda_n^{(i)}, \mathbf{t}_n^{(i)})$ with $\lambda_j^{(i)} \geq \lambda_{j+1}^{(i)}$. Since the intrinsic dimension of the data is d , one expects to find that

$$\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \dots \geq \lambda_d^{(i)} \gg \lambda_{d+1}^{(i)},$$

given that the sampling is sufficiently dense around x_i . Thus, one can determine the dimension d , and obtains an orthonormal basis $\{\mathbf{t}_1^{(i)}, \dots, \mathbf{t}_d^{(i)}\}$ estimating the tangent space at x_i . We denote the copy of $\mathbb{R}^d$ given by the span of these coordinates by $\overline{T_{x_i}M}$. In what follows, we assume such an estimation is obtained, and denote the basis vectors as above.

2.2 Approximation of metric tensor with linear local meshes

Given x_i , we aim to obtain a d -dimensional mesh for $K(i)$, which is x_i and its k -nearest neighbors. To simplify the discussion, we assume $d = 2$. In [18], they propose to solve this problem by first projecting $K(i)$ to the estimated tangent space:

$$T(i) := \left\{ \underbrace{\left(\mathbf{t}_1^{(i)} \cdot (x_{i_1} - x_{i_1}), \mathbf{t}_2^{(i)} \cdot (x_{i_1} - x_{i_1}) \right)}_{(0,0)=\vec{0}}, \dots, \left(\mathbf{t}_1^{(i)} \cdot (x_{i_k} - x_{i_1}), \mathbf{t}_2^{(i)} \cdot (x_{i_k} - x_{i_1}) \right) \right\} \subseteq \overline{T_{x_i}M}.$$

For simplicity of future notation, we define

$$\vec{v}_{i_r} = \begin{pmatrix} \mathbf{t}_1^{(i)} \cdot (x_{i_r} - x_{i_1}) \\ \mathbf{t}_2^{(i)} \cdot (x_{i_r} - x_{i_1}) \end{pmatrix}, \quad (1)$$

so that

$$T(i) = \{\vec{v}_{i_1} = \vec{0}, \vec{v}_{i_2}, \dots, \vec{v}_{i_k}\}.$$

A mesh on $T(i)$, denoted by $\mathbf{R}(i)$, is obtained using standard methods, e.g., the Delauney triangulation. Denote by $R(i)$ the first ring of this triangulation, that is, $R(i)$ is set of ℓ triangles of this mesh which include the basepoint $(0,0)$. These necessarily take the form

$$R(i) = \{T_{i,1}, \dots, T_{i,\ell}\} \subset \mathbf{R}(i),$$

where each $T_{i,r} = [\vec{0}, \vec{v}_{i_{j_r}}, \vec{v}_{i_{k_r}}]$. Consider any such triangle. This can be parameterized by the standard Barycentric parameterization with the canonical right triangle $T = \{(u_1, u_2) | u_1, u_2 \geq 0, u_1 + u_2 \leq 1\}$ via the map $\Phi_{T_{i,r}} : T \rightarrow T_{i,r} \subseteq \mathbb{R}^d$ defined by

$$(u_1, u_2) \mapsto u_1 \vec{v}_{i_{j_r}} + u_2 \vec{v}_{i_{k_r}}. \quad (2)$$

Since the first ring can be “lifted” from the tangent space to the manifold via the one-to-one correspondence between $T(i)$ and $K(i)$ given by

$$\vec{v}_{i_r} \mapsto x_{i_r}.$$

we obtain the map to the global coordinates, $\tilde{\Phi}_{T_{i,r}} : T \rightarrow \mathbb{R}^n$, defined by

$$(u_1, u_2) \mapsto u_1(x_{i_{j_r}} - x_i) + u_2(x_{i_{k_r}} - x_i) + x_i. \quad (3)$$

Let $y \in \tilde{\Phi}_{T_{i,r}}(T) \subset \mathbb{R}^n$. Based on the barycentric parameterization, there exists $(u_1, u_2) \in T$ such that

$$y = (1 - u_1 - u_2)x_i + u_1 x_{i_{j_r}} + u_2 x_{i_{k_r}} = x_i + u_1(x_i - x_{i_{j_r}}) + u_2(x_i - x_{i_{k_r}}) := \tilde{\Phi}_{T_{i,r}}(u_1, u_2),$$

which is effectively Eq (3) above. From this identity, it is clear that,

$$\underbrace{[\mathbf{t}_1^{(1)} \ \mathbf{t}_2^{(1)}]^\top}_{=\mathbf{T}^\top} (y - x_i) = u_1 \vec{v}_{i_{j_r}} + u_2 \vec{v}_{i_{k_r}} = \Phi_{T_{i,r}}(u_1, u_2).$$

Then

$$y - x_i = \mathbf{T}\mathbf{T}^\top (y - x_i) + \mathbf{t}_3^{(i)} \mathbf{t}_3^{(i)\top} (y - x_i),$$

so we can obtain the relation between $\tilde{\Phi}$ in terms of Φ ,

$$\tilde{\Phi}_{T_{i,r}}(u_1, u_2) = \mathbf{T}\Phi_{T_{i,r}}(u_1, u_2) + \mathbf{t}_3^{(i)} \mathbf{t}_3^{(i)\top} (y - x_i) + x_i. \quad (4)$$

In this case, the ‘lifted’ triangle can be understood as a linear interpolation in the normal component given by,

$$p_{i,r}(u_1, u_2) = \mathbf{t}_3^{(i)\top} (y - x_i) = u_1 \mathbf{t}_3^{(i)\top} (x_i - x_{i_{j_r}}) + u_2 \mathbf{t}_3^{(i)\top} (x_i - x_{i_{k_r}}),$$

where $p_{i,r} : T \rightarrow \mathbb{R}^{n-d}$ is a linear function.

The coordinates corresponding to the map $\tilde{\Phi}_{T_{i,r}}$ is then given by,

$$\left\{ \partial_{u_1} = x_{i_{j_r}} - x_i, \partial_{u_2} = x_{i_{k_r}} - x_i \right\}$$

in which the Riemannian metric takes the form

$$\mathbf{g}_{\tilde{\Phi}_{T_{i,r}}}(u_1, u_2) = \begin{bmatrix} \|x_{i_{j_r}} - x_i\|^2 & (x_{i_{j_r}} - x_i) \cdot (x_{i_{k_r}} - x_i) \\ (x_{i_{k_r}} - x_i) \cdot (x_{i_{j_r}} - x_i) & \|x_{i_{k_r}} - x_i\|^2 \end{bmatrix}. \quad (5)$$

We emphasize that in this formulation, the Riemannian metric is constant as a matrix valued function of (u_1, u_2) . The work in [18] leverages these coordinate representations, along with local coordinate invariant formulas, to approximate other geometric objects, such as the Laplace-Beltrami operator.

Remark 1. *Since the above metric formula is constant in (u_1, u_2) , it is clear that operator defines on the above local mesh parameterization cannot distinguish operators that are differed by higher order objects (e.g., the Hodge and Bochner Laplacians are differed by Ricci curvature). This issue, which arises from the trivial Cristoffel symbols in this linear mesh parameterization, motivates the use of curving (non-flat) local meshes as discussed in Section 2.4. In Section 3, we will demonstrate how such a general (curving) local mesh framework can be used to learn operators on general tensor fields.*

2.3 Generalized moving least squares approximation

Without loss of generality, we set $d = 2$ and $n = 3$ to simplify the discussion. Details for extending these ideas to the more general setting can be found in [21]. Given a point x_i , consider the basis $\{\mathbf{t}_1^{(i)}, \mathbf{t}_2^{(i)}, \mathbf{t}_3^{(i)}\}$, where $\{\mathbf{t}_1^{(i)}, \mathbf{t}_2^{(i)}\}$ is a basis for the approximated tangent space $T_{x_i}M$, and $\mathbf{t}_3^{(i)}$ approximates the normal direction. Recall that the k nearest neighbors $K(i)$, and the neighbors projected onto the tangent space $T(i)$. Denote by $N(i)$ the normal components:

$$N(i) := \left\{ \mathbf{t}_3^{(i)} \cdot (x_{i_1} - x_i), \dots, \mathbf{t}_3^{(i)} \cdot (x_{i_k} - x_i) \right\}.$$

Consider a polynomial $p_i : \overline{T_{x_i}M} \rightarrow \mathbb{R}^{n-d}$ defined by

$$p_i(v_1, v_2) = av_1^2 + bv_2^2 + cv_1v_2 + dv_1 + ev_2 + f, \quad (6)$$

where the coefficients are obtained by a weighted least squares fit to the data in $T(i)$ with labels given by $N(i)$. In particular, the coefficients are obtained via

$$(a, b, c, d, e, f) = \operatorname{argmin} \sum_{j=1}^k \omega(\|x_i - x_{ij}\|) \left(p_i(\vec{v}_{ij}) - \mathbf{t}_3^{(i)} \cdot (x_{ij} - x_{i_1}) \right)^2 \quad (7)$$

for some weight function ω , where $\vec{v}_{ij}$ is as defined in Equation (1). Following [20, 21], we will consider only the weight function

$$\omega(t) = \begin{cases} 1 & t = 0 \\ 1/k & \text{else,} \end{cases}$$

which is empirically found to be adequate for random data. After fitting such a polynomial, we obtain a local parameterization of the form

$$\alpha_{x_i} : (v_1, v_2) \mapsto (v_1, v_2, p_i(v_1, v_2)). \quad (8)$$

This parameterization induces the following coordinates:

$$\left\{ \partial_{v_1} = \begin{bmatrix} 1 \\ 0 \\ 2av_1 + cv_2 + d \end{bmatrix}, \partial_{v_2} = \begin{bmatrix} 0 \\ 1 \\ 2bv_2 + cv_1 + e \end{bmatrix} \right\}.$$

such that the Riemannian metric in these coordinates takes the form,

$$g_{x_i}(v_1, v_2) = \begin{bmatrix} 1 + (2av_1 + cv_2 + d)^2 & (2av_1 + cv_2 + d)(2bv_2 + cv_1 + e) \\ (2av_1 + cv_2 + d)(2bv_2 + cv_1 + e) & 1 + (2bv_2 + cv_1 + e)^2 \end{bmatrix}.$$

We remark that while the domain of the parameterization from the previous subsection is a standard simplex, the domain of the parameterization from this subsection is the estimated tangent space itself.

2.4 Approximation of metric tensor with local curved meshes

Local curved meshes conceptually combine the quadratic parameterization of the previous section with the local meshes of Section 2.2. As we shall see in Section 3, this approximation allows the flexibility and robustness in using random data in the weak approximation of differential operators while simultaneously avoiding the issues mentioned in Remark 1. To this end, consider a fixed point x_i , and its first ring $R(i)$. For an arbitrary triangle

$$T_{i,r} = [0, \vec{v}_{i_{jr}}, \vec{v}_{i_{kr}}] \in R(i),$$

recall the map $\Phi_{T_{i,r}}$ as defined in Equation 2. The codomain of this map is $\overline{T_{x_i}M}$, which is precisely the domain of the parameterization p_i from the previous section. Hence, it's natural to consider parameterizations $\tilde{\Phi}_{C_{i,r}} : T \rightarrow \mathbb{R}^n$ of the curved triangular patches of the locally approximated manifold, given by,

$$\tilde{\Phi}_{C_{i,r}}(u_1, u_2) = \mathbf{T}\Phi_{T_{i,r}}(u_1, u_2) + \mathbf{t}_3^{(i)} p_{i,r}(u_1, u_2) + x_i, \quad (9)$$

where we generalize (4) by setting the vertical component $p_{i,r} : T \rightarrow \mathbb{R}^n$ to be a nonlinear map,

$$p_{i,r}(u_1, u_2) = p_i \circ \Phi_{T_{i,r}}(u_1, u_2),$$

where p_i is obtained by the generalized moving least-squares procedure discussed in Section 2.3. The map $\tilde{\Phi}_{C_{i,r}}$ is effectively a generalization of $\tilde{\Phi}_{T_{i,r}}$, where we embed a curved (non-flat) mesh in $\mathbb{R}^n$ using a nonlinear map $p_{i,r}$.

Based on this map, one can construct the Riemannian metric $g_{\tilde{\Phi}_{C_{i,r}}}$ corresponding to the curved mesh, generalizing (5). Since the computation will be done locally for each based point x_i , it is more convenient to implement the construction without mapping to the global coordinates with the following,

$$\Phi_{C_{i,r}}(u_1, u_2) := \begin{pmatrix} \Phi_{T_{i,r}}(u_1, u_2) \\ p_i \circ \Phi_{T_{i,r}}(u_1, u_2) \end{pmatrix}. \quad (10)$$

This results in coordinates

$$\left\{ \partial_{u_1} = \begin{bmatrix} \vec{v}_{i_{jr}} \\ \nabla p_i(\Phi_{T_{i,r}}(u_1, u_2)) \cdot \vec{v}_{i_{jr}} \end{bmatrix}, \partial_{u_2} = \begin{bmatrix} \vec{v}_{i_{kr}} \\ \nabla p_i(\Phi_{T_{i,r}}(u_1, u_2)) \cdot \vec{v}_{i_{kr}} \end{bmatrix} \right\} \quad (11)$$

In these coordinates, the Riemannian metric takes the form

$$g_{\Phi_{C_{i,r}}}(u_1, u_2) = \begin{bmatrix} \|\vec{v}_{i_{jr}}\|^2 & \vec{v}_{i_{jr}} \cdot \vec{v}_{i_{kr}} \\ \vec{v}_{i_{jr}} \cdot \vec{v}_{i_{kr}} & \|\vec{v}_{i_{kr}}\|^2 \end{bmatrix} + \begin{bmatrix} \left(\nabla p_i(\Phi_{T_{i,r}}(u_1, u_2)) \cdot \vec{v}_{i_{jr}} \right)^2 & c_{j_r k_r}^i(u_1, u_2) \\ c_{j_r k_r}^i(u_1, u_2) & \left(\nabla p_i(\Phi_{T_{i,r}}(u_1, u_2)) \cdot \vec{v}_{i_{kr}} \right)^2 \end{bmatrix} \quad (12)$$

where $c_{j_r k_r}^i(u_1, u_2) = \left(\nabla p_i(\Phi_{T_{i,r}}(u_1, u_2)) \cdot \vec{v}_{i_{jr}} \right) \cdot \left(\nabla p_i(\Phi_{T_{i,r}}(u_1, u_2)) \cdot \vec{v}_{i_{kr}} \right).$

3 Operator Approximation in Weak Form

The local mesh method of [18], and hence the curved mesh method described in Section 2.4 lends itself to weak operator approximations, analogous to finite element frameworks. While the work in [18] outlines such approach in approximating the Laplace-Beltrami operator (on functions), the same framework can be generalized to differential operators on tensor fields. For instance, this framework can be used to estimate the Laplacian on vector fields. In this section, we outline this process. We begin by describing how the Laplace-Beltrami operator can be estimated, pointing out the key differences between [18] and the curved mesh method. We then generalize this, in the case of the curved mesh method, to arbitrary operators on tensor fields. Since operators on vector fields, such as the Bochner and Hodge Laplacians, are of particular interest, we supply additional details for these cases.

3.1 Weak approximation of the Laplace-Beltrami operator

The weak form Laplace-Beltrami eigenvalue problem is to find eigenpairs λ and $f \in H^1(M)$ that solves,

$$\int_M \langle \nabla f, \nabla h \rangle_g d\text{Vol} = \lambda \int_M f h d\text{Vol}, \quad (13)$$

for all $h \in H^1(M)$.

Following standard finite-element approach, consider a set of “hat” functions, $V_N = \{f = \sum_{i=1}^N f(x_i) e_i, e_i : M \rightarrow \mathbb{R}\} \subset H^1(M)$, where $\{e_i\}$ are nodal basis satisfying the property $e_i(x_j) = \delta_{ij}$. Specifically, if the true local parameterization $\gamma_{x_i} : \mathbf{R}(i) \rightarrow \mathbb{R}^3$ is denoted by,

$$\gamma_{x_i} : (v_1, v_2) \mapsto x_i + (v_1, v_2, z_{x_i}(v_1, v_2)), \quad (14)$$

using the canonical map $\Phi_{T_{i,r}} : T \rightarrow T_{i,r}$ as defined in (2), one can define the nodal basis as a hat function given as,

$$e_i \circ \gamma_{x_i} \circ \Phi_{T_{i,r}}(u_1, u_2) = 1 - u_1 - u_2.$$

Define a parameterization $\Upsilon_{i,r} := \gamma_{x_i} \circ \Phi_{T_{i,r}} : T \rightarrow \mathbb{R}^3$. For each pair $\{x_i, x_j\}$ consider the quantity

$$\int_M \langle \nabla e_i, \nabla e_j \rangle_g d\text{Vol} = \sum_{T_{i,r} \in R(i)} \int_T \langle \nabla_{\Upsilon_{i,r}} e_i, \nabla_{\Upsilon_{i,r}} e_j \rangle_{g_{\Upsilon_{i,r}}} \sqrt{\det(g_{\Upsilon_{i,r}})} du_1 du_2 := S_{ij}, \quad (15)$$

where $\nabla_{\Upsilon_{i,r}}$ and $g_{\Upsilon_{i,r}}$ denote the gradient and Riemannian metric, respectively, with each computed in the coordinates of the parameterization $\Upsilon_{i,r} : T \rightarrow \mathbb{R}^n$.

When the manifold is unknown as in our case, $\gamma_{i,r}$ is not available. If we consider approximating the linear mesh with parameterization $\tilde{\Phi}_{T_{i,r}}$ as defined in (3), then, the hat function e_i is defined not exactly on M , but on the linear triangulation $\tilde{\Phi}_{T_{i,r}}(T)$ that locally approximates M . In such a case, the function space $V_N \approx \hat{V}_N = \{f = \sum_{i=1}^N f(x_i) e_i, e_i : \tilde{\Phi}_{T_{i,r}}(T) \rightarrow \mathbb{R}\}$. Then,

$$S_{ij} \approx \hat{S}_{ij} := \sum_{T_{i,r} \in R(i)} \int_T \langle \nabla_{\tilde{\Phi}_{T_{i,r}}} e_i, \nabla_{\tilde{\Phi}_{T_{i,r}}} e_j \rangle_{g_{\tilde{\Phi}_{T_{i,r}}}} \sqrt{\det(g_{\tilde{\Phi}_{T_{i,r}}})} du_1 du_2.$$

The only nonzero terms in the above sum are when $i = j$ or when $\overline{x_i x_j}$ is a side in the lifted triangulation of $T(i)$. Such integrals can be computed analytically. In fact, when hat functions are used, this is formula results in the famous cotangent formula for the Laplace-Beltrami operator [13]. Denote by $\hat{\mathbf{S}}$ the $N \times N$ sparse matrix with (i, j) -th entry $(\hat{\mathbf{S}})_{ij} = \hat{S}_{ij}$. Similarly, we can approximate

$$\int_M e_i e_j d\text{Vol} \approx \hat{M}_{ij} := \sum_{T_{i,r} \in R(i)} \int_T e_i(u_1, u_2) e_j(u_1, u_2) \sqrt{\det(g_{\tilde{\Phi}_{T_{i,r}}})} du_1 du_2, \quad (16)$$

where the approximation occurs when we choose $e_i \in \hat{V}_N$ in $\hat{M}_{ij}$.

By definition, $\hat{\mathbf{S}}$ and $\hat{\mathbf{M}}$ are not necessarily symmetric. But since the left-hand-term in (15) and (16) are symmetric assuming we have access to the underlying metric g , we can approximate the stiffness and mass matrices by $\hat{\mathbf{A}} = \frac{1}{2}(\hat{\mathbf{S}} + \hat{\mathbf{S}}^T)$, and $\hat{\mathbf{B}} = \frac{1}{2}(\hat{\mathbf{M}} + \hat{\mathbf{M}}^T)$, following [18]. For convenience of discussion, we also define bilinear operators corresponding to these matrices,

$$a(e_i, e_j) = (\hat{\mathbf{A}})_{ij}, \quad b(e_i, e_j) = (\hat{\mathbf{B}})_{ij}.$$

Numerically, we approximate the weak form Laplace-Beltrami eigenvalue problem in (13), by finding λ and $f \in \hat{V}_N$, such that,

$$a(f, h) = \lambda b(f, h),$$

for any $h \in \widehat{V}_N$, which is analogous to solving the following eigenvalue problem,

$$\langle \widehat{\mathbf{A}}\mathbf{f}, \mathbf{h} \rangle = \lambda \langle \widehat{\mathbf{B}}\mathbf{f}, \mathbf{h} \rangle,$$

where $\mathbf{f} = (f_1, \dots, f_N)^\top$, $\mathbf{h} = (h_1, \dots, h_N)^\top$, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean dot product (which will be used accordingly for the remainder of this paper). The eigenpair $(\lambda, \mathbf{f})$ that solves the generalized eigenvalue problem $\widehat{\mathbf{A}}\mathbf{f} = \lambda \widehat{\mathbf{B}}\mathbf{f}$ is an estimate to the eigensolutions of the Laplace-Beltrami operator. Once $\mathbf{f}$ is estimated, the corresponding estimated eigenfunction is given by $f = \sum_{i=1}^N f_i e_i \in \widehat{V}_N$ that can be used on new (out-of-sample) data points.

Remark 2. In [18], the diagonal elements of $\widehat{\mathbf{S}}$ are chosen in terms of the off-diagonals to ensure that the resulting problem has non-negative eigenvalues, as well as a trivial eigenvalue with corresponding eigenvector being constant. Since we aim to estimate operators that do not necessarily have such properties, we did not employ such an adjustment in our formulation and implementation. Moreover, explicit simplification of the integrals (16) and (15) can be found in [18] in the case of flat local meshes with the parameterization $\Phi_{T_{i,r}}$ in (3). We approximate the integral in the stiffness and mass matrices, $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{M}}$, with a first-order quadrature based on evaluating the nodes of the triangle T .

3.2 Generalization to operators on arbitrary tensor fields

In this section, we will generalize the framework from the previous subsection to find eigenvalue λ and eigenvector field $W \in H^1(TM)$ that solves,

$$\langle DW, DV \rangle_U = \lambda \langle W, V \rangle_{T^{(1,0)}M}, \quad (17)$$

for all $V \in H^1(TM)$. Here the left-hand inner product is defined for appropriate range space of $D: H^1(TM) \rightarrow U$. For Bochner Laplacian, $D := \text{grad}_g: H^1(TM) \rightarrow U = T^{(2,0)}M$ is gradient of vector field. For Hodge Laplacian, $D: H^1(TM) \rightarrow U = T^{(2,0)}M \oplus C^\infty(M)$ defined as $D = (\nabla_g, d^*\flat)$, where the first component is the gradient of vector field and d^* in the second component denotes the codifferential of the exterior derivative, d , acting to 1-form, defined as, $d^* = (-1)^{d-1} \text{sign}(g) \star d \star$, where $\star: \Omega^k(M) \rightarrow \Omega^{d-k}(M)$ is the standard Hodge star operator, and the musical notation $\flat$ denotes the “lowering” operation that takes vector field to its covector field (1-form in our case).

For clarity, we present the case of $d = 2$. However, this setup can be easily extended to estimate operators on arbitrary tensor fields over manifolds of higher dimension. First, we introduce a finite element space. A natural choice is to consider,

$$\mathcal{W}_N = \left\{ W = \sum_{i=1}^N \sum_{\ell=1}^d W_\ell^{(i)} e_i \mathbf{t}_\ell^{(i)}, e_i: M \rightarrow \mathbb{R} \right\} \subset H^1(TM), \quad (18)$$

in analogous V_N in previous section. where the orthonormal basis $\{\mathbf{t}_\ell^{(i)}\}$ of local tangent space at x_i are defined in Section 2.1. Here, the coefficients are given as, $W_\ell^{(i)} = (\mathbf{t}_\ell^{(i)})^\top W(x_i)$.

To facilitate the analysis, we consider an intermediate step, where we approximate the left-and right hand-sides of (17) with symmetric bilinear forms, respectively,

$$a(W_i, W_j) = \frac{1}{2} (\langle DW_i, DW_j \rangle_U + \langle DW_j, DW_i \rangle_U), \quad (19)$$

$$b(W_i, W_j) = \frac{1}{2} (\langle W_i, W_j \rangle_{T^{(1,0)}M} + \langle W_j, W_i \rangle_{T^{(1,0)}M}), \quad (20)$$

for $W_i, W_j \in \mathcal{W}_N$, where each of the inner products in the right-hand-terms may not be symmetric when they are realized by a local parameterization, $\Upsilon_{i,r}$, that is also used in (15). Theoretically, for any $V, W \in H^1(TM)$ and the underlying Riemannian metric g that is unknown in our case, $a(W, V) = \langle DW, DV \rangle_U$ and $b(W, V) = \langle W, V \rangle_{T^{(1,0)}M}$.

Our goal is to approximate (17) by finding λ and $W \in \mathcal{W}_N$ that satisfy,

$$a(W, V) = \lambda b(W, V), \quad (21)$$

for all $V \in \mathcal{W}_N$, which is equivalent to solving the generalized eigenvalue problem,

$$\mathbf{A}\mathbf{W} = \lambda \mathbf{B}\mathbf{W}, \quad (22)$$

where λ denotes the eigenvalue corresponding to eigenvector field $\mathbf{W} \in \mathbb{R}^{Nd}$ with entries given by

$$\begin{pmatrix} (\mathbf{W})_{2i+1} \\ (\mathbf{W})_{2i+2} \end{pmatrix} = \begin{pmatrix} (\mathbf{t}_1^{(i)})^\top W(x_i) \\ (\mathbf{t}_2^{(i)})^\top W(x_i) \end{pmatrix} = \begin{pmatrix} W_1^{(i)} \\ W_2^{(i)} \end{pmatrix}, \quad (23)$$

and $\mathbf{A} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^\top)$ and $\mathbf{B} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^\top)$ are the symmetrization of the stiffness and mass matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. In this case, the mass and stiffness matrices are sparse, $dN \times dN$ matrices. It is convenient to define these matrices in terms of their $d \times d$ block submatrices. For $d = 2$, the mass matrix $\mathbf{M}$ has 2×2 block entries given by,

$$\begin{bmatrix} [\mathbf{M}]_{2i-1,2j-1} & [\mathbf{M}]_{2i-1,2j} \\ [\mathbf{M}]_{2i,2j-1} & [\mathbf{M}]_{2i,2j} \end{bmatrix} = \sum_{T_{i,r} \in R(i)} \int_T e_i e_j \begin{bmatrix} \langle \mathbf{t}_1^{(i)}, \mathbf{t}_1^{(j)} \rangle_{g_{Y_{i,r}}} & \langle \mathbf{t}_1^{(i)}, \mathbf{t}_2^{(j)} \rangle_{g_{Y_{i,r}}} \\ \langle \mathbf{t}_2^{(i)}, \mathbf{t}_1^{(j)} \rangle_{g_{Y_{i,r}}} & \langle \mathbf{t}_2^{(i)}, \mathbf{t}_2^{(j)} \rangle_{g_{Y_{i,r}}} \end{bmatrix} \sqrt{\det(g_{Y_{i,r}})} du_1 du_2, \quad (24)$$

where the integrals above are taken entrywise, and the inner-products are Riemannian inner-products between vector fields. The stiffness matrix $\mathbf{S}$ has entries,

$$\begin{bmatrix} [\mathbf{S}]_{2i-1,2j-1} & [\mathbf{S}]_{2i-1,2j} \\ [\mathbf{S}]_{2i,2j-1} & [\mathbf{S}]_{2i,2j} \end{bmatrix} = \sum_{T_{i,r} \in R(i)} \int_T \begin{bmatrix} \langle De_i \mathbf{t}_1^{(i)}, De_j \mathbf{t}_1^{(j)} \rangle_{g_{Y_{i,r}}} & \langle De_i \mathbf{t}_1^{(i)}, De_j \mathbf{t}_2^{(j)} \rangle_{g_{Y_{i,r}}} \\ \langle De_i \mathbf{t}_2^{(i)}, De_j \mathbf{t}_1^{(j)} \rangle_{g_{Y_{i,r}}} & \langle De_i \mathbf{t}_2^{(i)}, De_j \mathbf{t}_2^{(j)} \rangle_{g_{Y_{i,r}}} \end{bmatrix} \sqrt{\det(g_{Y_{i,r}})} du_1 du_2, \quad (25)$$

where the Riemannian inner product operation in each component will be defined according to the codomain of D . In the case of Bochner Laplacian, the Riemannian inner product is over $(2,0)$ tensor field, whereas for the Hodge Laplacian, it will be defined over $(2,0)$ tensor fields and functions for the first and second components of the co-domain, respectively.

We should again point out that the formulation above is only useful when the true local parameterization $Y_{i,r}$ is known. In our case, we consider using the local curved mesh parameterization $\tilde{\Phi}_{C_{i,r}}$ as defined in (9). We also note that we choose to employ such a parameterization over the flat local mesh parameterization $\tilde{\Phi}_{T_{i,r}}$ used in previous section since the latter introduces significant errors in the stiffness matrix calculation with zero Christoffel symbols.

With this parameterization, we effectively solve (21) on the following space,

$$\widehat{\mathcal{W}}_N = \left\{ W = \sum_{i=1}^N \sum_{\ell=1}^d W_\ell^{(i)} e_i \mathbf{t}_\ell^{(i)}, e_i : \tilde{\Phi}_{C_{i,r}}(T) \rightarrow \mathbb{R} \right\},$$

which differs from $\mathcal{W}_N$ on the parameterized domain of e_i . We will denote the resulting eigenvalue problem,

$$\widehat{\mathbf{A}}\mathbf{W} = \lambda \widehat{\mathbf{B}}\mathbf{W}, \quad (26)$$

where $\mathbf{W}$ is defined as in (23) except that $W \in \widehat{\mathcal{W}}_N$. Also, the symmetrized stiffness and mass matrices $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ are defined exactly as $\mathbf{A}$ and $\mathbf{B}$, respectively, where the local curved mesh parameterization $\tilde{\Phi}_{C_{i,r}}$ is used in placed of $Y_{i,r}$ in (24) and (25).

Detailed computations for the mass matrix for vector fields, as well as for the stiffness matrices for the Bochner and Hodge Laplacians in the case of the local curved mesh method are provided in the Appendices A.1, A.2, and A.3, respectively. Remaining details for computing the mass matrix are summarized in the following remark.

Remark 3. A subtlety in the above block formula is in computing $\langle \mathbf{t}_{k_1}^{(i)}, \mathbf{t}_{k_2}^{(j)} \rangle_{g_{\Phi_{C_{i,r}}}}$ since the local parameterizations of these pair of tangent vectors are different. Particularly, $\mathbf{t}_{k_1}^{(i)}$ is constructed at base point x_i , whereas the latter, $\mathbf{t}_{k_2}^{(j)}$ is constructed at base point x_j , we need to represent these two vectors with the same coordinates to have a well defined inner product. To accomplish this, we first represent $\mathbf{t}_{k_2}^{(j)}$ in terms of $\{\mathbf{t}_1^{(i)}, \dots, \mathbf{t}_n^{(i)}\}$:

$$\mathbf{t}_{k_2}^{(j)} = \sum_{k=1}^n b_k \mathbf{t}_k^{(i)}$$

Let $\vec{b} = (b_1, \dots, b_n)^\top$. Similarly, let $\vec{a}$ denote a vector of all zeros with a one in the k_1 -th position, which is the analogous representation for $\mathbf{t}_{k_1}^{(i)}$. We then regress the above vectors onto the natural basis for the tangent space corresponding to the parameterization, given by $\{\partial_{u_1} \Phi_{C_{i,r}}, \partial_{u_2} \Phi_{C_{i,r}}\}$ to obtain coefficient vector $\mathbf{a}_{k_1}^{ii}, \mathbf{a}_{k_2}^{ij}$ satisfying

$$\begin{aligned} \vec{a} &= (\mathbf{a}_{k_1}^{ii})_1 \partial_{u_1} \Phi_{C_{i,r}} + (\mathbf{a}_{k_1}^{ii})_2 \partial_{u_2} \Phi_{C_{i,r}} \\ \vec{b} &= (\mathbf{a}_{k_2}^{ij})_1 \partial_{u_1} \Phi_{C_{i,r}} + (\mathbf{a}_{k_2}^{ij})_2 \partial_{u_2} \Phi_{C_{i,r}}. \end{aligned} \quad (27)$$

The two linear problems above can be written as follows,

$$\begin{aligned} \mathbf{T}^\top \mathbf{t}_{k_1}^{(i)} &= \mathbf{R}_{i,r} \mathbf{a}_k^{ii} \\ \mathbf{T}^\top \mathbf{t}_{k_2}^{(j)} &= \mathbf{R}_{i,r} \mathbf{a}_k^{ij}, \end{aligned}$$

where $\mathbf{T} = [\mathbf{t}_1^{(i)}, \dots, \mathbf{t}_n^{(i)}] \in \mathbb{R}^{n \times n}$ and $\mathbf{R}_{i,r} = [\partial_{u_1}, \partial_{u_2}] \in \mathbb{R}^{n \times d}$ whose columns form a tangent space for the local curved mesh of triangle $T_{i,r}$, as defined in (11). The regression solutions of these two problems are given by,

$$\mathbf{a}_k^{i\ell} = (\mathbf{R}_{i,r}^\top \mathbf{R}_{i,r})^{-1} \mathbf{R}_{i,r}^\top \mathbf{T}^\top \mathbf{t}_k^{(\ell)},$$

for both $\ell = i$ and $\ell = j$.

Since $\mathbf{a}_k^{ii}$ and $\mathbf{a}_k^{ij}$ are the components of vectors $\mathbf{t}_{k_1}^{(i)}$ and $\mathbf{t}_{k_2}^{(i)}$, respectively, under the same coordinates $\{\partial_{u_1}, \partial_{u_2}\}$, then it is clear that

$$\langle \mathbf{t}_{k_1}^{(i)}, \mathbf{t}_{k_2}^{(j)} \rangle_{g_{\Phi_{C_{i,r}}}} = (\mathbf{a}_{k_1}^{ii})^\top g_{\Phi_{C_{i,r}}} \mathbf{a}_{k_2}^{ij}. \quad (28)$$

4 Convergence of Eigenvalues

In this section, we study the convergence of eigenvalues of a Bochner Laplacian discretization using the local curved mesh method, obtained from the generalized eigenvalue problem (26), to the true eigenvalues of the Bochner Laplacian (17).

In Section 4.1, we introduce some geometric preliminaries and define local charts which serve as the analytic object that is estimated by the local curved mesh method. We also state convergence results that follow from standard GMLS results adapted to our setup. In Section 4.2, we will discuss the spectral error due to variational approximation on $\mathcal{W}_N$. The main result in this section is the spectral consistency of (22) to (17) stated in Lemma 4.4. In Section 4.3, we quantify the spectral error between (22) and (26) induced by the approximation of the local chart, formalized in Lemma 4.7. In Section 4.4, we state the main theorem based on Lemmas 4.4 and 4.7 with some discussion on the spectral gap condition for convergence.

Throughout this section, we let $X = \{x_1, \dots, x_N\}$ denote a dataset of N points sampled uniformly from a manifold M . We remark that, for simplicity, we assume that the estimated tangent space $\overline{T_{x_i} M}$ is exact for each point i . In practice, high-order approximations of the tangent space can be obtained, and other error terms investigated in this analysis dominate. As in previous sections, we assume that the manifold is of dimension $d = 2$ embedded in $\mathbb{R}^3$ to simplify computations. However, generalizing the computations of this section to higher dimensional manifolds of higher codimension is straightforward.

4.1 Local metric approximation

Fix a point x_i and consider its k -nearest neighbors $K(i) = \{x_{i_1}, \dots, x_{i_k}\}$, as well as $T(i)$, which is $K(i)$ projected to $T_{x_i} M$. Let $r_i = \max_{\tilde{v}_j \in T(i)} \|\tilde{v}_j\|$. In the previous equation, and the remainder of this paper, $\|\cdot\|$ denotes the Euclidean norm (specifically, for vector in $\mathbb{R}^d$ in the above). We remark that k , the number of nearest neighbors, which depends on N , is chosen to scale sufficiently slowly so that $r_i \rightarrow 0$ as $N \rightarrow \infty$. To this end, we assume that N is sufficiently large so that $r_i \leq r_M$, where r_M denotes the injectivity radius of the manifold.

To facilitate the analysis below, we define the local fill and separation distances, respectively:

$$h_i := \sup_{\tilde{v} \in B_{\tilde{0}}(r_i)} \min_{\tilde{v}_j \in T(i)} \|\tilde{v} - \tilde{v}_j\|, \quad q_i := \frac{1}{2} \min_{\tilde{v}_j, \tilde{v}_k \in T(i), i \neq j} \|\tilde{v}_j - \tilde{v}_k\|.$$

Consider also the global fill and separation distances, respectively:

$$h_{X,M} := \sup_{x \in M} \min_{x_j \in X} \|x - x_j\|, \quad q_X := \frac{1}{2} \min_{x_i, x_j \in X, i \neq j} \|x_i - x_j\|.$$

We assume that the dataset X is quasi-uniform in the sense that

$$q_X \leq h_{X,M} \leq c_q q_X, \quad (29)$$

for some constant c_q . It is easy to see that the projected data is also quasi-uniform same sense (i.e., that $q_i \leq h_i \leq c_{q_i} q_i$, for some constant $c_{q_i} > 0$). Since the points $\tilde{v} \in B_{\tilde{0}}(r_i)$ correspond to projected points in M , one can verify that

$$h_i \leq h_{X,M} \quad \text{for all } i \in \{1, \dots, N\}. \quad (30)$$

Let $\gamma_{x_i} : B_{\tilde{0}}(r_i) \rightarrow \mathbb{R}^n$ denote a local parameterization of the form

$$\gamma_{x_i} : (v_1, v_2) \mapsto x_i + (v_1, v_2, z_{x_i}(v_1, v_2)), \quad (31)$$

for some smooth map z_{x_i} , which is obtained by projecting points nearby x_i to $T_{x_i}M$.

In this section, we quantify the error of estimating the coordinates induced by γ_{x_i} with the coordinates corresponding to α_{x_i} of (8). Such a result follows from general approximation error results from the GMLS, which have been derived in [22]. We state their results in our context below.

Lemma 4.1. *Let $X \subseteq M$ be such that $h_{X,M} \leq h_0$ for some constant $h_0 > 0$. Define $\Omega_i^* = \bigcup_{\tilde{v} \in B_{\tilde{0}}(r_i)} B_{\tilde{v}}(C_2 h_0)$ for some constant $C_2 > 0$. Then for any $f \in C^{m+1}(\Omega_i^*)$ and a multi index ℓ that satisfies $|\ell| \leq m$, we have*

$$|D^\ell f(\tilde{v}) - D^\ell \hat{f}(\tilde{v})| \leq C h_i^{m+1-|\ell|} |f|_{C^{m+1}(\Omega_i^*)} \leq C h_{X,M}^{m+1-|\ell|} |f|_{C^{m+1}(\Omega_i^*)}$$

for any $\tilde{v} \in B_{\tilde{0}}(r_i)$, where $\hat{f}$ is a GMLS approximation to f using polynomial of up to degree- m . Here, D^ℓ is a shorthand for multivariate derivative of order- ℓ .

In our case, we estimate the $d = 2$ dimensional smooth function z_{x_i} in (31) with a polynomial p_i as in (6) of degree $m = 2$ such that,

$$|D^\ell z_{x_i}(\tilde{v}) - D^\ell p_i(\tilde{v})| = O(h_{X,M}^{3-|\ell|}) \quad (32)$$

for any multi-index ℓ with $|\ell| \leq 2$. We remark that since $z_{x_i} : B_{\tilde{0}}(r_i) \rightarrow \mathbb{R}^n$ is smooth, it can certainly be extended to a smooth function on Ω_i^* .

Using this result, we can obtain bounds on estimating the necessary local quantities. The proof is simple, and can be found in Appendix B.

Lemma 4.2. *Let $M \subset \mathbb{R}^n$ be a 2-dimensional C^3 manifold. Denote by $\{\partial_{v_1}(\gamma_{x_i}), \partial_{v_2}(\gamma_{x_i})\}$ and $\{\partial_{v_1}(\alpha_{x_i}), \partial_{v_2}(\alpha_{x_i})\}$ the coordinates induced from γ_{x_i} defined in (31) and α_{x_i} defined in (8), respectively. Denote the Riemannian metric associated with each set of coordinates by $g_{\gamma_{x_i}}$ and $g_{\alpha_{x_i}}$, and the components of their inverse as $g_{\gamma_{x_i}}^{qs}$ and $g_{\alpha_{x_i}}^{qs}$. Let $\Gamma_{st}^q(\gamma_{x_i}), \Gamma_{st}^q(\alpha_{x_i})$ denote the Christoffel symbols associated with $g_{\gamma_{x_i}}, g_{\alpha_{x_i}}$, respectively. Fix any triangle $T_{i,r} \in R(i)$. Then for any $\tilde{v} \in T_{i,r}$,*

$$\begin{aligned} \|\partial_{v_k}(\gamma_{x_i})(\tilde{v}) - \partial_{v_k}(\alpha_{x_i})(\tilde{v})\| &= O(h_{X,M}^2), \quad k = 1, 2 \\ \left| \sqrt{\det(g_{\gamma_{x_i}})(\tilde{v})} - \sqrt{\det(g_{\alpha_{x_i}})(\tilde{v})} \right| &= O(h_{X,M}) \\ \left| \left(g_{\gamma_{x_i}}^{qs}(\tilde{v}) - g_{\alpha_{x_i}}^{qs}(\tilde{v}) \right) \right| &= O(h_{X,M}) \\ \left| \Gamma_{st}^q(\gamma_{x_i})(\tilde{v}) - \Gamma_{st}^q(\alpha_{x_i})(\tilde{v}) \right| &= O(h_{X,M}). \end{aligned}$$

where the constant term in the big-oh notation for the first three lines above depend on the C^2 norm of α_{x_i} , while the constant associated with the Christoffel symbol result depends on the C^3 norm of α_{x_i} .

4.2 Variational approximation

The results of the previous subsection quantify the error between estimating the metric and its derivatives. Here, we quantify the error coming from the estimation of a vector-field W on the space $\mathcal{W}_N$ defined in (18).

To this end, consider a vector field $W : M \rightarrow TM$. While the computation is done via the representation of the vector field under the basis $\{\partial_{u_1} \Phi_{C_{i,r}}, \partial_{u_2} \Phi_{C_{i,r}}\}$ evaluated at $(\Phi_{i,r})^{-1}(\tilde{v}_{i,r})$ as discussed in Remark 3, it is theoretically more convenient to work in the tangent space $T_{x_i}M$, leveraging the error bounds induced by the estimated coordinate system $\{\partial_{v_1}(\alpha_{x_i}), \partial_{v_2}(\alpha_{x_i})\}$ as deduced in Lemma 4.2. Analytically, due to the diffeomorphism outlined in Section 4.1, any vector field $W : M \rightarrow TM$ can be written in the local coordinates as

$$W(v_1, v_2) = c_1^{(i,r)}(v_1, v_2) \partial_{v_1}(\gamma_{x_i}) + c_2^{(i,r)}(v_1, v_2) \partial_{v_2}(\gamma_{x_i}). \quad (33)$$

Definition 4.1. *Define the local interpolation $I_N : H^1(TM) \rightarrow \mathcal{W}_N$ where $\mathcal{W}_N$ is defined in (18),*

$$I_N W = I_N c_1^{(i,r)}(\gamma_{x_i}) + I_N c_2^{(i,r)}(v_1, v_2)(\gamma_{x_i}), \quad (34)$$

Here, we abused the notation I_N to also acts on any function f ,

$$I_N f = \sum_i f(x_i) e_i,$$

where $\{e_i\}$ is the nodal basis.

With this definition, we can show that.

Lemma 4.3. *For any $W \in H^2(TM)$, we have,*

$$|I_N W - W|_{H^k(TM)} \leq c h_{X,M}^{2-k} |W|_{H^2(TM)},$$

for $k = 0, 1$ and some constant $c > 0$ that can depend on $q_{X,M}$ and the regularity of g_{γ_i} .

Proof. First, let us derive a bound for the norm of $D(W - I_N W)$ represented in ambient coordinates. Particularly,

$$\begin{aligned} \|D(W - I_N W)\|^2 &= \left\| \sum_{j=1}^n (D(W - I_N W) \cdot \mathbf{e}_j) \mathbf{e}_j \right\|^2 \\ &= \sum_{j=1}^n \left(\sum_{k=1}^2 D(c_k^{(i,r)} - I_N c_k^{(i,r)}) \partial_{v_k}(\gamma_{x_i}) \cdot \mathbf{e}_j + (c_k^{(i,r)} - I_N c_k^{(i,r)}) D \partial_{v_k}(\gamma_{x_i}) \cdot \mathbf{e}_j \right)^2 \\ &\leq 4 \sum_{j=1}^n \sum_{k=1}^2 \left(D(c_k^{(i,r)} - I_N c_k^{(i,r)}) \partial_{v_k}(\gamma_{x_i}) \cdot \mathbf{e}_j \right)^2 + \left(c_k^{(i,r)} - I_N c_k^{(i,r)} \right)^2 \left(D \partial_{v_k}(\gamma_{x_i}) \cdot \mathbf{e}_j \right)^2 \\ &\leq 4nC \sum_{k=1}^2 \left(D(c_k^{(i,r)} - I_N c_k^{(i,r)}) \right)^2 + \left(c_k^{(i,r)} - I_N c_k^{(i,r)} \right)^2, \end{aligned} \quad (35)$$

where we used Young's inequality to bound the mixed terms and set $C > 0$ as an upper bound for the basis expansion.

Here, we deduce the case of $k = 1$ since the other case follows similar argument.

$$\begin{aligned} |I_N W - W|_{H^1(TM)}^2 &= \int_M \langle D(I_N W - W), D(I_N W - W) \rangle_g d\text{Vol} \\ &= \sum_{T_{i,r} \in R(i)} \int_{T_{i,r}} \langle D(I_N W - W), D(I_N W - W) \rangle_{g_{\gamma_{i,r}}} \sqrt{\det(g_{\gamma_{i,r}})} dv_1 dv_2 \\ &\leq g_{\max} \sum_{T_{i,r} \in R(i)} \int_{T_{i,r}} \langle D(I_N W - W), D(I_N W - W) \rangle_{g_{\gamma_{i,r}}} dv_1 dv_2 \\ &\leq 4nC g_{\max} \sum_{T_{i,r} \in R(i)} \sum_{k=1}^2 \int_{T_{i,r}} \left(D(c_k^{(i,r)} - I_N c_k^{(i,r)}) \right)^2 + \left(c_k^{(i,r)} - I_N c_k^{(i,r)} \right)^2 dv_1 dv_2, \end{aligned} \quad (36)$$

where we have used the local parameterization defined in (31) in the second equality, the regularity of the metric tensor $(g_{\gamma_i})_{k\ell}$ in the third line above, that is, $0 < g_{\min} \delta_{k\ell} \leq (g_{\gamma_i})_{k\ell} \leq \delta_{k\ell} g_{\max} < \infty$.

Consider the following standard finite element result. For any polynomial of degree $m - 1$ interpolant $I_N f$ of a function $f \in W^{m,p}(T_{i,r})$, we have the following local interpolation error bound (see e.g. Theorem 4.4.4, in [6]):

$$|I_N f - f|_{W^{\ell,p}(T_{i,r})} \leq c h_i^{m-\ell} q_i^\ell |f|_{W^{m,p}(T_{i,r})}$$

$1 \leq p \leq \infty$, $0 \leq \ell \leq m$. It follows that, for $m = 2$, $p = 2$, $\ell = 1$, we have

$$|I_N f - f|_{H^1(T_{i,r})} \leq c h_i q_i |f|_{H^2(T_{i,r})},$$

Inserting this bound to (36), we obtain,

$$\begin{aligned} |I_N W - W|_{H^1(TM)} &\leq C \sqrt{n g_{\max}} h_{X,M} \left(\sum_{T_{i,r} \in R(i)} \sum_{k=1}^2 \int_{T_{i,r}} (D^2 f)^2 dv_1 dv_2 \right)^{1/2} \\ &\leq C \sqrt{n g_{\max}} h_{X,M} \left(\sum_{T_{i,r} \in R(i)} \sum_{j=1}^n \sum_{k=1}^2 \int_{T_{i,r}} (D^2 f \partial_{v_k}(\gamma_{x_i}) \cdot \mathbf{e}_j)^2 dv_1 dv_2 \right)^{1/2} \\ &\leq C \frac{\sqrt{g_{\max}}}{\sqrt{g_{\min}}} h_{X,M} \sum_{T_{i,r} \in R(i)} \int_{T_{i,r}} \langle D^2 W, D^2 W \rangle_{g_{\gamma_{i,r}}} \sqrt{\det(g_{\gamma_{i,r}})} dv_1 dv_2 \\ &\leq C h_{X,M}^2 |W|_{H^2(TM)}, \end{aligned} \quad (37)$$

where we used the regularity of metric tensor g_{γ_i} and abuse the constant notation $C > 0$ that are different after each inequality. $\square$

For technical analysis below, we define the Ritz projection.

Definition 4.2 (Ritz projection). *Let $R_N : H^1(TM) \rightarrow \mathcal{W}_N$ be the Ritz projection, which is the orthogonal projection in the following sense,*

$$a(W - R_N W, V_N) = 0, \quad \forall V_N \in \mathcal{W}_N. \quad (38)$$

Following the standard argument in a priori analysis for FEM solution to finding $W \in H^1(TM)$ such that $a(W, V) = b(V, V)$ for all $V \in H^1(TM)$, with $R_N W$ be the solution in $\mathcal{W}_N$, one can deduce the well-known a priori error bound,

$$\|W - R_N W\|_{H^1(TM)} \leq Ch_{X,M} |W|_{H^2(TM)}. \quad (39)$$

For the L^2 -norm error bound, consider the following auxiliary problem,

$$a(E, \Phi) = b(E, E) = \|E\|_{L^2(TM)},$$

where $E = W - R_N W$ and $\Phi \in H^1(TM)$. Note that,

$$\|E\|_{L^2(TM)}^2 = a(E, \Phi - I_N \Phi) \leq M \|E\|_{H^1(TM)} \|\Phi - I_N \Phi\|_{H^1(TM)} \leq Ch_{X,M}^2 |W|_{H^2(TM)} \|\Phi\|_{H^2(TM)},$$

where we have used the orthogonality condition in (38) in the first equality, continuity of a in the second equality, and finally, the interpolation error in Lemma 4.3 and the a priori error bound (39). By the standard regularity result for the auxiliary problem above, $\|\Phi\|_{H^2(TM)} \leq C \|E\|_{L^2(TM)}$, we conclude the L^2 -error bound for the Ritz projection,

$$\|W - R_N W\|_{L^2(TM)} \leq Ch_{X,M}^2 |W|_{H^2(TM)}. \quad (40)$$

With this preliminary, we can deduce the following bound for the eigenvalues.

Lemma 4.4. *Let λ_ℓ and $\lambda_{\ell,N}$ be the ℓ th eigenvalues of (17) and (21), respectively. Then, there exists $C > 0$ such that*

$$\lambda_\ell \leq \lambda_{\ell,N} \leq \lambda_\ell + Ch_{X,M}^2.$$

Proof. The first equality inequality above is clear since

$$\lambda_\ell = \min_{S_\ell \subset H^1(TM)} \max_{W \in S_\ell} \frac{a(W, W)}{b(W, W)}, \quad \lambda_{\ell,N} = \min_{S_\ell \subset \mathcal{W}_N} \max_{W \in S_\ell} \frac{a(W, W)}{b(W, W)},$$

where $\dim(S_\ell) = \ell$, and $\mathcal{W}_N \subset H^1(TM)$.

To prove the second inequality, let E_ℓ be the space spanned by the leading eigenvector fields, $\Phi_1, \dots, \Phi_\ell$. Denote $E_{\ell,N} = R_N E_\ell$, where R_N denotes the Ritz projection as defined above. Then,

$$\lambda_{\ell,N} \leq \max_{W \in E_{\ell,N}} \frac{a(W, W)}{b(W, W)} = \max_{W \in E_\ell} \frac{a(R_N W, R_N W)}{b(R_N W, R_N W)} \leq \max_{W \in E_\ell} \frac{a(W, W)}{\|R_N W\|_{L^2(TM)}^2}, \quad (41)$$

where in the last inequality, we use the fact that $a(R_N W, R_N W) + a(R_N W - W, R_N W - W) = a(W, W)$. For $W \in E_\ell$, (40) becomes,

$$\|W - R_N W\|_{L^2(TM)} \leq Ch_{X,M}^2 |W|_{H^2(TM)} \leq Ch_{X,M}^2 |D^2 W|_{L^2(TM)} \leq C \lambda_\ell \|W\|_{L^2(TM)},$$

and it is clear that,

$$\|R_N W\|_{L^2(TM)} \geq \|W\|_{L^2(TM)} - \|W - R_N W\|_{L^2(TM)} \geq \|W\|_{L^2(TM)} (1 - Ch_{X,M}^2).$$

Inserting this to (41), we obtain,

$$\lambda_{\ell,N} \leq \max_{W \in E_\ell} \frac{a(W, W)}{\|W\|_{L^2(TM)}^2} (1 + Ch_{X,M}^2) = \lambda_\ell + Ch_{X,M}^2,$$

and the proof is completed. $\square$

4.3 Local vector field approximation

In this section, we will derive a bound between eigenvalues between the generalized eigenvalue problems in (22) and (26). To achieve this, we first derive the absolute error between components of $\mathbf{A}$ and $\hat{\mathbf{A}}$ and components of $\mathbf{B}$ and $\hat{\mathbf{B}}$, which amount to bound errors between $\mathbf{S}$ and $\hat{\mathbf{S}}$ and $\mathbf{M}$ and $\hat{\mathbf{M}}$.

Technically, we will need the bound the error between estimators $W_N \in \mathcal{W}_N$ and $\widehat{W}_N \in \widehat{\mathcal{W}}_N$.

For convenience, we represent W_N and $\widehat{W}_N$ on $T_{i,r}$ as follows,

$$W_N(v_1, v_2) = b_1^{(i,r)}(v_1, v_2)\partial_{v_1}(\gamma_{x_i}) + b_2^{(i,r)}(v_1, v_2)\partial_{v_2}(\gamma_{x_i}) \quad (42)$$

$$\widehat{W}_N(v_1, v_2) = \widehat{b}_1^{(i,r)}(v_1, v_2)\partial_{v_1}(\alpha_{x_i}) + \widehat{b}_2^{(i,r)}(v_1, v_2)\partial_{v_2}(\alpha_{x_i}) \quad (43)$$

where the estimated coefficients are functions of $T_{i,r}$. Here, we recall that γ_{x_i} and α_{x_i} denote the true local parameterization in (31) and the curved mesh parameterization in (8), respectively.

Lemma 4.5. *Let M be a two-dimensional C^3 -manifold. For a given $x_i \in X$, consider any triangle $T_{i,r} \in R(i)$, and fix any $(v_1, v_2) \in T_{i,r}$. Denote by $W_N(v_1, v_2) \in \mathcal{W}_N$ in (42) as an approximation to the solution W of (21) in the analytic charts induced by γ_{x_i} . Let $\widehat{W}(v_1, v_2)$ denote the estimation defined in (43), where the parameterizations $\{\alpha_{x_i}\}_{i=1}^N$ are estimated using polynomials of degree-2 as in (6). Then*

$$\|D^\ell W_N(v_1, v_2) - D^\ell \widehat{W}_N(v_1, v_2)\| = O(h_{X,M}), \quad (44)$$

where $h_{X,M}$ is the global fill distance defined as before and ℓ is an index of order $|\ell| \leq 1$, and the constant depends on the local separation distance q_i .

Proof. Fix $(v_1, v_2) \in T_{i,r}$. We introduce an intermediate estimator:

$$\widetilde{W}_N(v_1, v_2) = \widehat{b}_1^{(i,r)}(v_1, v_2)\partial_{v_1}(\gamma_{x_i})(v_1, v_2) + \widehat{b}_2^{(i,r)}(v_1, v_2)\partial_{v_2}(\gamma_{x_i})(v_1, v_2),$$

where $\widehat{c}_k^{(i,r)}$ is an interpolation of $c_k^{(i,r)}$ using nodal basis functions and the values on the vertices of $T_{i,r}$. We split the error as follows:

$$\|D^\ell W_N(v_1, v_2) - D^\ell \widehat{W}_N(v_1, v_2)\| \leq \|D^\ell W_N(v_1, v_2) - D^\ell \widetilde{W}_N(v_1, v_2)\| + \|D^\ell \widetilde{W}_N(v_1, v_2) - D^\ell \widehat{W}_N(v_1, v_2)\|$$

The error of the first term can be analyzed in the basis of the ambient space:

$$\begin{aligned} \|D^\ell W_N(v_1, v_2) - D^\ell \widetilde{W}_N(v_1, v_2)\| &= \left\| \sum_{j=1}^n \left((D^\ell W_N(v_1, v_2) - D^\ell \widetilde{W}_N(v_1, v_2)) \cdot \mathbf{e}_j \right) \mathbf{e}_j \right\| \\ &= \sqrt{\sum_{j=1}^n \left(\sum_{k=1}^d D^\ell (b_k^{(i,r)} - \widehat{b}_k^{(i,r)}) \partial_{v_k}(\gamma_{x_i}) \cdot \mathbf{e}_j + (b_k^{(i,r)} - \widehat{b}_k^{(i,r)}) D^\ell \partial_{v_k}(\gamma_{x_i}) \cdot \mathbf{e}_j \right)^2}. \end{aligned}$$

For simplicity of notation, we have suppressed the evaluation on (v_1, v_2) above, as we do in what follows. We note that, for quasi-uniform data, we have $h_i/q_i \leq c_{q_i}$.

Clearly, it suffices to bound $|D^\ell (b_k^{(i,r)} - \widehat{b}_k^{(i,r)})|$. Recall that $b_k^{(i,r)}$ is interpolated by hat functions based on the values $c_k^{(i,r)}(x_i), c_k^{(i,r)}(x_{i_{jr}}), c_k^{(i,r)}(x_{i_{kr}})$ written in the basis $\{\partial_{v_1}(\gamma_{x_i}), \partial_{v_2}(\gamma_{x_i})\}$ (see (33)). Similarly for $\widehat{b}_k^{(i,r)}$, but in the basis $\{\partial_{v_1}(\alpha_{x_i}), \partial_{v_2}(\alpha_{x_i})\}$.

Let $\vec{v}_{i_{jr}}$ denote an arbitrary vertex in $T_{i,r}$. Define the following:

$$\begin{aligned} \mathbf{D} &= \begin{bmatrix} \partial_{v_1}(\gamma_{x_i})(\vec{v}_{i_{jr}}), \partial_{v_2}(\gamma_{x_i})(\vec{v}_{i_{jr}}) \end{bmatrix}, \quad \widehat{\mathbf{D}} = \begin{bmatrix} \partial_{v_1}(\alpha_{x_i})(\vec{v}_{i_{jr}}), \partial_{v_2}(\alpha_{x_i})(\vec{v}_{i_{jr}}) \end{bmatrix}, \\ \mathbf{b} &= \begin{pmatrix} b_1^{(i,r)}(\vec{v}_{i_{jr}}), b_2^{(i,r)}(\vec{v}_{i_{jr}}) \end{pmatrix}^\top, \quad \widehat{\mathbf{b}} = \begin{pmatrix} \widehat{b}_1^{(i,r)}(\vec{v}_{i_{jr}}), \widehat{b}_2^{(i,r)}(\vec{v}_{i_{jr}}) \end{pmatrix}^\top. \end{aligned}$$

We remark that $W(\vec{v}_{i_{jr}}) = \mathbf{D}\mathbf{b}$, and $W(\vec{v}_{i_{jr}}) = \widehat{\mathbf{D}}\widehat{\mathbf{b}} + \mathbf{r}$, where the first regression problem has no residual as $\widehat{\mathbf{b}} = \mathbf{c} = (c_1^{(i,r)}(\vec{v}_{i_{jr}}), c_2^{(i,r)}(\vec{v}_{i_{jr}}))^\top$ on the vertex of the $T_{i,r}$, whereas the second regression problem may have a residual due to the approximation of the basis representation. Since $\|\partial_{v_k}(\gamma_{x_i}) - \partial_{v_k}(\alpha_{x_i})\| = O(h_i^2)$ (by Lemma 4.2), it follows that $\|\mathbf{D} - \widehat{\mathbf{D}}\|_2 \leq \|\mathbf{D} - \widehat{\mathbf{D}}\|_F = O(h_i^2)$, where $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the spectral and Frobenius matrix norms, respectively.

Since the condition number of $\kappa_2(\mathbf{C}) = O(1)$, for any small perturbation $h_i^2 \ll 1$, it follows from standard least squares perturbation results (e.g., see Theorem 3.4 of [11]) that $\|\mathbf{b} - \widehat{\mathbf{b}}\| \leq h_i^2 2\kappa_2(\mathbf{D})\|\mathbf{b}\|$. Hence,

$$\left| b_k^{(i,r)}(\tilde{v}_{i_{jr}}) - \widehat{b}_k^{(i,r)}(\tilde{v}_{i_{jr}}) \right| = O(h_i^2).$$

Since $\tilde{v}_{i_{jr}}$ is an arbitrary vertex of $T_{i,r}$, it follows immediately from definition of hat functions that

$$\left| D \left(b_k^{(i,r)}(v_1, v_2) - \widehat{b}_k^{(i,r)}(v_1, v_2) \right) \right| = O(h_i), \quad \forall (v_1, v_2) \in T_{i,r},$$

where the above constant in the big-oh notation depends on the condition number of $\mathbf{D}$.

Hence,

$$\left\| D^\ell W_N(v_1, v_2) - D^\ell \widetilde{W}_N(v_1, v_2) \right\| = O(h_i).$$

For the term $\left\| D^\ell \widetilde{W}_N(v_1, v_2) - D^\ell \widehat{W}_N(v_1, v_2) \right\|$, we again analyze the error in the basis of the ambient space:

$$\begin{aligned} \left\| D^\ell \widetilde{W}_N(v_1, v_2) - D^\ell \widehat{W}_N(v_1, v_2) \right\| &= \left\| \sum_{j=1}^n \left((D^\ell \widetilde{W}_N(v_1, v_2) - D^\ell \widehat{W}_N(v_1, v_2)) \cdot \mathbf{e}_j \right) \mathbf{e}_j \right\| \\ &= \sqrt{\sum_{j=1}^n \left(\sum_{k=1}^d (D^\ell \widehat{b}^{(i,r)})(\partial_{v_k}(\gamma_{x_i}) - \partial_{v_k}(\alpha_{x_i})) \cdot \mathbf{e}_j + \widehat{b}^{(i,r)} D^\ell (\partial_{v_k}(\gamma_{x_i}) - \partial_{v_k}(\alpha_{x_i})) \cdot \mathbf{e}_j \right)^2}. \end{aligned}$$

We proved in Lemma 4.2 that $\left\| D^\ell (\partial_{v_k}(\gamma_{x_i}) - \partial_{v_k}(\alpha_{x_i})) \right\| = O(h_i^{2-\ell})$ for $\ell = 0, 1$.

Hence,

$$\left\| D^\ell \widetilde{W}_N(v_1, v_2) - D^\ell \widehat{W}_N(v_1, v_2) \right\| = O(h_i),$$

where the constant in the big-oh notation depend of C^3 -norm of α_{x_i} . Finally, we remark that each above bound in h_i has a global bound in terms of $h_{X,M}$. This completes the proof. $\square$

The following local integration convergence result follows immediately from Lemmas 4.2 and 4.5. Its proof can be found in Appendix B

Lemma 4.6. Denote by $V_N, W_N \in \mathcal{W}_N$ and $\widehat{V}_N, \widehat{W}_N \in \widehat{\mathcal{W}}_N$ defined as in the Lemma 4.5. Let $DW_N = (\nabla W_N)_{\gamma_{x_i}}$ and $D\widehat{W}_N = (\nabla \widehat{W}_N)_{\alpha_{x_i}}$ be represented in coordinates γ_{x_i} and α_{x_i} , respectively. Then

$$\left| \int_{T_{i,r}} \left(\left\langle (\nabla U_N)_{\gamma_{x_i}}, (\nabla W_N)_{\gamma_{x_i}} \right\rangle_{g_{\gamma_{x_i}}} \sqrt{\det(g_{\gamma_{x_i}})} - \left\langle (\nabla \widehat{U}_N)_{\alpha_{x_i}}, (\nabla \widehat{W}_N)_{\alpha_{x_i}} \right\rangle_{g_{\alpha_{x_i}}} \sqrt{\det(g_{\alpha_{x_i}})} \right) dv_1 dv_2 \right| = O(h_{X,M}).$$

and

$$\left| \int_{T_{i,r}} \left(\left\langle (U_N)_{\gamma_{x_i}}, (W_N)_{\gamma_{x_i}} \right\rangle_{g_{\gamma_{x_i}}} \sqrt{\det(g_{\gamma_{x_i}})} - \left\langle (\widehat{U}_N)_{\alpha_{x_i}}, (\widehat{W}_N)_{\alpha_{x_i}} \right\rangle_{g_{\alpha_{x_i}}} \sqrt{\det(g_{\alpha_{x_i}})} \right) dv_1 dv_2 \right| = O(h_{X,M}).$$

To finalize the result, we review the following perturbation theory for generalized eigenvalue problem (Theorem 8.7.3 in [14] that was originally derived in [27].

Proposition 4.1. Suppose $\mathbf{A} - \lambda \mathbf{B} \in \mathbb{R}^{2N \times 2N}$ is symmetric-definite pencil with eigenvalues $\lambda_{1,N} \geq \lambda_{2,N} \geq \dots \geq \lambda_{2N,N}$. Let $\widehat{\mathbf{A}} - \lambda \widehat{\mathbf{B}}$ be symmetric definite pencil with eigenvalues $\widehat{\lambda}_{1,N} \geq \widehat{\lambda}_{2,N} \geq \dots \geq \widehat{\lambda}_{2N,N}$. Suppose that $\widehat{\mathbf{A}} = \mathbf{A} + \mathbf{E}_A$ and $\widehat{\mathbf{B}} = \mathbf{B} + \mathbf{E}_B$ satisfy,

$$\epsilon^2 := \|\mathbf{E}_A\|_2^2 + \|\mathbf{E}_B\|_2^2 < c(\mathbf{A}, \mathbf{B}),$$

where

$$c(\mathbf{A}, \mathbf{B}) := \min_{\|\mathbf{x}\|_2=1} (\mathbf{x}^\top \mathbf{A} \mathbf{x})^2 + (\mathbf{x}^\top \mathbf{B} \mathbf{x})^2 > 0.$$

is known as the Crawford number of the pencil $\mathbf{A} - \lambda \mathbf{B}$. Then,

$$\frac{|\lambda_{\ell,N} - \widehat{\lambda}_{\ell,N}|}{|1 + \lambda_{\ell,N} \widehat{\lambda}_{\ell,N}|} \leq |\tan(\tan^{-1}(\lambda_{\ell,N}) - \tan^{-1}(\widehat{\lambda}_{\ell,N}))| \leq \frac{\epsilon}{c(\mathbf{A}, \mathbf{B})}.$$

The condition $\epsilon^2 < c(\mathbf{A}, \mathbf{B})$ ensures the perturbation size does not exceed the gap that introduces indefinite matrix. Since $\mathbf{A}$ and $\mathbf{B}$ are symmetric positive definite, they are simultaneously diagonalized (see Corollary 8.7.2 in [14]), and one can write $\lambda_{\ell,N} = \frac{a_\ell}{b_\ell}$, where a_ℓ and b_ℓ are eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, respectively. Hence, the spectral gap condition simply means,

$$\epsilon^2 < \min_{\ell=1,\dots,2N} a_\ell^2 + b_\ell^2.$$

We can now finalize this section with the following result.

Lemma 4.7. *Suppose $\min_{\ell=1,\dots,N} a_\ell^2 + b_\ell^2 > C^2 h_{X,M}^2$, where the constant C corresponds to the constant in the big-oh notations in Lemma 4.6. Then,*

$$|\lambda_{\ell,N} - \hat{\lambda}_{\ell,N}| \leq \frac{Ch_{X,M}}{c(\mathbf{A}, \mathbf{B})} |1 + \lambda_{\ell,N} \hat{\lambda}_{\ell,N}|.$$

Proof. Recall that we can reparameterize the integrands in Lemma 4.6 with $\Phi_{i,r} : T \rightarrow T_{i,r}$ such that,

$$\begin{aligned} \int_{T_{i,r}} \left\langle (\nabla \hat{U}_N)_{\gamma_{x_i}}, (\nabla \hat{W}_N)_{\gamma_{x_i}} \right\rangle_{g_{\gamma_{x_i}}} \sqrt{\det(g_{\gamma_{x_i}})} dv_1 dv_2 &= \int_T \left\langle (\nabla \hat{U}_N)_{\gamma_{i,r}}, (\nabla \hat{W}_N)_{\gamma_{i,r}} \right\rangle_{g_{\gamma_{i,r}}} \sqrt{\det(g_{\gamma_{i,r}})} du_1 du_2. \\ \int_{T_{i,r}} \left\langle (\nabla \hat{U}_N)_{\alpha_{x_i}}, (\nabla \hat{W}_N)_{\alpha_{x_i}} \right\rangle_{g_{\alpha_{x_i}}} \sqrt{\det(g_{\alpha_{x_i}})} dv_1 dv_2 &= \int_T \left\langle (\nabla \hat{U}_N)_{\Phi_{C_{i,r}}}, (\nabla \hat{W}_N)_{\Phi_{C_{i,r}}} \right\rangle_{g_{\Phi_{C_{i,r}}}} \sqrt{\det(g_{\Phi_{C_{i,r}}})} du_1 du_2. \end{aligned}$$

Letting $\hat{U}_N = e_i \mathbf{t}_k^{(i)}$ and $\hat{W}_N = e_j \mathbf{t}_\ell^{(j)}$ for any $i, j = \{1, \dots, N\}$ and $k, \ell \in \{1, 2\}$. From the definition of the stiffness matrix in (25), it is clear that $\|\mathbf{S} - \hat{\mathbf{S}}\|_2 \leq \|\mathbf{S} - \hat{\mathbf{S}}\|_F = O(h_{X,M})$. This implies that, $\|\mathbf{E}_A\|_2 = \|\hat{\mathbf{A}} - \mathbf{A}\|_2 = O(h_{X,M})$. Using similar argument, we also have, $\|\mathbf{E}_B\|_2 = \|\hat{\mathbf{B}} - \mathbf{B}\|_2 = O(h_{X,M})$. By the assumption and Proposition 4.1, the proof is complete. $\square$

4.4 Main result

Collecting the assumptions and results from Lemma 4.4 and (4.7), we obtain.

Theorem 4.1. *Let $M \subset \mathbb{R}^n$ on a d -dimensional C^3 manifold. Let λ_ℓ and $\hat{\lambda}_{\ell,N}$ be the ℓ th eigenvalues of (17) and (26), respectively. Denote eigenvalues of $\mathbf{A}$ as $a_1 \geq a_2 \geq \dots \geq a_{2N} > 0$ and eigenvalues of $\mathbf{B}$ as $b_1 \geq b_2 \geq \dots \geq b_{2N} > 0$, where these matrices are defined in (22). Suppose that the following spectral gap condition holds,*

$$\min_{\ell=1,\dots,2N} a_\ell^2 + b_\ell^2 > C^2 h_{X,M}^2,$$

where $h_{X,M}$ is the fill distance of a quasi-uniform point cloud data, $X = \{x_1, \dots, x_N\} \subset M$ as defined in Section 4.1. Then,

$$|\lambda_\ell - \hat{\lambda}_{\ell,N}| \leq Ch_{X,M},$$

where the constant $C = \sqrt{C_1^2 + C_2^2} > 0$, where $\|\mathbf{A} - \hat{\mathbf{A}}\|_2 \leq C_1 h_{X,M}$ and $\|\mathbf{B} - \hat{\mathbf{B}}\|_2 \leq C_2 h_{X,M}$.

In the result above, one can interpret the result in Lemma 4.4 as bias error due to restricting the solution of the hypothesis space $\mathcal{W}_N$. On the other hand, the result in Lemma 4.7 corresponds to manifold learning error. The spectral gap condition is described in terms of eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ corresponds to the generalized eigenvalue problem,

$$\mathbf{A}\mathbf{W} = \lambda \mathbf{B}\mathbf{W}, \tag{45}$$

restricting the variational eigenvalue problem on the hypothesis space $\mathcal{W}_N$.

We first note that $h_{X,M} \rightarrow 0$ as $N \rightarrow \infty$. For example, if the data are independent and identically uniformly distributed, one can show that $h_{X,M} \leq C \left(\frac{\log N}{N} \right)^{1/d}$ (see Lemma B.2 in [15]). In such a context, the gap condition always be can be satisfied for sufficiently large N . Finally, we note that while the result above does not allow one to mathematically describe the gap condition in terms of eigenvalues of the underlying variational problem (17) since (45) is only a finite-dimensional projection, the bias error converges at a faster rate $h_{X,M}^2$ compared to the manifold learning error of order $h_{X,M}$, which suggests that the gap condition is representing how much manifold learning error is allowed relative to the bias error, even when the Crawford number is undefined as $N \rightarrow \infty$.

Finally, we note that the convergence of eigenvector field can be achieved with the same proof technique used our previous work [15, 24]. Based on the previous results, we believe that the error of the estimated eigenvector field measured in term of the L2 norm over the point cloud data X is bounded under the same error rate, $h_{X,M}$. In the next section, we will numerically verify this conjecture on several examples.

5 Numerical Results

In this section, we numerically implement the local curved mesh method estimation of the Bochner and Hodge Laplacians on some example manifolds. In this study, we numerically demonstrate the convergence of eigenvalues and eigenvectors of the generalized eigenvalue problem in (22) obtained from randomly sampled data to the analytic eigenvalues and eigenvector fields of the Bochner and Hodge Laplacians on the 2D sphere and the latter on a 2D torus. In Section 5.1, we demonstrate the convergence of eigenvalues and eigenvector fields for both the Hodge and Bochner Laplacian on the sphere, since the analytic eigenvalues and eigenvector-fields are known. Since analytic eigensolutions for the Bochner and Hodge Laplacian on a torus are unknown, we demonstrate the convergence of eigenvalues of the estimated Hodge Laplacian to semi-analytic solutions in Section 5.2. Finally, since in practical situations, one often has access to noisy observations that do not lie precisely on a manifold, we numerically investigate the robustness of this algorithm to noise in Section 5.3. For convenience, we refer to the estimations obtained from the curved mesh method by CMM.

5.1 Bochner and Hodge Laplacian Estimations on Sphere

In this section, we demonstrate the numerical convergence of the Hodge and Bochner Laplacians on the 2D unit sphere, $M = \{x \in \mathbb{R}^3 : \|x\| = 1\}$, embedded in $\mathbb{R}^3$. To simulate uniform random sampling on this surface, we take the following approach: first, we generate a vector in $\mathbb{R}^3$ via $\tilde{w}_i \sim \mathcal{N}(\vec{0}, \mathbf{I})$. We then normalize each sample: $x_i = \tilde{w}_i / \|\tilde{w}_i\|$. An example of the resulting dataset is shown in Figure 1a.

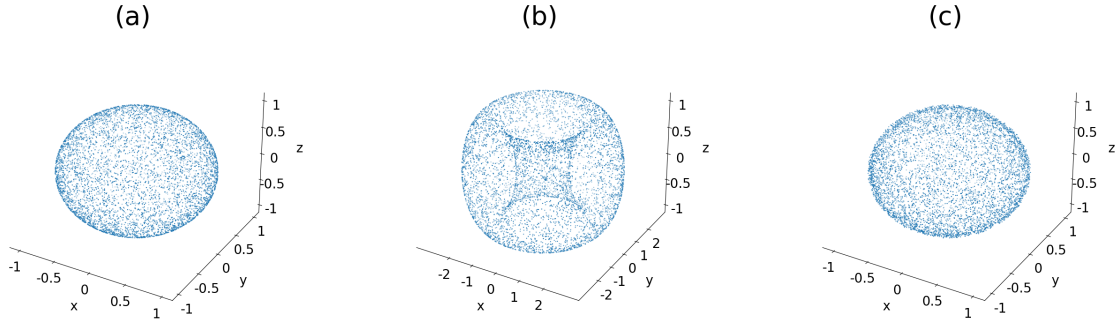


Figure 1: Example datasets used for numerical experiments. Displayed is a single trial with $N = 4000$ points for (a) sphere (b) torus, and (c) noisy sphere (each data point is perturbed in the normal direction with uniform noise of size 0.10).

Analytic eigensolutions for the Bochner and Hodge Laplacians on this manifold are well known. For a detailed description, see, for instance Section 5.4 of [15]. To summarize, the distinct eigenvalues of the Hodge and Bochner are given by the following:

$$\begin{aligned} \text{Hodge: } \lambda_\ell &= (\ell + 1)\ell, \\ \text{Bochner: } \lambda_\ell &= (\ell + 1)\ell - 1, \quad \ell \in \mathbb{N}^+. \end{aligned}$$

Since the sphere is of constant curvature, the Hodge and Bochner Laplacians share eigenvector fields. Eigenvector fields for the first 3 distinct eigenspaces are given by $\{\mathbf{n} \times \nabla_{\mathbb{R}^3} f, \mathbf{P} \nabla_{\mathbb{R}^3} f\}$, where $\mathbf{n}$ denotes the outward facing normal vector, $\mathbf{P}$ denotes the tangential projection tensor projecting $(\nabla_{\mathbb{R}^3} f)(x)$ to $T_x M$, and f is given by one of the spherical harmonics:

$$\begin{aligned} f(\vec{v}) &= (\vec{v})_i, \quad \text{for } \ell = 1, \\ f(\vec{v}) &= 3(\vec{v})_i(\vec{v})_k - \delta_{ik}, \quad \text{for } \ell = 2, \\ f(\vec{v}) &= 15(\vec{v})_i(\vec{v})_j(\vec{v})_k - 3\delta_{ij}(\vec{v})_k - 3\delta_{ki}(\vec{v})_j - 3\delta_{jk}(\vec{v})_i, \quad \text{for } \ell = 3, \end{aligned}$$

where $i, j, k \in \{1, 2, 3\}$. We remark that the above formulas imply the dimension of the first three eigenspaces, where color indicates pointwise vector norm.

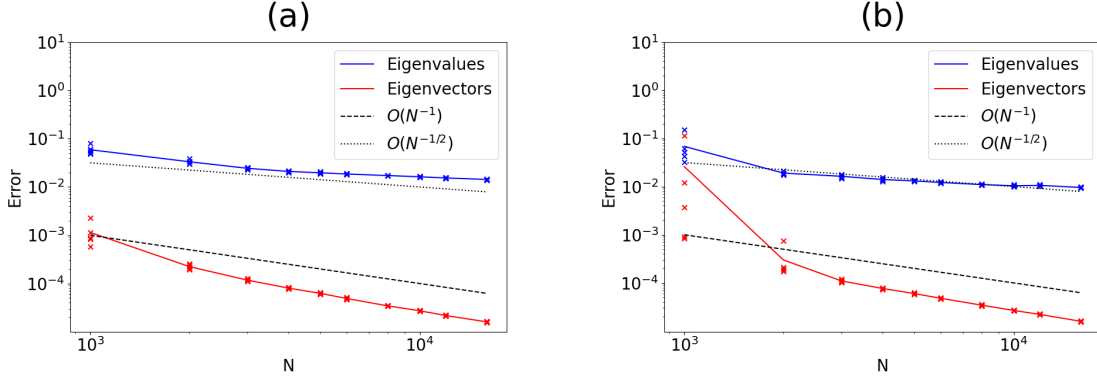


Figure 2: Operator estimation on sphere with uniform sampling distribution. Average of relative error (eigenvalues, leading $L = 48$ modes) and Average of mean-square-error (eigenvectors, leading $L = 6$ modes) as defined in Equation (46), (47) respectively: (a) Bochner Laplacian; (b) Hodge Laplacian.

Figure 2 displays the convergence of CMM for the estimation of eigenvalues and eigenvector fields of the Bochner and Hodge Laplacians as functions of N . The error metrics reported for convergence are given by

$$\text{Error of eigenvalues} := \frac{1}{L} \sum_{j=1}^L \frac{|\lambda_j - \hat{\lambda}_{j,N}|}{\lambda_j} \quad (46)$$

$$\text{Error of eigenvector fields} := \frac{1}{NL} \sum_{i=1}^N \sum_{j=1}^L \left\| W^{(j)}(x_i) - \hat{W}^{(j)}(x_i) \right\|^2, \quad (47)$$

where $W^{(j)}(x_i)$ denotes the j -th analytic vector field described above, and $\hat{W}^{(j)}$ denotes the least-squares linear combination of eigenvectors of the CMM estimator associated with the corresponding estimated eigenspace. We note that the above enumeration of eigenvalues is not distinct (i.e., includes multiplicities). In Figure 2a, results for estimating the Bochner Laplacian are shown, while in Figure 2b the corresponding results for the Hodge Laplacian are shown. We remark that, in either case, Theorem 4.1 is verified with a convergence rate of $O(N^{-1/d})$. In particular, one can show (see Appendix B, Lemma B.2 of [15]) that as $N \rightarrow \infty$, $h_{X,M} = O(N^{-1/d})$ almost surely, and $d = 2$ in our case. As for the eigenvector fields, the empirical convergence rates agree with our conjectures. For visual comparison, Figure 3 shows estimations (CMM estimation of Bochner Laplacian) of the first 4 eigenvector fields from a single trial with $N = 6000$. Overall, this example demonstrates that CMM can be used to accurately approximate multiple operators acting on vector fields, with competitive convergence rates.

5.2 Hodge Laplacian Estimation on Torus

In this section, we demonstrate numerical convergence of CMM on the 2D torus. In particular, consider $M = \left\{ \left((2 + \cos \theta) \cos \phi, (2 + \cos \theta) \sin \phi, \sin \theta \right) : 0 \leq \theta, \phi \leq 2\pi \right\}$. To sample uniformly from the torus, we impose a rejection sampling scheme. Namely, we sample a pair $(\theta_i, \phi_i) \sim \text{Unif}([0, 2\pi]^2)$, along with a rejection parameter $w_i \sim \text{Unif}([0, 1])$. We accept the pair (θ_i, ϕ_i) if $w_i \leq (2/3 + \cos \theta_i/3)$, in which case we obtain a point $x_i = ((2 + \cos \theta_i) \cos \phi_i, (2 + \cos \theta_i) \sin \phi_i, \sin \theta_i)$. Otherwise, we reject. A dataset of size $N =$ generated in this way is shown in Figure 1b.

We remark that, on the torus, analytic eigen-solutions to the eigenvalue problem are not available for the Bochner and Hodge Laplacians. However, we can leverage the following result, along with semi-analytic eigen-solutions for the Laplace-Beltrami operator, to obtain semi-analytic eigenvalues for the Hodge Laplacian.

Theorem 5.1. *Let M denote a 2-dimensional manifold embedded in $\mathbb{R}^3$. Then the nontrivial eigenvalues of the Hodge Laplacian are identical to the nontrivial eigenvalues of the Laplace-Beltrami operator where the multiplicities of eigenvalues of the Hodge Laplacian doubles those of the Laplace-Beltrami operator.*

The above is a standard result. A detailed proof can be found in [15]. Hence, to evaluate eigenvalue error, we combine the above result with semi-analytic eigen-solutions to the Laplace-Beltrami eigenvalue problem on the torus (following [15]). In particular, the Laplace-Beltrami eigenvalue problem on the 2D torus can be written in

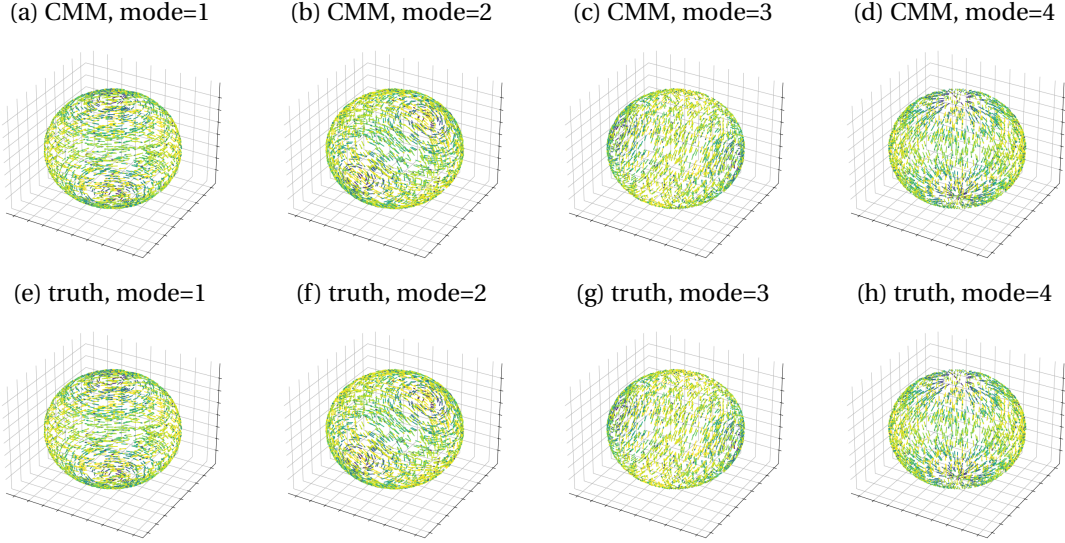


Figure 3: Visual comparison of the first four eigenvector fields for the Bochner Laplacian obtained using CMM (row 1) and analytic methods (row 2). The estimates are obtained from a single trial of $N = 6000$ randomly sampled points. Color indicates vector norm. Vectors visualized correspond to on a random subset of size 2000.

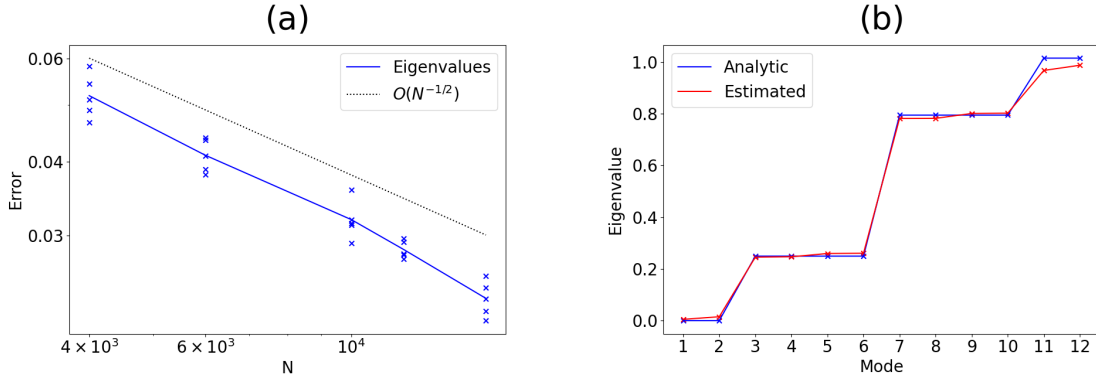


Figure 4: Operator estimation on torus with uniform sampling (with rejection) distribution. (a) Average of relative error (over the first 10 nontrivial) eigenvalues of the Hodge Laplacian estimator obtained from CMM as a function of N . (b) Mode by mode estimation for of spectrum for a single trial with $N = 16000$.

intrinsic coordinates as,

$$\Delta_M f_\ell = -\frac{1}{(2 + \cos\theta)^2} \frac{\partial^2 f_\ell}{\partial \phi^2} - \frac{\partial^2 f_\ell}{\partial \theta^2} + \frac{\sin\theta}{2 + \cos\theta} \frac{\partial f_\ell}{\partial \theta} = \lambda_\ell f_\ell.$$

Using separation of variables (assuming that $f_\ell(\theta, \phi) = \Phi_\ell(\phi)\Theta_\ell(\theta)$), we reduce the above to the following problems:

$$\Phi_\ell'' = m_\ell \Phi_\ell, \quad \Theta_\ell'' - \frac{\sin\theta}{2 + \cos\theta} \Theta_\ell' - \frac{m_\ell^2}{(2 + \cos\theta)^2} \Theta_\ell = \lambda_\ell \Theta_\ell.$$

The first can be solved analytically, while the second is solved numerically using a standard finite difference scheme on a fine grid. The resulting values for λ_ℓ are used to evaluate the error in Figure 4a. Here, we demonstrate the relative error in the estimation of the eigenvalues of the Hodge Laplacian estimated using CMM which numerically verifies the convergence reported in Theorem 4.1. Figure 4b shows the semi-analytic and estimated spectrum mode-by-mode for a single trial with $N = 16000$. CMM produces an accurate estimation, especially for modes 0-9.

5.3 Robustness to Noise

To investigate the effect of noise, we turn again to the unit sphere example. Consider the dataset of points N given by $x_i = (1 + \epsilon_i)\tilde{x}_i$ where $\tilde{x}_i \sim \text{Unif}(M)$, and $\epsilon_i \sim \text{Unif}([- \eta/2, \eta/2])$. Here, $\text{Unif}(M)$ denotes uniform sampling with respect to the volume measure of the manifold. To generate such samples, we follow the approach of Section 5.1. Numerical results for the computation of eigenvalues and eigenvectors for various values of η is shown in Figure 5.

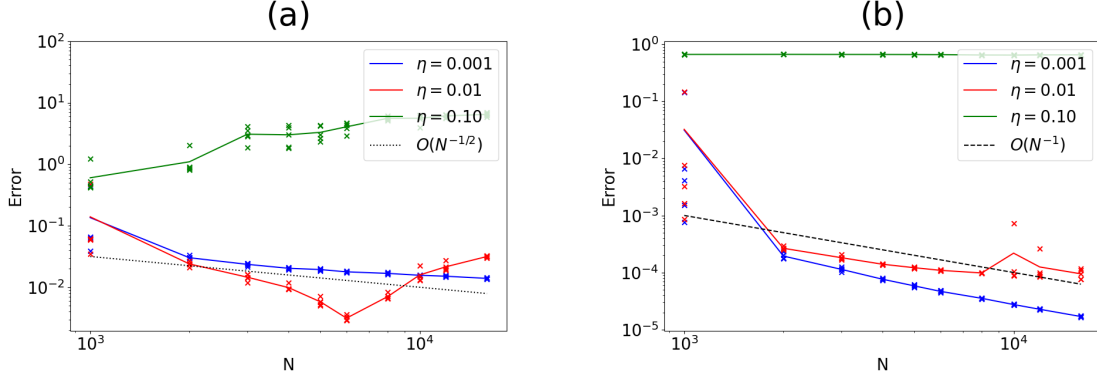


Figure 5: Operator estimation on a noisy sphere with uniform sampling distribution as functions of N . (a) Mean of percent error of eigenvalues. (b) Mean of mean-square-error of eigenvectors. Metrics are as defined in Equations (46) and (47). Color indicates noise level.

We see that when the noise is sufficiently small (0.1% of the radius of the sphere), convergence for both eigenvalues and eigenvectors is still realized. When the noise is increased to 1.0%, while convergence still occurs for the eigenvector fields (red, Figure 5b), eigenvalue convergence for large N does not occur, though the error in this case is still relatively small. When the noise is increased to 10.0%, the results for both the eigenvalues and eigenvectors are inaccurate, and do not converge. Based on these results as well as the theoretical analysis, we suspect that the algorithm is robust to noise so long as the size of the noise ($\|x_i - \tilde{x}_i\|$ in this example) does not exceed the error induced by this numerical approximation.

Given this analysis, there are many interesting future directions for modifying CMM to be more robust to noise. For instance, CMM hinges on the accurate estimation of tangent spaces (Section 2.1), which is in general sensitive to noise. One could employ more robust methods for approximating the tangent space on potentially noisy data (see, for instance, [9, 31]) and investigate how this affects the spectral error.

6 Summary

In this paper, we developed a numerical framework for approximating arbitrary differential operators in the weak form on Riemannian manifolds identified only by randomly sampled point cloud data. We proved, for the case of estimating the Bochner Laplacian, the convergence of eigenvalues, and provided several numerical examples validating this convergence in practice. This framework employs curved (nonlinear) local meshes, which are constructed by projecting point cloud data to the estimated tangent spaces [18] and the moving least squares parameterization [21], thus, allowing the curvature information such as Christoffel symbols to be appropriately accounted in the operator estimation. This approach is convenient as it avoids the curse of high ambient dimension, the construction of global parameterizations and meshes, and the tuning of various hyperparameters. Moreover, this method is flexible and is not restrictive to estimating only Laplacians. Theoretical and observed convergence rates, which for uniformly sampled data coincide with $O(N^{-1/d})$ for $d \geq 2$, are competitive with other estimators in the literature in the context of eigenvalue estimation for operators acting on vector fields on unknown manifolds. For instance, the spectral convergence of the RBF approximation to Bochner Laplacian was reported in [15] with convergence rate $O(N^{-1/d})$. As for the Graph Laplacian type method, the spectral convergence for the vector diffusion maps to the Bochner (or connection) Laplacian was established with no convergence rate as reported in [26]. Instead, they reported the pointwise convergence to the Bochner Laplacian acting on vector fields with convergence rate $O(N^{-\frac{1}{d/2+4}})$.

The ideas proposed in this paper suggest many potential directions for future work. First, the theoretical work in this paper assumes that the underlying manifold is without boundary, however, the weak formulation is flexible for

complex geometry with arbitrary boundary conditions. Nevertheless, details regarding local parameterization error analysis for points near the boundary remain to be seen. Second, throughout this manuscript, we emphasize on solving 1-Laplacian eigenvalue problems. The proposed formulation offers a generic method that can be applied to approximate arbitrary covariant derivatives on general tensor fields for PDE modeling applications. Lastly, as we have shown in the last section, more work needs to be done to overcome noisy data.

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A Example Computations for Local Curved Mesh Method

Throughout this appendix, local computations are done in terms of the metric computed in Section 2.4. For simplicity, we refer to the coordinates of this metric, the entries of this metric, the entries of its inverse, and Christoffel symbols in this metric as $\{\partial_{u_1}, \partial_{u_2}\}$, g_{ij} , g^{ij} , and Γ_{jk}^i , respectively. Explicit formulas for all quantities above can be deduced for the metric (12). For simplicity, we present the computations for $d = 2$.

The following notation will be necessary throughout. Fix a block i . Let $\mathbf{T} = [\mathbf{t}_1^{(i)}, \mathbf{t}_2^{(i)}, \dots, \mathbf{t}_n^{(i)}]$, the matrix whose columns form an orthonormal basis for the tangent space (first d components) and normal directions (last $n - d$ components) at i . Similarly, fix an arbitrary triangle $T_{i,r} \in R(i)$, and let $\mathbf{R}_{i,r} = [\partial_{u_1}, \partial_{u_2}]$ denote the matrix whose columns form a tangent space for the local curved mesh of triangle $T_{i,r}$, as defined in (11). We remark that $\mathbf{R}_{i,r}$ depends on the point (u_1, u_2) . Define

$$\mathbf{a}_k^{ij} = \mathbf{R}_{i,r}^{LS} \mathbf{T}^\top \mathbf{t}_k^{(j)}, \quad (48)$$

the vector of length d corresponding to the representation of $\mathbf{t}_k^{(j)}$ in the local coordinates, where the dependence of $\mathbf{a}_k^{ij}$ on $T_{i,r}$ is understood. We note that $\mathbf{R}_{i,r}^{LS} = (\mathbf{R}_{i,r}^\top \mathbf{R}_{i,r})^{-1} \mathbf{R}_{i,r}^\top$ is the least-squares solution for a map that takes the tangent vector $\mathbf{t}_k^{(j)}$ in the coordinates induced by columns of $\mathbf{T}$ to the canonical $\mathbb{R}^2$ coordinates, which we denoted as $\mathbf{a}_k^{ij}$. With this representation, its clear that

$$\langle \mathbf{t}_{k_1}^{(i)}, \mathbf{t}_{k_2}^{(j)} \rangle_g = \langle g \mathbf{a}_{k_1}^{ii}, \mathbf{a}_{k_2}^{ij} \rangle = \left(\mathbf{a}_{k_1}^{ii} \right)^\top g \left(\mathbf{a}_{k_2}^{ij} \right), \quad (49)$$

as noted in (28) in Remark 3, where $g = g_{\Phi_{C_{i,r}}}$ corresponds to the 2×2 metric tensor defined in (12).

In practice, since a smooth vector field $W : M \rightarrow TM$ can be represented locally on a neighborhood of $T_{x_i} M$, and for sufficiently large N (such that the curvature between neighboring points of data is sufficiently small), $R(i)$ is contained in this neighborhood, instead of $\mathbf{T}$, one can use $\mathbf{T}_i = [\mathbf{t}_1^{(i)}, \dots, \mathbf{t}_d^{(i)}]$, and subsequently ignore the last $(n - d)$ columns of $\mathbf{R}_{i,r}$, which reduces computation time as the first d rows of $\mathbf{R}_{i,r}$ do not depend on the point (u_1, u_2) ; see (11). We observe the difference in estimated eigenvalues and eigenvector fields when using the full matrices as in (48) and the reduced approach outlined above is negligible.

We remark that computations corresponding to off-diagonal matrix elements simplify significantly. In particular, integrals corresponding to the (i, j) -th entry for $i \neq j$ vanish for each $T_{i,r} \in R(i)$ except those for which the edge connecting $\vec{0}$ and $\mathbf{T}_i^\top(x_j - x_i)$, is a side of $T_{i,r}$. Since $T_{i,r}$ is a triangle with vertices $\{\vec{0}, \vec{v}_{i_{jr}}, \vec{v}_{i_{kr}}\}$, this is simply stating that $\mathbf{T}_i^\top(x_j - x_i)$ is a node of $T_{i,r}$. Therefore, for future convenience, we denote

$$Q(i, j) = \{T_{i,r} \in R(i) : \mathbf{T}_i^\top(x_j - x_i) \text{ is a node of } T_{i,r}\}.$$

A.1 Mass matrix for operators on vector fields

Here we explicitly compute the mass matrix for vector-fields, $\mathbf{M}$. The diagonal block formula from before reads

$$\begin{bmatrix} [\mathbf{M}]_{2i-1,2j-1} & [\mathbf{M}]_{2i-1,2j} \\ [\mathbf{M}]_{2i,2j-1} & [\mathbf{M}]_{2i,2j} \end{bmatrix} = \sum_{T_{i,r} \in R(i)} \int_T e_i e_j \begin{bmatrix} \langle \mathbf{t}_1^{(i)}, \mathbf{t}_1^{(j)} \rangle_g & \langle \mathbf{t}_1^{(i)}, \mathbf{t}_2^{(j)} \rangle_g \\ \langle \mathbf{t}_2^{(i)}, \mathbf{t}_1^{(j)} \rangle_g & \langle \mathbf{t}_2^{(i)}, \mathbf{t}_2^{(j)} \rangle_g \end{bmatrix} \sqrt{\det(g)} du_1 du_2,$$

Since we compute each integral with a quadrature rule on the vertices, it suffices to compute the integrand at an arbitrary point (u_1, u_2) .

To begin, we compute the diagonal block components. When $i = j$, we have

$$\begin{bmatrix} [\mathbf{M}]_{2i-1,2i-1} & [\mathbf{M}]_{2i-1,2i} \\ [\mathbf{M}]_{2i,2i-1} & [\mathbf{M}]_{2i,2i} \end{bmatrix} = \sum_{T_{i,r} \in R(i)} \int_T (1-u_1-u_2)^2 \begin{bmatrix} \langle g\mathbf{a}_1^{ii}, \mathbf{a}_1^{ii} \rangle & \langle g\mathbf{a}_1^{ii}, \mathbf{a}_2^{ii} \rangle \\ \langle g\mathbf{a}_2^{ii}, \mathbf{a}_1^{ii} \rangle & \langle g\mathbf{a}_2^{ii}, \mathbf{a}_2^{ii} \rangle \end{bmatrix} \sqrt{\det(g)} du_1 du_2,$$

where we have used (49). We remark that the quantities above indeed depend on each triangle $T_{i,r}$.

When $j \neq i$, we recall that the only two triangles for which the integration is nonzero are those in $Q(i, j)$ (which consists of 0 or 2 triangles). In both of these triangles, we can reorder the vertices so that $(1, 0) \rightarrow \mathbf{T}_i^\top(x_j - x_i)$. Hence, a nonzero of diagonal block of the mass matrix takes the form

$$\begin{bmatrix} [\mathbf{M}]_{2i-1,2j-1} & [\mathbf{M}]_{2i-1,2j} \\ [\mathbf{M}]_{2i,2j-1} & [\mathbf{M}]_{2i,2j} \end{bmatrix} = \sum_{T_{i,r} \in Q(i,j)} \int_T (1-u_1-u_2)u_1 \begin{bmatrix} \langle g\mathbf{a}_1^{ii}, \mathbf{a}_1^{ij} \rangle & \langle g\mathbf{a}_1^{ii}, \mathbf{a}_2^{ij} \rangle \\ \langle g\mathbf{a}_2^{ii}, \mathbf{a}_1^{ij} \rangle & \langle g\mathbf{a}_2^{ii}, \mathbf{a}_2^{ij} \rangle \end{bmatrix} \sqrt{\det(g)} du_1 du_2.$$

A.2 Stiffness Matrix for Bochner Laplacian

The Bochner Laplacian $\Delta_B : TM \rightarrow TM$ can be (weakly) defined by the following formula:

$$\langle \Delta_B w, v \rangle_g = \langle \text{grad}_g w, \text{grad}_g v \rangle_g \quad (50)$$

where the gradient grad_g of a vector field $v = v^i \frac{\partial}{\partial u^i}$ is a $(2, 0)$ tensor field given by

$$\text{grad}_g v = g^{kj} \left(\frac{\partial v^i}{\partial u^k} + v^p \Gamma_{pk}^i \right) \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}.$$

Notice that (50) denotes the Riemmanian inner product of two $(2, 0)$ tensor fields. Recall that for tensor fields $a = a_{ij} \frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j}$ and $b = b_{ij} \frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j}$, the Riemannian inner product is defined as,

$$\langle a, b \rangle_g = \sum_{i,j,k,\ell} a_{ij} g_{ik} b_{k\ell} g_{j\ell}.$$

Interpreting a, b , and g as matrices written in the basis $\{\frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j}\}_{i,j=1}^d$, it is not difficult to show that this can be rewritten

$$\langle a, b \rangle_g = \text{trace}(agb^\top g).$$

In this section, we compute the integrands necessary for employing the local mesh method with $D = \text{grad}_g$. Recall the blockwise formula for the stiffness matrix:

$$\begin{bmatrix} [\mathbf{S}]_{2i-1,2j-1} & [\mathbf{S}]_{2i-1,2j} \\ [\mathbf{S}]_{2i,2j-1} & [\mathbf{S}]_{2i,2j} \end{bmatrix} = \sum_{T_{i,r} \in R(i)} \int_T \begin{bmatrix} \langle \text{grad}_g e_i \mathbf{t}_1^{(i)}, \text{grad}_g e_j \mathbf{t}_2^{(j)} \rangle_g & \langle \text{grad}_g e_i \mathbf{t}_1^{(i)}, \text{grad}_g e_j \mathbf{t}_2^{(j)} \rangle_g \\ \langle \text{grad}_g e_i \mathbf{t}_2^{(i)}, \text{grad}_g e_j \mathbf{t}_1^{(j)} \rangle_g & \langle \text{grad}_g e_i \mathbf{t}_2^{(i)}, \text{grad}_g e_j \mathbf{t}_2^{(j)} \rangle_g \end{bmatrix} \sqrt{\det(g)} du_1 du_2.$$

We begin by computing the diagonal case, $\langle \text{grad}_g e_i \mathbf{t}_{k_1}^{(i)}, \text{grad}_g e_i \mathbf{t}_{k_2}^{(i)} \rangle_g$.

Simplifying the abuse of notation, we have

$$\langle \text{grad}_g e_i \mathbf{t}_{k_1}^{(i)}, \text{grad}_g e_i \mathbf{t}_{k_2}^{(j)} \rangle_g = \text{trace} \left(\mathbf{A}_{k_1}^{ii} g (\mathbf{A}_{k_2}^{ii})^\top g \right). \quad (51)$$

where $\mathbf{A}_{k_1}^{ii}, \mathbf{A}_{k_2}^{ii}$ correspond to the 2×2 matrix representations of $(2, 0)$ tensor fields

$$\begin{aligned} \mathbf{A}_{k_1}^{ii} &= \text{grad}_g \left[(1-u_1-u_2)(\mathbf{a}_{k_1}^{ii})_1 \partial_{u_1} + (1-u_1-u_2)(\mathbf{a}_{k_1}^{ii})_2 \partial_{u_2} \right], \\ \mathbf{A}_{k_2}^{ii} &= \text{grad}_g \left[(1-u_1-u_2)(\mathbf{a}_{k_2}^{ii})_1 \partial_{u_1} + (1-u_1-u_2)(\mathbf{a}_{k_2}^{ii})_2 \partial_{u_2} \right] \end{aligned}$$

in the basis of $\{\partial_{u_i} \otimes \partial_{u_j}\}_{i,j}$. By linearity, we have

$$\begin{aligned} \mathbf{A}_{k_1}^{ii} &= (\mathbf{a}_{k_1}^{(ii)})_1 \text{grad}_g [(1-u_1-u_2) \partial_{u_1}] + (\mathbf{a}_{k_1}^{(ii)})_2 \text{grad}_g [(1-u_1-u_2) \partial_{u_2}] \\ \mathbf{A}_{k_2}^{ii} &= (\mathbf{a}_{k_2}^{(ii)})_1 \text{grad}_g [(1-u_1-u_2) \partial_{u_1}] + (\mathbf{a}_{k_2}^{(ii)})_2 \text{grad}_g [(1-u_1-u_2) \partial_{u_2}]. \end{aligned}$$

It remains to compute the matrices $\text{grad}_g[(1 - u_1 - u_2)\partial_{u_1}]$, $\text{grad}_g[(1 - u_1 - u_2)\partial_{u_2}]$, which amounts to plugging in to the local formula for grad_g . In our case, gradient of a vector field $v = v^i \frac{\partial}{\partial u_i}$, where $v^1 = 1 - u_1 - u_2$ and $v^2 = 0$, is given as,

$$\text{grad}_g v = g^{kj} \left(\frac{\partial v^i}{\partial u_k} + v^p \Gamma_{pk}^i \right) \partial_{u_i} \otimes \partial_{u_j}.$$

Expanding each component, we have

$$\begin{aligned} \text{grad}_g[(1 - u_1 - u_2)\partial_{u_1}] &= g^{k1} \left(\frac{\partial(1 - u_1 - u_2)}{\partial u_k} + (1 - u_1 - u_2)\Gamma_{1k}^1 \right) \partial_{u_1} \otimes \partial_{u_1} \\ &\quad + g^{k2} \left(\frac{\partial(1 - u_1 - u_2)}{\partial u_k} + (1 - u_1 - u_2)\Gamma_{1k}^1 \right) \partial_{u_1} \otimes \partial_{u_2} \\ &\quad + g^{k1} ((1 - u_1 - u_2)\Gamma_{1k}^2) \partial_{u_2} \otimes \partial_{u_1} + g^{k2} ((1 - u_1 - u_2)\Gamma_{1k}^2) \partial_{u_2} \otimes \partial_{u_2} \\ &= (-g^{11} - g^{21} + (1 - u_1 - u_2)(\Gamma_{11}^1 g^{11} + \Gamma_{12}^1 g^{21})) \partial_{u_1} \otimes \partial_{u_1} \\ &\quad + (-g^{12} - g^{22} + (1 - u_1 - u_2)(\Gamma_{11}^1 g^{12} + \Gamma_{12}^1 g^{22})) \partial_{u_1} \otimes \partial_{u_2} \\ &\quad + ((1 - u_1 - u_2)(\Gamma_{11}^2 g^{11} + \Gamma_{12}^2 g^{21})) \partial_{u_2} \otimes \partial_{u_1} \\ &\quad + ((1 - u_1 - u_2)(\Gamma_{11}^2 g^{12} + \Gamma_{12}^2 g^{22})) \partial_{u_2} \otimes \partial_{u_2} \end{aligned}$$

which can be written in matrix form as follows:

$$\text{grad}_g[(1 - u_1 - u_2)\partial_{u_1}] = \left((1 - u_1 - u_2) \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \right) g^{-1}$$

Following the same calculation, we have:

$$\text{grad}_g[(1 - u_1 - u_2)\partial_{u_2}] = \left((1 - u_1 - u_2) \begin{bmatrix} \Gamma_{21}^1 & \Gamma_{22}^1 \\ \Gamma_{21}^2 & \Gamma_{22}^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \right) g^{-1}.$$

This completes the computation for $i = j$. When $i \neq j$, we instead have

$$\langle \text{grad}_g e_i \mathbf{t}_{k_1}^{(i)}, \text{grad}_g e_i \mathbf{t}_{k_2}^{(j)} \rangle_g = \text{trace}(\mathbf{A}_{k_1}^{ii} g(\mathbf{A}_{k_2}^{ij})^\top g),$$

where

$$\mathbf{A}_{k_2}^{ij} = (\mathbf{a}_{k_2}^{ij})_1 \text{grad}_g[u_1 \partial_{u_1}] + (\mathbf{a}_{k_2}^{ij})_2 \text{grad}_g[u_1 \partial_{u_2}].$$

where u_1 in the above is obtained by reordering the vertices of a single triangle. The final quantities to compute are given by

$$\begin{aligned} \text{grad}_g[u_1 \partial_{u_1}] &= \left(u_1 \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) g^{-1} \\ \text{grad}_g[u_1 \partial_{u_2}] &= \left(u_1 \begin{bmatrix} \Gamma_{21}^1 & \Gamma_{22}^1 \\ \Gamma_{21}^2 & \Gamma_{22}^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) g^{-1}. \end{aligned}$$

In summary, when $i = j$,

$$\begin{bmatrix} [\mathbf{S}]_{2i-1,2i-1} & [\mathbf{S}]_{2i-1,2i} \\ [\mathbf{S}]_{2i,2i-1} & [\mathbf{S}]_{2i,2i} \end{bmatrix} = \sum_{T_{i,r} \in R(i)} \int_T \begin{bmatrix} \text{trace}(\mathbf{A}_1^{ii} g(\mathbf{A}_1^{ii})^\top g) & \text{trace}(\mathbf{A}_1^{ii} g(\mathbf{A}_2^{ii})^\top g) \\ \text{trace}(\mathbf{A}_2^{ii} g(\mathbf{A}_1^{ii})^\top g) & \text{trace}(\mathbf{A}_2^{ii} g(\mathbf{A}_2^{ii})^\top g) \end{bmatrix} \sqrt{\det(g)} du_1 du_2,$$

while for $i \neq j$, we have

$$\begin{bmatrix} [\mathbf{S}]_{2i-1,2j-1} & [\mathbf{S}]_{2i-1,2j} \\ [\mathbf{S}]_{2i,2j-1} & [\mathbf{S}]_{2i,2j} \end{bmatrix} = \sum_{T_{i,r} \in Q(i,j)} \int_T \begin{bmatrix} \text{trace}(\mathbf{A}_1^{ii} g(\mathbf{A}_1^{ij})^\top g) & \text{trace}(\mathbf{A}_1^{ii} g(\mathbf{A}_2^{ij})^\top g) \\ \text{trace}(\mathbf{A}_2^{ii} g(\mathbf{A}_1^{ij})^\top g) & \text{trace}(\mathbf{A}_2^{ii} g(\mathbf{A}_2^{ij})^\top g) \end{bmatrix} \sqrt{\det(g)} du_1 du_2,$$

where in each case, $\mathbf{A}_k^{ij}$ are as defined above.

A.3 Stiffness Matrix for Hodge Laplacian

The Hodge Laplacian is typically defined on a k -form ω by the formula

$$\Delta_H \omega = d^* d \omega + d d^* \omega$$

where d denotes the total differential, and d^* is the codifferential. The Hodge Laplacian can also be defined on vector fields by leveraging the musical isomorphisms:

$$\Delta_H v := \sharp(d^* d \omega + d d^* \omega) \flat v,$$

For the purposes of this paper, only the weak form is necessary:

$$\langle \Delta_H v, w \rangle_g = \langle d \flat v, d \flat w \rangle_g + \langle d^* \flat v, d^* \flat w \rangle_g, \quad (52)$$

for any $v, w \in \mathfrak{X}(M)$, so the first inner-product is defined over 2-forms (or $(0,2)$ -tensors) whereas the second inner product is defined over 0-forms (or functions). Based on the above formula, all that is needed for computations is the action of d and d^* on 1-forms (on manifolds of dimension 2). These formulas are given by

$$\begin{aligned} d(\omega_i du^i) &= \frac{\partial \omega_i}{\partial u^j} du^j, \\ d^*(\omega_i du^i) &= (-1) \star d \star (\omega_i du^i), \end{aligned}$$

where $\star$ denotes the hodge star operator, which is defined locally on a k -form ω to be the unique $(d-k)$ form $\star \omega$ satisfying the formula

$$\omega' \wedge \star \omega = \langle \omega', \omega \rangle_g \sqrt{\det g} du_1 \wedge du_2$$

for any k -form ω' . Since (52), as well as the above formula, contains inner-products of k -forms, we remind the reader that the inner product of two k forms is given by

$$\langle \omega^1 \wedge \dots \wedge \omega^k, \eta^1 \wedge \dots \wedge \eta^k \rangle = \det(\langle \omega^i, \eta^j \rangle_g).$$

The definition for the stiffness matrix can be initially simplified as follows:

$$\begin{aligned} &\begin{bmatrix} [\mathbf{S}]_{2i-1,2j-1} & [\mathbf{S}]_{2i-1,2j} \\ [\mathbf{S}]_{2i,2j-1} & [\mathbf{S}]_{2i,2j} \end{bmatrix} = \\ &\sum_{T_{i,r} \in R(i)} \int_T \begin{bmatrix} \langle d \flat e_i \mathbf{t}_1^{(i)}, d \flat e_j \mathbf{t}_1^{(j)} \rangle_g + \langle d^* \flat e_i \mathbf{t}_1^{(i)}, d^* \flat e_j \mathbf{t}_1^{(j)} \rangle_g & \langle d \flat e_i \mathbf{t}_1^{(i)}, d \flat e_j \mathbf{t}_2^{(j)} \rangle_g + \langle d^* \flat e_i \mathbf{t}_1^{(i)}, d^* \flat e_j \mathbf{t}_2^{(j)} \rangle_g \\ \langle d \flat e_i \mathbf{t}_2^{(i)}, d \flat e_j \mathbf{t}_1^{(j)} \rangle_g + \langle d^* \flat e_i \mathbf{t}_2^{(i)}, d^* \flat e_j \mathbf{t}_1^{(j)} \rangle_g & \langle d \flat e_i \mathbf{t}_2^{(i)}, d \flat e_j \mathbf{t}_2^{(j)} \rangle_g + \langle d^* \flat e_i \mathbf{t}_2^{(i)}, d^* \flat e_j \mathbf{t}_2^{(j)} \rangle_g \end{bmatrix} \sqrt{\det(g)} du_1 du_2. \end{aligned} \quad (53)$$

We'll begin by computing $d \flat v$ and $d^* \flat v$ on basis functions in each case. Note that in general, the nodal basis functions are a linear combination of

$$v = c_1 \hat{\phi} \frac{\partial}{\partial u_1} + c_2 \hat{\phi} \frac{\partial}{\partial u_2},$$

where $\hat{\phi} = 1 - u_1 - u_2$, or

$$v = c_1 u_1 \frac{\partial}{\partial u_1} + c_2 u_1 \frac{\partial}{\partial u_2}.$$

Hence, we handle the following four cases:

$$\begin{aligned} \text{case 1:} & \quad v = (1 - u_1 - u_2) \frac{\partial}{\partial u_1} \\ \text{case 2:} & \quad v = (1 - u_1 - u_2) \frac{\partial}{\partial u_2} \\ \text{case 3:} & \quad v = u_1 \frac{\partial}{\partial u_1} \\ \text{case 4:} & \quad v = u_1 \frac{\partial}{\partial u_2} \end{aligned}$$

For case 1, we have $\flat v = (1 - u_1 - u_2) g_{11} du_1 + (1 - u_1 - u_2) g_{12} du_2$. Hence,

$$\begin{aligned} d \flat v &= \frac{\partial(g_{11} \hat{\phi})}{\partial u_2} du_2 \wedge du_1 + \frac{\partial(g_{12} \hat{\phi})}{\partial u_1} du_1 \wedge du_2 = \left(\frac{\partial(g_{12} \hat{\phi})}{\partial u_1} - \frac{\partial(g_{11} \hat{\phi})}{\partial u_2} \right) du_1 \wedge du_2 \\ &= \left(g_{11} - g_{12} + \hat{\phi} \frac{\partial g_{12}}{\partial u_1} - \hat{\phi} \frac{\partial g_{11}}{\partial u_2} \right) du_1 \wedge du_2 := A_1 du_1 \wedge du_2. \end{aligned}$$

Similarly, we have that

$$d^* \flat v = (-1) \star d \star (\hat{\phi} g_{11} du_1 + \hat{\phi} g_{12} du_2).$$

The formula for $\star$ shows that $\star(du_1) = -g^{12} \sqrt{\det(g)} du_1 + g^{11} \sqrt{\det(g)} du_2$, $\star(du_2) = -g^{22} \sqrt{\det(g)} du_1 + g^{12} \sqrt{\det(g)} du_2$, and $\star(du_1 \wedge du_2) = \det(g^{-1}) \sqrt{\det(g)} = (\det(g))^{-1/2}$. Hence,

$$\begin{aligned} d^* \flat v &= (-1) \star d \left(\hat{\phi} \sqrt{\det(g)} ((-g_{11} g^{12} - g_{12} g^{22}) du_1 + (g_{12} g^{12} + g_{11} g^{11}) du_2) \right) \\ &= (-1) \star d \left(\hat{\phi} \sqrt{\det(g)} (1 du_2) \right) \\ &= (-1) \star \left(\frac{\partial}{\partial u_1} \left(\hat{\phi} \sqrt{\det(g)} \right) du_1 \wedge du_2 \right) \\ &= \frac{-1}{\sqrt{\det(g)}} \left(\frac{\partial}{\partial u_1} \left(\hat{\phi} \sqrt{\det(g)} \right) \right) \\ &= 1 - \frac{\hat{\phi}}{\sqrt{\det g}} \frac{\partial \sqrt{\det g}}{\partial u_1} := A_3 \end{aligned}$$

Similar computations yield the results for other three cases. The results are as follows:

$$\begin{aligned} \text{case 2:} \quad d \flat v &= \left(g_{12} - g_{22} + \hat{\phi} \frac{\partial g_{22}}{\partial u_1} - \hat{\phi} \frac{\partial g_{12}}{\partial u_2} \right) du_1 \wedge du_2 := A_2 du_1 \wedge du_2 \\ d^* \flat v &= 1 - \frac{\hat{\phi}}{\sqrt{\det g}} \frac{\partial \sqrt{\det g}}{\partial u_2} := A_4 \\ \text{case 3:} \quad d \flat v &= \left(g_{21} + u_1 \frac{\partial g_{21}}{\partial u_1} - u_1 \frac{\partial g_{11}}{\partial u_2} \right) du_1 \wedge du_2 := B_1 du_1 \wedge du_2 \\ d^* \flat v &= -1 - \frac{u_1}{\sqrt{\det g}} \frac{\partial \sqrt{\det g}}{\partial u_1} := B_3 \\ \text{case 4:} \quad d \flat v &= \left(g_{22} + u_1 \frac{\partial g_{22}}{\partial u_1} - u_1 \frac{\partial g_{12}}{\partial u_2} \right) du_1 \wedge du_2 := B_2 du_1 \wedge du_2 \\ d^* \flat v &= -\frac{u_1}{\sqrt{\det g}} \frac{\partial \sqrt{\det g}}{\partial u_2} := B_4. \end{aligned}$$

We can now compute the first term completely in the two cases: the diagonal blocks, and the off diagonal blocks. On the diagonal blocks, (meaning $i = j$), we have

$$\begin{aligned} \langle d \flat e_i \mathbf{t}_r^{(i)}, d \flat e_i \mathbf{t}_s^{(i)} \rangle_g &= (\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ii})_1 A_1^2 \langle du_1 \wedge du_2, du_1 \wedge du_2 \rangle_g + (\mathbf{a}_r^{ii})_2 (\mathbf{a}_s^{ii})_1 A_1 A_2 \langle du_1 \wedge du_2, du_1 \wedge du_2 \rangle_g \\ &\quad + (\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ii})_2 A_1 A_2 \langle du_1 \wedge du_2, du_1 \wedge du_2 \rangle_g + (\mathbf{a}_r^{ii})_2 (\mathbf{a}_s^{ii})_2 A_2^2 \langle du_1 \wedge du_2, du_1 \wedge du_2 \rangle_g \\ &= \frac{1}{\det g} \left((\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ii})_1 A_1^2 + (\mathbf{a}_r^{ii})_2 (\mathbf{a}_s^{ii})_1 A_1 A_2 + (\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ii})_2 A_1 A_2 + (\mathbf{a}_r^{ii})_2 (\mathbf{a}_s^{ii})_2 A_2^2 \right) \end{aligned} \quad (54)$$

Similarly,

$$\langle d^* \flat e_i \mathbf{t}_r^{(i)}, d^* \flat e_i \mathbf{t}_s^{(i)} \rangle_g = \left((\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ii})_1 A_3^2 + (\mathbf{a}_r^{ii})_2 (\mathbf{a}_s^{ii})_1 A_3 A_4 + (\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ii})_2 A_3 A_4 + (\mathbf{a}_r^{ii})_2 (\mathbf{a}_s^{ii})_2 A_4^2 \right). \quad (55)$$

Whence, its clear that the formula for the stiffness matrix on the diagonal elements is given by (53), plugging in the computed values from (54), (55).

On the off diagonal blocks, (meaning $i \neq j$), we have

$$\begin{aligned} \langle d \flat e_i \mathbf{t}_r^{(i)}, d \flat e_j \mathbf{t}_s^{(j)} \rangle_g &= (\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ij})_1 A_1 B_1 \langle du_1 \wedge du_2, du_1 \wedge du_2 \rangle_g + (\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ij})_2 A_1 B_2 \langle du_1 \wedge du_2, du_1 \wedge du_2 \rangle_g \\ &\quad + (\mathbf{a}_r^{ij})_2 (\mathbf{a}_s^{ij})_1 A_2 B_1 \langle du_1 \wedge du_2, du_1 \wedge du_2 \rangle_g + (\mathbf{a}_r^{ij})_2 (\mathbf{a}_s^{ij})_2 A_2 B_2 \langle du_1 \wedge du_2, du_1 \wedge du_2 \rangle_g \\ &= \frac{1}{\det g} \left((\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ij})_1 A_1 B_1 + (\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ij})_2 A_1 B_2 + (\mathbf{a}_r^{ij})_2 (\mathbf{a}_s^{ij})_1 A_2 B_1 + (\mathbf{a}_r^{ij})_2 (\mathbf{a}_s^{ij})_2 A_2 B_2 \right). \end{aligned} \quad (56)$$

Similarly,

$$\langle d^* \flat e_i \mathbf{t}_r^{(i)}, d^* \flat e_j \mathbf{t}_s^{(j)} \rangle_g = \left((\mathbf{a}_r^{ii})_1 (\mathbf{a}_s^{ij})_1 A_3 B_3 + (\mathbf{a}_r^{ii})_2 (\mathbf{a}_s^{ij})_1 B_3 A_4 + (\mathbf{a}_r^{ij})_1 (\mathbf{a}_s^{ij})_2 A_3 B_4 + (\mathbf{a}_r^{ij})_2 (\mathbf{a}_s^{ij})_2 A_4 B_4 \right). \quad (57)$$

From this, its clear that when $i = j$, the formula for the stiffness matrix is given by (53), plugging in the computed values from (56), (57).

B Proofs of Technical Lemmas

Proof of Lemma 4.2. For simplicity of notation, we do not explicitly write evaluation of each quantity on $\vec{v}$. First, notice that

$$\begin{aligned}\partial_{v_1}(\gamma_{x_i}) &= \left(1, 0, \frac{\partial z_{x_i}}{\partial v_1}\right)^\top, \\ \partial_{v_2}(\gamma_{x_i}) &= \left(0, 1, \frac{\partial z_{x_i}}{\partial v_2}\right)^\top.\end{aligned}$$

Similarly,

$$\begin{aligned}\partial_{v_1}(\alpha_{x_i}) &= \left(1, 0, \frac{\partial p_i}{\partial v_1}\right)^\top, \\ \partial_{v_2}(\alpha_{x_i}) &= \left(0, 1, \frac{\partial p_i}{\partial v_2}\right)^\top.\end{aligned}$$

Using (32), we see that

$$\|\partial_{v_k}(\gamma_{x_i})(\vec{v}) - \partial_{v_k}(\alpha_{x_i})(\vec{v})\| = O(h_{X,M}^2).$$

Moreover, adding and subtracting $(\partial p_i / \partial v_q)(\partial z_{x_i} / \partial v_s)$, we find that

$$\begin{aligned}|(g_{\alpha_{x_i}})_{qs} - (g_{\gamma_{x_i}})_{qs}| &\leq |(\partial p_i / \partial v_q)(\partial p_i / \partial v_s) - (\partial p_i / \partial v_q)(\partial z_{x_i} / \partial v_s)| \\ &\quad + |(\partial p_i / \partial v_q)(\partial z_{x_i} / \partial v_s) - (\partial z_{x_i} / \partial v_q)(\partial z_{x_i} / \partial v_s)| \\ &= O(h_{X,M}^2).\end{aligned}$$

Similarly, adding and subtracting $(g_{\alpha_{x_i}})_{11}(g_{\gamma_{x_i}})_{22} + (g_{\alpha_{x_i}})_{12}(g_{\gamma_{x_i}})_{21}$, we have that

$$\begin{aligned}|\det(g_{\gamma_{x_i}}) - \det(g_{p_i})| &\leq \left| (g_{\gamma_{x_i}})_{11}(g_{\gamma_{x_i}})_{22} - (g_{\alpha_{x_i}})_{11}(g_{\gamma_{x_i}})_{22} \right| \\ &\quad + \left| (g_{\alpha_{x_i}})_{11}(g_{\gamma_{x_i}})_{22} - (g_{\alpha_{x_i}})_{11}(g_{\alpha_{x_i}})_{22} \right| \\ &\quad + \left| (g_{\gamma_{x_i}})_{12}(g_{\gamma_{x_i}})_{21} - (g_{\alpha_{x_i}})_{12}(g_{\gamma_{x_i}})_{21} \right| \\ &\quad + \left| (g_{\alpha_{x_i}})_{12}(g_{\gamma_{x_i}})_{21} - (g_{\alpha_{x_i}})_{12}(g_{\alpha_{x_i}})_{21} \right| \\ &= O(h_{X,M}^2).\end{aligned}$$

Since $\det(g_{\gamma_{x_i}})$ is bounded away from 0 independent of x_i , we have that

$$\left| \sqrt{\det(g_{\gamma_{x_i}})} - \sqrt{\det(g_{p_i})} \right| = O(h_{X,M}).$$

Using the above facts along with the entrywise formula for the coefficients of the inverse (see, for instance, Chapter 5, Theorem 7 of [19]), it is immediate to see that,

$$\left| g_{\gamma_{x_i}}^{qs} - g_{p_i}^{qs} \right| = O(h_{X,M}).$$

For the Christoffel symbols, recall that p_i can estimate two derivatives of z_{x_i} up to order $h_{X,M}$. Since

$$\begin{aligned}\frac{\partial(g_{\gamma_{x_i}})_{qs}}{\partial v_t} &= (\partial^2 z_{x_i} / \partial v_q \partial v_t)(\partial z_{x_i} / \partial v_s) + (\partial z_{x_i} / \partial v_q)(\partial^2 z_{x_i} / \partial v_s \partial v_t) \\ \frac{\partial(g_{\alpha_{x_i}})_{qs}}{\partial v_t} &= (\partial^2 p_i / \partial v_q \partial v_t)(\partial p_i / \partial v_s) + (\partial p_i / \partial v_q)(\partial^2 p_i / \partial v_s \partial v_t),\end{aligned}$$

adding and subtracting $(\partial^2 z_{x_i}/\partial v_q v_t)(\partial p_i/\partial v_s) + (\partial z_{x_i}/\partial v_q)(\partial^2 p_i/\partial v_s v_t)$, we see that

$$\begin{aligned}
\left| \partial(g_{\gamma_{x_i}})_{qs}/\partial v_t - \partial(g_{p_i})_{qs}/\partial v_t \right| &\leq \left| (\partial^2 z_{x_i}/\partial v_q v_t)(\partial z_{x_i}/\partial v_s) - (\partial^2 z_{x_i}/\partial v_q v_t)(\partial p_i/\partial v_s) \right| \\
&+ \left| (\partial^2 z_{x_i}/\partial v_q v_t)(\partial p_i/\partial v_s) - (\partial^2 p_i/\partial v_q v_t)(\partial p_i/\partial v_s) \right| \\
&+ \left| (\partial z_{x_i}/\partial v_q)(\partial^2 z_{x_i}/\partial v_s v_t) - (\partial z_{x_i}/\partial v_q)(\partial^2 p_i/\partial v_s v_t) \right| \\
&+ \left| (\partial z_{x_i}/\partial v_q)(\partial^2 p_i/\partial v_s v_t) - (\partial p_i/\partial v_q)(\partial^2 p_i/\partial v_s v_t) \right| \\
&= O(h_{X,M}).
\end{aligned}$$

Plugging into the well known formula for the Christoffel symbols in terms of the metric tensor and leveraging estimates above, it follows immediately that

$$|\Gamma_{st}^q(p_i) - \Gamma_{st}^q(\gamma_{x_i})| = O(h_{X,M}).$$

This completes the proof. $\square$

Proof of Lemma 4.6. Note that,

$$(\nabla W_N)_{\gamma_{x_i}} := g_{\gamma_{x_i}}^{sj} \left(\frac{\partial W_N^t}{\partial v_s} + W_N^p \Gamma_{ps}^t(\gamma_{x_i}) \right) \frac{\partial}{\partial v_t} \otimes \frac{\partial}{\partial v_j}.$$

Similarly,

$$(\nabla \widehat{W}_N)_{\alpha_{x_i}} := g_{\alpha_{x_i}}^{sj} \left(\frac{\partial \widehat{W}_N^t}{\partial v_s} + \widehat{W}_N^p \Gamma_{ps}^t(\alpha_{x_i}) \right) \frac{\partial}{\partial v_t} \otimes \frac{\partial}{\partial v_j}.$$

It follows from Lemmas 4.2 and 4.5 that

$$\begin{aligned}
|g_{\gamma_{x_i}}^{sj} - g_{\alpha_{x_i}}^{sj}| &= O(h_{X,M}), & |(g_{\gamma_{x_i}})_{sj} - (g_{\alpha_{x_i}})_{sj}| &= O(h_{X,M}) \\
|W_N^p - \widehat{W}_N^p| &= O(h_{X,M}), & \left| \frac{\partial W_N^t}{\partial v_s} - \frac{\partial \widehat{W}_N^t}{\partial v_s} \right| &= O(h_{X,M}) \\
|\Gamma_{ps}^t(\gamma_{x_i}) - \Gamma_{ps}^t(\alpha_{x_i})| &= O(h_{X,M}), & \left| \sqrt{\det(g_{\gamma_{x_i}})} - \sqrt{\det(g_{\alpha_{x_i}})} \right| &= O(h_{X,M}).
\end{aligned}$$

Combining these estimates, simple algebraic manipulations show that

$$\left\langle (\nabla U)_{\gamma_{x_i}}, (\nabla W)_{\gamma_{x_i}} \right\rangle_{g_{\gamma_{x_i}}} \sqrt{\det(g_{\gamma_{x_i}})} - \left\langle (\nabla \widehat{U})_{\alpha_{x_i}}, (\nabla \widehat{W})_{\alpha_{x_i}} \right\rangle_{g_{\alpha_{x_i}}} \sqrt{\det(g_{\alpha_{x_i}})} = O(h_{X,M}),$$

and therefore that

$$\left| \int_{T_{i,r}} \left[\left\langle (\nabla U)_{\gamma_{x_i}}, (\nabla W)_{\gamma_{x_i}} \right\rangle_{g_{\gamma_{x_i}}} \sqrt{\det(g_{\gamma_{x_i}})} - \left\langle (\nabla \widehat{U})_{\alpha_{x_i}}, (\nabla \widehat{W})_{\alpha_{x_i}} \right\rangle_{g_{\alpha_{x_i}}} \sqrt{\det(g_{\alpha_{x_i}})} \right] dv_1 dv_2 \right| = O(\text{Vol}(T_{i,r}) h_{X,M}).$$

This completes the proof for the first bound. The second bound follows the same argument. $\square$

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