



Two-way coupled stabilized finite element method for transient thermal convection in irregular geometries with electromagnetic force fields

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Abstract

A stabilized mixed finite element method is presented for a two-way coupled system of partial differential equations governing the transient thermal convection of nanofluids. The method employs a scale separation of the velocity and temperature fields into coarse and fine scales, facilitated by the Variational Multiscale (VMS) framework. This, combined with the consistent linearization of the fine-scale variational equations, yields a stabilized formulation with improved stability and accuracy. The explicit structure of the stabilization terms is derived through the direct application of a bubble function-based approach to the fine-scale variational equations. The formulation ensures full coupling between the mechanical and thermal phases and satisfies the *inf-sup* condition, enabling the use of equal-order Lagrange basis functions for all unknown fields. The resulting numerical method not only yields quadratic convergence of the nonlinear coupled equations, it also yields optimal convergence rates, both in space and time, for different element types and time marching schemes. The method is applied to transient thermal convection problems that include natural convection in irregular geometries and forced convection under the influence of electromagnetic force field.

Keywords Variational multiscale · Finite element method · Stabilized method · Transient thermal convection · Lorentz force · Wavy wall

1 Introduction

Since Maxwell [1] first proposed that suspending solid particles in fluids can enhance their electrical conductivity, extensive research has been conducted—ranging from the development of stable suspensions to the investigation of the thermophysical properties of the resulting fluids [2]. The term nanofluids, introduced by Choi et al. [3], refers to fluids containing suspended nanoscale solid particles, primarily oxides, intended to enhance thermal conductivity. Due to their superior thermal properties, nanofluids have found applications in a variety of areas including energy storage systems such as solar collectors, thermal management systems like heat exchangers, and cooling devices such as refrigerators and air conditioners. Additionally, nanofluids are employed in lab-on-a-chip technologies, drug delivery systems, and as lubricants in automotive engineering [2, 4]. Despite these advances, there remain significant challenges in the preparation of nanofluids and in understanding their thermal behavior. Existing theoretical models often lack generalizability and are insufficient for fully explaining the

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observed phenomena [5]. Computational modeling and simulation thus offer a powerful framework for uncovering the underlying physical mechanisms and enhancing the design and performance of nanofluidic systems.

Buoyancy-driven natural convection is a fundamental mechanism in heat transfer and is often preferred over forced or mixed convection due to its lower energy consumption and reduced maintenance requirements in practical systems [6]. The evolution of unsteady natural convection typically proceeds through three stages: the initial stage, the transition stage, and the fully developed steady state [7]. While the steady state has been well studied [8], the initial and transitional stages remain relatively underexplored. Key areas requiring further investigation include the development time of the thermal boundary layer, the duration of the forcing period, and the filling time of enclosures [9, 10].

Numerical modeling of nanofluids generally follows two main approaches: single-phase and two-phase models. In single-phase models, the base fluid and nanoparticles are treated as a homogeneous continuum characterized by effective thermophysical properties. These properties can be temperature-dependent, yielding improved accuracy compared to temperature-independent formulations [10]. Beyond basic homogeneous assumptions, more advanced single-phase models, such as the thermal disruption model and Bourgnino's model, have also been proposed [11, 12]. In contrast, two-phase models represent the base fluid and nanoparticles as distinct phases with interfacial interactions. Although two-phase models offer higher fidelity in comparison with experimental data, they are significantly more computationally intensive [13]. Given the balance between accuracy and computational cost, single-phase homogeneous models, such as the one employed in this work, have been widely adopted and shown to perform adequately for oxide-based nanofluids [10].

A variety of numerical methods have been applied to simulate natural convection in nanofluids at scales ranging from the macroscale to the microscale. These include the finite difference method [14], finite volume method [15, 16], finite element method [17], control volume finite element method [18], lattice Boltzmann method [19], and molecular dynamics simulations [20]. The main challenge in the application of these methods to the modeling of nanofluids lies in bridging the scale gap between micro- and macroscale physics [13], and in accurately representing microscale interactions between nanoparticles and the base fluid to capture the overall hydrodynamic behavior [21].

Within the realm of finite element methods (FEM), several formulations have been developed to address natural convection problems. General error estimates are provided in [22], while a least-squares Petrov-Galerkin mixed FEM was proposed in [23]. A modified Gauge-Uzawa FEM scheme was also introduced in [24]. Huang et al. [25] presented a quadratic equal-order stabilized FEM for

advection–diffusion equations; however, it does not satisfy the discrete *inf-sup* condition. Additionally, multigrid and multistep methods have been explored [17, 26, 27]. In multigrid methods, nonlinear equations are initially solved on a coarse mesh, and the solution is used to initialize the linear problem on a finer mesh. In multistep methods, the mesh remains unchanged, but multiple intermediate time steps are employed to resolve nonlinearities. Due to the staggered nature of these methods, the convergence rate is typically first order, and significant computational resources are often required. More recently, characteristic-based Variational Multiscale (VMS) methods have been developed for transient natural convection problems [28, 29]. These methods incorporate artificial diffusion for stabilization and use a first-order linear approximation for the nonlinear inertial terms. Furthermore, they implement a decoupling strategy based on Taylor-Hood elements to satisfy the discrete *inf-sup* condition, although the governing equations remain nonlinearly coupled.

Stabilized FEMs have demonstrated superior stability properties over standard Galerkin formulations in both fluid and solid mechanics, with substantial success in solving practical problems [30]. The development of VMS methods [31] provided a rigorous theoretical framework for stabilization, grounded in variational principles. A significant advancement was made by Masud and co-workers, who developed a new class of VMS methods for fluids [32] and solids [33] by leveraging bubble function approach to resolve the fine-scale variational equations. Elements of the Residual-Free Bubbles (RFB) method [34] were also incorporated to aid in the construction of bubble functions. This methodology was formalized in [35], offering a systematic approach for deriving the structure of stabilization terms and associated tensors necessary for enhanced stability and accuracy. Subsequent developments extended the VMS-based stabilized FEM to the Navier–Stokes equations [36] and to fluid systems involving scalar field equations for modeling moving embedded interfaces [37] and moving phase boundaries [38]. Mathematical analysis in the context of diffusion–reaction systems was presented in [39], and a generalization of the framework to finite strain thermomechanical problems was proposed in [40].

In this paper, we propose a robust and efficient numerical method based on the VMS framework that addresses the aforementioned challenges in a unified manner. Thermal stratification is incorporated via a buoyancy force term in the momentum equations, while the thermal phase is modeled using an advection–diffusion equation. Through scale separation and consistent linearization, a stabilized formulation is developed that fully couples the mechanical and thermal fields. The stabilization tensor is derived using a systematic procedure. The resulting variationally consistent formulation satisfies the discrete *inf-sup* condition for equal-order

elements and demonstrates quadratic convergence rate in the nonlinear iterative solution.

An outline of rest of the paper is as follows: Sect. 2 presents the governing transient equations for incompressible thermal fluids. Section 3 details the derivation of the fully coupled stabilized formulation using the VMS approach. Section 4 describes the numerical implementation of the method, including explicit expressions for the stabilization tensors for 2- and 3-noded elements. Section 5 evaluates the method numerically through four test cases of increasing complexity in flow physics. Section 5.1 examines the variational consistency of the method via a rigorous spatiotemporal convergence study involving different element types, polynomial orders, and time integration schemes. Section 5.3 explores natural convection of a nanofluid within a sinusoidal cavity, assessing the impact of geometric and flow parameters on hydrothermal behavior to determine optimal configurations for heat transfer. Section 5.4 investigates the influence of electromagnetic (Lorentz) forces on ferrofluid convection inside a cavity. Finally, concluding remarks are presented in Sect. 6. Additional implementation details, including the explicit formulation of stabilization tensors for 2-noded and 3-noded elements, are provided in Appendix 1.

2 Incompressible thermal fluid with Boussinesq buoyancy

2.1 Strong form

The influence of the spatiotemporally varying active temperature field $T(\mathbf{x}, t)$ results in a spatiotemporally varying density field $\rho(\mathbf{x}, t)$. In the context of incompressible Newtonian fluids, the effect of the change in density is accounted for by introducing a buoyancy force to the momentum balance equation, while the continuity equation acts as the incompressibility constraint. Furthermore, in addition to any local heat source, a background thermal gradient serves as the driving mechanism for the thermal phase. With these assumptions, the governing system also known as the Boussinesq equations [41] for density-stratified flows is written as:

$$\mathbf{u}_{,t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla \cdot (2\nu \nabla^s \mathbf{u}) - \mathbf{g}\beta\theta + \mathbf{f}_b \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

$$\theta_{,t} + \mathbf{u} \cdot \nabla \theta - \nabla \cdot (\alpha \nabla \theta) = f \quad (3)$$

where $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ are the velocity and kinematic pressure fields, respectively; $\theta = T - T_0$ is the relative temperature, T is the absolute temperature and T_0 is the reference temperature; ν is the kinematic viscosity of the fluid, β is the

thermal expansion coefficient, $\mathbf{g}$ is the gravitational acceleration vector, and α is the coefficient of thermal diffusivity; $\mathbf{f}_b$ is the non-gravitational body force (e.g., Lorentz force) and f is the heat source/sink. (1) is the momentum balance equation with a buoyancy term accounting for the thermal effects, (2) is the continuity equation to enforce the incompressibility condition, and (3) is the energy conservation equation in the form of the convection-diffusion of the relative temperature field.

The initial conditions in the domain Ω and boundary conditions on the domain boundary Γ are:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega \times \{0\} \quad (4)$$

$$\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}) \quad \text{in } \Omega \times \{0\} \quad (5)$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{g}_M \quad \text{on } \Gamma_g^M \times]0, T[\quad (6)$$

$$\theta(\mathbf{x}, t) = g_E \quad \text{on } \Gamma_g^E \times]0, T[\quad (7)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = (2\nu \nabla^s \mathbf{u} - p\mathbf{I}) \cdot \mathbf{n} = \mathbf{h}_M \quad \text{on } \Gamma_h^M \times]0, T[\quad (8)$$

$$\phi \cdot \mathbf{n} = \alpha \nabla \theta \cdot \mathbf{n} = h_E \quad \text{on } \Gamma_h^E \times]0, T[\quad (9)$$

where $\mathbf{u}_0$ and θ_0 are the initial conditions for velocity and relative temperature fields, respectively, $\mathbf{g}_M$ and g_E are the Dirichlet boundary conditions for the mechanical and thermal phases, $\mathbf{h}_M$ and h_E are the Neumann boundary conditions for the mechanical and thermal phases. $\boldsymbol{\sigma}$ is the total stress in the fluid and ϕ is the heat flux. $\mathbf{n}$ is the outward normal vector at the boundary. Specifically, these boundaries satisfy the following conditions: $\Gamma_g^M \cup \Gamma_h^M = 0$, $\Gamma_g^M \cup \Gamma_h^M = \Gamma$, $\Gamma_g^E \cap \Gamma_h^E = 0$ and $\Gamma_g^E \cup \Gamma_h^E = \Gamma$.

As described previously, we have adopted the single-phase modeling approach in this work. The continuity equation is assumed to be valid for the nanofluid with uniformly dispersed nanoparticles. The thermophysical properties of the resulting nanofluid are assumed to be temperature independent and are homogenized based on the volume fraction of nanoparticles. The homogenization equations for different thermophysical properties of the nanofluid are discussed below. The density ρ and heat capacity ρC_p are modeled via weighted averaging.

$$\begin{aligned} \rho_{nf} &= (1 - \phi)\rho_{bf} + \phi\rho_{np} \\ (\rho C_p)_{nf} &= (1 - \phi)(\rho C_p)_{bf} + \phi(\rho C_p)_{np} \end{aligned} \quad (10)$$

where the subscripts $(\cdot)_{nf}$, $(\cdot)_{bf}$, and $(\cdot)_{np}$ denote the corresponding material parameters of the nanofluid, the base fluid, and the nanoparticles, respectively. Brinkman's model

[42] is used to calculate the dynamic viscosity μ of the mixture, which is suitable for small rigid spherical particles.

$$\mu_{nf} = \frac{\mu_{bf}}{(1 - \phi)^{5/2}} \quad (11)$$

The thermal conductivity k of the nanofluid is modeled using Maxwell-Garnett model [1] as follows.

$$\frac{k_{nf}}{k_{bf}} = \frac{(k_{np} + 2k_{bf}) - 2\phi(k_{bf} - k_{np})}{(k_{np} + 2k_{bf}) + \phi(k_{bf} - k_{np})} \quad (12)$$

Accordingly, the kinematic viscosity ν and thermal diffusivity ρC_p are calculated as follows.

$$\begin{aligned} \nu_{nf} &= \frac{\mu_{nf}}{\rho_{nf}} \\ \alpha_{nf} &= \frac{k_{nf}}{(\rho C_p)_{nf}} \end{aligned} \quad (13)$$

The thermal expansion β_{nf} is scaled as follows [16].

$$\beta_{nf} = \frac{(1 - \phi)(\rho\beta)_{bf} + \phi(\rho\beta)_{np}}{\rho_{nf}} \quad (14)$$

There are two dimensionless parameters, namely, Prandtl number (Pr) and Rayleigh number (Ra) that are used to non-dimensionalize the governing equations (1)–(3). By introducing the characteristic length scale L_0 , temperature scale ΔT , and velocity scale U_0 , the two classic non-dimensional numbers are defined as,

$$Pr = \frac{\nu}{\alpha} \quad \text{Prandtl Number} \quad (15)$$

$$Ra = \frac{\beta \Delta T L_0^3 |g|}{\nu \alpha} \quad \text{Rayleigh Number} \quad (16)$$

Using the above defined parameters along with the homogenized thermophysical properties of a nanofluid, the governing equations (1)–(3) can be non-dimensionalized by defining kinematic viscosity in (1) as $\nu = Pr \nu_{nf}/\nu_{bf}$, the thermal diffusivity in (3) as $\alpha = \alpha_{nf}/\alpha_{bf}$, and the thermal expansion coefficient in (1) as $\beta = Ra Pr \beta_{nf}/\beta_{bf}$. The velocity fields are non-dimensionalized using the characteristic scale L/α , while time is non-dimensionalized using L^2/α .

2.2 Standard weak form

The appropriate functional spaces $\mathcal{U}_t, \mathcal{P}_t, \mathcal{T}_t$ for the velocity, pressure, and temperature trial solutions are time dependent counterparts of (17)–(19):

$$\mathcal{U} = \left\{ \mathbf{u} \mid \mathbf{u} \in (H^1(\Omega))^{n_{sd}}, \mathbf{u} = \mathbf{g}_M \text{ on } \Gamma_g^M \right\} \quad (17)$$

$$\mathcal{P} = \{p \mid p \in (L_2(\Omega))^{n_{sd}}\} \quad (18)$$

$$\mathcal{T} = \left\{ \theta \mid \theta \in (H^1(\Omega))^{n_{sd}}, \theta = g_E \text{ on } \Gamma_g^E \right\} \quad (19)$$

where $L_2(\Omega)$ and $H^1(\Omega)$ are the standard Sobolev spaces, and n_{sd} is the number of spatial dimensions. Let $\mathbf{w}(\mathbf{x}) \in \mathcal{W} = (H_0^1(\Omega))^{n_{sd}}$, $q(\mathbf{x}) \in \mathcal{Q} = C^0(\Omega) \cap L^2(\Omega)$, and $\eta(\mathbf{x}) \in \mathcal{H} = H_0^1(\Omega)$ be the weighting functions for the velocity $\mathbf{u}$, kinematic pressure p , and relative temperature θ , respectively. The weighting function spaces $\mathcal{W}, \mathcal{Q}, \mathcal{H}$ are the corresponding spaces of (17)–(19) that satisfy the homogeneous part of the essential boundary conditions.

The standard weak form of the problem is: Find $\mathbf{V} = \{\mathbf{u}, p, \theta\} \in \mathcal{U}_t \times \mathcal{P}_t \times \mathcal{T}_t$, such that $\forall \mathbf{W} = \{\mathbf{w}, q, \eta\} \in \mathcal{W} \times \mathcal{Q} \times \mathcal{H}$

$$\begin{aligned} (\mathbf{w}, \mathbf{u}_{,t}) + (\mathbf{w}, \mathbf{u} \cdot \nabla \mathbf{u}) + (\nabla \mathbf{w}, 2\nu \nabla^s \mathbf{u}) \\ - (\nabla \cdot \mathbf{w}, p) + (\mathbf{w}, \mathbf{g}\beta\theta) = (\mathbf{w}, \mathbf{h}_M)_{\Gamma_h^M} + (\mathbf{w}, \mathbf{f}_b) \end{aligned} \quad (20)$$

$$(q, \nabla \cdot \mathbf{u}) = 0 \quad (21)$$

$$(\eta, \theta_{,t}) + (\eta, \mathbf{u} \cdot \nabla \theta) + (\nabla \eta, \alpha \nabla \theta) = (\eta, h_E)_{\Gamma_h^E} + (\eta, f) \quad (22)$$

where $(\cdot, \cdot) = \int_{\Omega} (\cdot) d\Omega$ denotes the $L_2(\Omega)$ inner product.

3 Variationally derived stabilized method

The bounded domain Ω is discretized into non-overlapping subdomains $\tilde{\Omega}$ with subdomain boundary $\tilde{\Gamma}$, such that $\Omega = \bigcup_1^n \tilde{\Omega}$ and n is the total number of subdomains that comprise Ω . The union of interior of subdomains is denoted as $\Omega' = \bigcup_1^n (\text{int}) \tilde{\Omega}$, and the union of subdomain boundaries is $\Gamma' = \bigcup_1^n \tilde{\Gamma}$. Using the scale split feature of the VMS framework, we decompose the trial solution fields $\{\mathbf{u}, \theta\}$ into coarse and fine scales. In the context of nonlinear problems, this decomposition is to be viewed as a projection of sub-grid scales onto resolved scales, i.e., $\mathbf{u}' = \mathbb{P}(\mathbf{u} - \bar{\mathbf{u}})$.

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}) \quad \theta(\mathbf{x}, t) = \bar{\theta}(\mathbf{x}, t) + \theta'(\mathbf{x}) \quad (23)$$

The spaces of coarse scale functions are linearly independent of the space of functions for the fine scales, i.e., $\bar{\mathbf{u}}(\mathbf{x}, t) \in \bar{\mathcal{U}}_t$ and $\mathbf{u}'(\mathbf{x}, t) \in \mathcal{U}'_t$ where $\bar{\mathcal{U}}_t \oplus \mathcal{U}'_t = \mathcal{U}_t$ and $\bar{\mathcal{U}}_t \cap \mathcal{U}'_t = \emptyset$. The same holds for the relative temperature field. Similarly, the corresponding weighting functions $(\mathbf{w}, \eta)$ are decomposed as:

$$\mathbf{w}(\mathbf{x}) = \bar{\mathbf{w}}(\mathbf{x}) + \mathbf{w}'(\mathbf{x}) \quad \eta(\mathbf{x}) = \bar{\eta}(\mathbf{x}) + \eta'(\mathbf{x}) \quad (24)$$

By the additive nature of the decomposition of weighting function in (24), we can split the weak form into coarse-scale and fine-scale problems through grouping the terms depending on the weighting functions of either scale:

Coarse-scale coupled problem

$$(\bar{\mathbf{w}}, \mathbf{u}_t) + (\bar{\mathbf{w}}, \mathbf{u} \cdot \nabla \mathbf{u}) + (\nabla \bar{\mathbf{w}}, 2\nu \nabla^s \mathbf{u}) - (\nabla \cdot \bar{\mathbf{w}}, p) + (\bar{\mathbf{w}}, \mathbf{g}\beta\theta) = (\bar{\mathbf{w}}, \mathbf{h}_M)_{\Gamma_h^M} + (\bar{\mathbf{w}}, \mathbf{f}_b) \quad (25)$$

$$(q, \nabla \cdot \mathbf{u}) = 0 \quad (26)$$

$$(\bar{\eta}, \theta_t) + (\bar{\eta}, \mathbf{u} \cdot \nabla \theta) + (\nabla \bar{\eta}, \alpha \nabla \theta) = (\bar{\eta}, h_E)_{\Gamma_h^E} + (\bar{\eta}, f) \quad (27)$$

Fine-scale coupled problem

$$(\mathbf{w}', \mathbf{u}_t) + (\mathbf{w}', \mathbf{u} \cdot \nabla \mathbf{u}) + (\nabla \mathbf{w}', 2\nu \nabla^s \mathbf{u}) - (\nabla \cdot \mathbf{w}', p) + (\mathbf{w}', \mathbf{g}\beta\theta) = (\mathbf{w}', \mathbf{f}_b) \quad (28)$$

$$(\eta', \theta_t) + (\eta', \mathbf{u} \cdot \nabla \theta) + (\nabla \eta', \alpha \nabla \theta) = (\eta', f) \quad (29)$$

where fine-scale fields are assumed to be zero on the element boundary.

3.1 Derivation of fine scale models for velocity and temperature fields

The key idea at this point is to resolve (28) and (29) locally and extract the models for the time-dependent fine-scale velocity and temperature fields. These models when embedded in the coarse scale formulation provides stability in the sense of *inf-sup* condition as well as high advection velocity. These sub-grid models also project missing physics onto the resolved scales and improve the accuracy of the formulation. The fine-scale sub-problem presented in equations (28) and (29) yields a nonlinear coupled system. Therefore, we first linearize the fine-scale variational equations with respect to the fine-scale velocity and relative temperature fields as follows:

$$(\mathbf{w}', \delta \mathbf{u}'_t) + (\mathbf{w}', \delta \mathbf{u}' \cdot \nabla \bar{\mathbf{u}}) + (\mathbf{w}', \bar{\mathbf{u}} \cdot \nabla \delta \mathbf{u}') + (\nabla \mathbf{w}', 2\nu \nabla^s \delta \mathbf{u}') + (\mathbf{w}', \mathbf{g}\beta \delta \theta') = -(\mathbf{w}', \bar{\mathbf{r}}_M) \quad (30)$$

$$(\eta', \delta \theta'_t) + (\eta', \delta \mathbf{u}' \cdot \nabla \bar{\theta}) + (\eta', \bar{\mathbf{u}} \cdot \nabla \delta \theta') + (\nabla \eta', \alpha \nabla \delta \theta') = -(\eta', \bar{r}_E) \quad (31)$$

where $\bar{\mathbf{r}}_M = \mathbf{r}_M(\bar{\mathbf{u}}, p, \bar{\theta})$ is the residual of the Euler-Lagrange equations of the coarse-scale conservation of momentum, and $\bar{r}_E = r_E(\bar{\mathbf{u}}, \bar{\theta})$ is the residual of the Euler-Lagrange

equation of the coarse-scale conservation of energy. These are defined as,

$$\mathbf{r}_M(\bar{\mathbf{u}}, p, \bar{\theta}) = \bar{\mathbf{u}}_t + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \nabla p - \nabla \cdot (2\nu \nabla^s \bar{\mathbf{u}}) + \mathbf{g}\beta \bar{\theta} - \mathbf{f}_b \quad (32)$$

$$r_E(\bar{\mathbf{u}}, \bar{\theta}) = \bar{\theta}_t + \bar{\mathbf{u}} \cdot \nabla \bar{\theta} - \nabla \cdot (\alpha \nabla \bar{\theta}) - f \quad (33)$$

In addition, to make the fine scale problem tractable, we assume that the fine-scale weighting functions and trial solutions vanish at the subdomain boundaries, namely, $\mathbf{w}' = \mathbf{u}' = \mathbf{0}$ on Γ' and $\theta' = \eta' = 0$ on Γ' . By resolving the fine-scale problem, we construct the projection from the residual of the coarse-scale Euler-Lagrange equations to the fine-scale solution fields $\mathbf{V}' = \{\mathbf{u}', \theta'\}$. We follow along the lines of [35] and employ bubble functions $b^{\tilde{\Omega}}$ defined over $\tilde{\Omega}$ to interpolate the fine-scale trial solutions and weighting functions as follows.

$$\delta \mathbf{u}' = b^{\tilde{\Omega}} \mathbf{u}'^{\tilde{\Omega}} \quad \mathbf{w}' = b^{\tilde{\Omega}} \mathbf{w}'^{\tilde{\Omega}} \quad \delta \theta' = b^{\tilde{\Omega}} \theta'^{\tilde{\Omega}} \quad \eta' = b^{\tilde{\Omega}} \eta'^{\tilde{\Omega}} \quad (34)$$

We substitute the above defined fine-scale trial solutions into (30) and (31), apply mean value theorem, and utilize backward Euler time marching scheme to get fine scale solution that is scaled by the stabilization tensor. For ease of notation, we set subdomain based bubble functions $b^{\tilde{\Omega}}$ as b and remove the superscript $\tilde{\Omega}$ in the subsequent formulation.

$$\begin{aligned} \begin{bmatrix} \delta \mathbf{u}' \\ \delta \theta' \end{bmatrix}_{n+1}^{\tilde{\Omega}} &= b \begin{bmatrix} \delta \mathbf{u}' \\ \delta \theta' \end{bmatrix}_{n+1}^{\tilde{\Omega}} \\ &= \underbrace{-b(b, 1) \left\{ \frac{\mathbf{M}}{\Delta t} + \begin{bmatrix} \hat{\tau}_M & \hat{\tau}_{ME} \\ \hat{\tau}_{EM} & \hat{\tau}_E \end{bmatrix} \right\}^{-1}}_{\boldsymbol{\tau}} \begin{bmatrix} \bar{\mathbf{r}}_M \\ \bar{r}_E \end{bmatrix}_{n+1}^{\tilde{\Omega}} \end{aligned} \quad (35)$$

where $\mathbf{M}$ is the mass matrix and Δt is the time step. The explicit form of various terms in (35) are:

$$\hat{\tau}_{M(n_{sd} \times n_{sd})} = \int_{\Omega^e} (b \bar{\mathbf{u}} \cdot \nabla b + (b)^2 \nabla^T \bar{\mathbf{u}} + \nu \nabla b \cdot \nabla b + \nu \nabla b \otimes \nabla b) d\Omega \quad (36)$$

$$\hat{\tau}_{E(1 \times 1)} = \int_{\Omega^e} (b \bar{\mathbf{u}} \cdot \nabla b + \alpha \nabla b \cdot \nabla b) d\Omega \quad (37)$$

$$\hat{\tau}_{ME(n_{sd} \times 1)} = \int_{\Omega^e} \mathbf{g}\beta(b)^2 d\Omega \quad (38)$$

$$\hat{\tau}_{EM(1 \times n_{sd})} = \int_{\Omega^e} (b)^2 \nabla^T \bar{\theta} d\Omega \quad (39)$$

where $\boldsymbol{\tau}$ is the discrete system that relates the fine-scale velocity and temperature with the residual of coarse scale momentum balance and energy conservation equations. The physical interpretation of $\hat{\tau}_M$ includes the sweeping effect

(i.e., the fine-scale velocity transported by the coarse-scale velocity), the distortion effect (i.e., the coarse-scale velocity transported by the coarse-scale velocity), and the fine-scale diffusion; $\hat{\tau}_E$ is the discrete system that relates the fine-scale temperature and the residual of the coarse-scale equation of energy conservation. They are form identical to the discrete fine-scale system of linearized Navier–Stokes equations and a generic advection–diffusion equation from our earlier works [32, 36, 37].

Remark 1 An explicit form of the stabilization tensor in (35) is presented in Appendix 1. for 2-noded and 3-noded linear elements. Quadratic and cubic bubble functions are used for 2-noded and 3-noded elements, respectively. To compute the stabilization tensor of 4-noded bilinear elements and 3-noded linear triangles, 4×4 Gaussian quadrature rule is employed. To avoid the zero value in the diagonal term due to the orthogonality of standard bubble and its derivatives, we adopt the idea of advection bubble in the weighting slot of the skew advection term, as presented in [32, 36, 37].

3.2 Stabilized formulation

Similarly, we linearize the nonlinear coarse-scale sub-problems in (25)–(27) with respect to the fine-scale velocity and temperature fields, combine the resulting formulation and group the terms that depend on the fine scale fields.

$$\begin{aligned} & (\bar{\mathbf{w}}, \bar{\mathbf{u}}_t) + (\bar{\mathbf{w}}, \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}) + (\nabla \bar{\mathbf{w}}, 2\nu \nabla^s \bar{\mathbf{u}}) + (q, \nabla \cdot \bar{\mathbf{u}}) \\ & - (\nabla \cdot \bar{\mathbf{w}}, p) + (\bar{\eta}, \bar{\theta}_t) + (\bar{\eta}, \bar{\mathbf{u}} \cdot \nabla \bar{\theta}) + (\nabla \bar{\eta}, \alpha \nabla \bar{\theta}) \\ & + (\bar{\mathbf{w}}, \mathbf{g}\beta\bar{\theta}) + (\chi_M, \delta \mathbf{u}') + (\chi_E, \delta \theta') = (\bar{\mathbf{w}}, \mathbf{h}_M)_{\Gamma_h^M} \\ & + (\bar{\mathbf{w}}, \mathbf{f}_b) + (\bar{\eta}, h_E)_{\Gamma_h^E} + (\bar{\eta}, f) \end{aligned} \quad (40)$$

where,

$$\begin{aligned} \chi_M &= -\bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{w}} + \bar{\mathbf{w}} \cdot \nabla^T \bar{\mathbf{u}} - \nu(\nabla(\nabla \cdot \bar{\mathbf{w}}) + \Delta \bar{\mathbf{w}}) - \nabla q + \bar{\eta} \nabla^T \bar{\theta} \\ \chi_E &= \mathbf{g}\beta \bar{\mathbf{w}} - \bar{\mathbf{u}} \cdot \nabla \bar{\eta} - \alpha \Delta \bar{\eta} \end{aligned}$$

Remark 2 Equation (40) is fully represented in terms of the coarse-scale fields, therefore, we drop the superposed bar from the coarse-scale weighting function and trial solutions and perturbation notation δ before fine-scale trial solution fields.

By substituting the fine-scale solutions defined in (35) into the corresponding slots in coarse-scale formulation, we arrive at the final stabilized form that is expressed in the residual form as follows. Formal

Table 1 Thermophysical properties of the base fluid and the nanoparticles

Property	Water	Cu	Fe ₃ O ₄	Al ₂ O ₃
Specific heat, C_p (J/kgK)	4179	385	670	686.2
Thermal conductivity, k (W/mK)	0.613	401	6	8.9538
Thermal diffusivity, α (10^{-7} m ² /s)	1.47	1163.1	17.22	30.02
Thermal expansion, β (10^{-5} K ⁻¹)	21	1.67	1.30	0.85
Density, ρ (kg/m ³)	997.1	8933	5200	3970

statement is: Find $\mathbf{V} = \{\mathbf{u}, p, \theta\} \in \mathcal{U}_t \times \mathcal{P}_t \times \mathcal{T}_t$, such that $\forall \mathbf{W} = \{\mathbf{w}, q, \eta\} \in \mathcal{W} \times \mathcal{Q} \times \mathcal{H}$, the following holds

$$\mathcal{R}^{Gal}(\mathbf{w}, q, \eta; \mathbf{u}, p, \theta) + \mathcal{R}^{VMS}(\mathbf{w}, q, \eta; \mathbf{u}, p, \theta) = \mathcal{F}^{Gal}(\mathbf{w}, q, \eta) \quad (41)$$

where,

$$\begin{aligned} \mathcal{R}^{Gal}(\mathbf{w}, q, \eta; \mathbf{u}, p, \theta) &= \mathcal{R}_M^{Gal}(\mathbf{w}; \mathbf{u}, \theta) \\ &\quad + \mathcal{R}_C^{Gal}(q; \mathbf{u}) + \mathcal{R}_E^{Gal}(\eta; \mathbf{u}, \theta) \end{aligned}$$

$$\mathcal{R}^{VMS}(\mathbf{w}, \eta; \mathbf{u}, \theta) = \left([\chi_M, \chi_E], \tau[\mathbf{r}_M, r_E] \right)^T$$

$$\begin{aligned} \mathcal{R}_M^{Gal}(\mathbf{w}; \mathbf{u}, \theta) &= (\mathbf{w}, \mathbf{u}_t) + (\mathbf{w}, \mathbf{u} \cdot \nabla \mathbf{u}) + (\nabla \mathbf{w}, 2\nu \nabla^s \mathbf{u}) \\ &\quad - (\nabla \cdot \mathbf{w}, p) + (\mathbf{w}, \mathbf{g}\beta\theta) \end{aligned}$$

$$\mathcal{R}_E^{Gal}(\eta; \mathbf{u}, \theta) = (\eta, \theta_t) + (\eta, \mathbf{u} \cdot \nabla \theta) + (\nabla \eta, \alpha \nabla \theta)$$

$$\mathcal{R}_C^{Gal}(q; \mathbf{u}) = (q, \nabla \cdot \mathbf{u})$$

$$\mathcal{F}^{Gal}(\mathbf{w}, q, \eta) = (\mathbf{w}, \mathbf{h}_M)_{\Gamma_h^M} + (\mathbf{w}, \mathbf{f}_b) + (\eta, h_E)_{\Gamma_h^E} + (\eta, f)$$

Remark 3 For computational expediency, in the development of the discrete algebraic problem, we set $\tilde{\Omega} \cong \Omega^e$ where Ω^e is the domain of any generic element in the mesh. In this case, $b^{\tilde{\Omega}}$ is represented by polynomial bubble b^e defined over Ω^e such that $b^e = 0$ on the boundary of Ω^e .

4 Numerical implementation of the stabilized finite element method

Variational consistency of the model lends itself to consistent linearization that results in quadratic convergence of the solution in the Newton–Raphson loop within each load step (Tables 3 and 4). A direct solver (SuperLU) is employed to solve the discretized linear system in each iteration within the Newton–Raphson loop where the tolerance criterion for the initial residual is set as 10^{-12} . The algorithmic form of the solution procedure is presented as Reference for Algorithm 1.

Algorithm 1 Implementation of the proposed method using generalized- α time marching scheme.

Input: initial guess $\{\mathbf{u}_0, p_0, \theta_0\}$
Time Step Loop $n \leftarrow n + 1$
 Initialize the iteration counter $i \leftarrow 0$
 Initialize the current timestep w.r.t last converged solution: $\mathbf{u}_i^{n+1} = \mathbf{u}^n$; $p_i^{n+1} = p^n$; $\theta_i^{n+1} = \theta^n$
 Predictors: $(\mathbf{u}, t)_i^{n+1} = (\gamma - 1)/\gamma (\mathbf{u}, t)^n$; $(\theta, t)_i^{n+1} = (\gamma - 1)/\gamma (\theta, t)^n$
 Compute initial residual $R = \|\mathbf{R}_0\|_2$ from (41)
Newton-Raphson Loop $i \leftarrow i + 1$
while: $\|\mathbf{R}_i^{n+1}\|_2 \geq \text{tol}$. R_0 do
 Compute stabilization tensor $\boldsymbol{\tau}_i$ in (35) using (36)–(39), where i is the iteration counter.
 Compute the residual $\mathbf{R}_i^{n+1}$ using (41).
 Compute consistent tangent $\mathbf{K}_i^{n+1} = -\partial \mathbf{R}_i^{n+1} / \partial \mathbf{V}_i^{n+1}$, $\mathbf{V}$ is the vector of incremental velocity, pressure, and temperature fields.
 Solve linearized system: $\mathbf{K}_i^{n+1} [\Delta \mathbf{u}_i^{n+1}; \Delta p_i^{n+1}; \Delta \theta_i^{n+1}] = \mathbf{R}_i^{n+1}$.
 Update solution fields: $\mathbf{u}_{i+1}^{n+1} = \mathbf{u}_i^{n+1} + \Delta \mathbf{u}_i^{n+1}$; $p_{i+1}^{n+1} = p_i^{n+1} + \Delta p_i^{n+1}$; $\theta_{i+1}^{n+1} = \theta_i^{n+1} + \Delta \theta_i^{n+1}$.
 Correctors: $(\mathbf{u}, t)_{i+1}^{n+1} = (\mathbf{u}, t)_i^{n+1} + 1/(\gamma \Delta t) \Delta \mathbf{u}_i^{n+1}$; $(\theta, t)_{i+1}^{n+1} = (\theta, t)_i^{n+1} + 1/(\gamma \Delta t) \Delta \theta_i^{n+1}$.
 Update the iteration counter: $i \leftarrow i + 1$
end
Output: $\mathbf{u}^{n+1}, p^{n+1}, \theta^{n+1}, (\mathbf{u}, t)^{n+1}, (\theta, t)^{n+1}$ at current time step $n + 1$

The parameter γ in generalized α method is computed as $\gamma = 0.5 - \alpha_m - \alpha_f$, where $\alpha_m = (2\rho_\infty - 1)/(\rho_\infty + 1)$, $\alpha_f = \rho_\infty/(\rho_\infty + 1)$, and $0 \leq \rho_\infty \leq 1$.

5 Numerical tests

This section presents numerical test cases that establish variational consistency and accuracy of the method, and provides validation of the computed physics. The unique feature of stabilized methods for mixed field problems is that they are free of the constraint imposed by the *inf-sup* stability conditions on the admissible discrete spaces of functions for the various fields. The stabilized formulation developed herein results in a fully coupled method where all the unknown fields are solved in a monolithic fashion while using equal-order interpolation functions for the velocity, pressure, and temperature fields.

We investigate different types of nanofluids. Table 1 shows thermophysical properties of water and various nanoparticles used in this work. The properties of the nanofluids used are homogenized using (10)–(14) and the non-dimensionalized equations are used as described in Sect. 2.1. For all the nanofluids used in the numerical section, water is considered as the base fluid.

5.1 Beltrami flow with analytical solution

This test problem investigates the variational consistency of the proposed stabilized formulation via convergence rate study using a sub-class of the Beltrami flow [43]. The analytical solution of velocity, pressure and the relative temperature field of the flow is as follows:

$$\mathbf{u}(x, y, t) = [-\cos(\pi x) \sin(\pi y) \sin(2t), \sin(\pi x) \cos(\pi y) \sin(2t)]^T \quad (42)$$

$$p(x, y, t) = -\frac{1}{4}(\cos(2\pi x) + \cos(2\pi y))(\sin(2t))^2 \quad (43)$$

$$\theta(x, y, t) = \cos(\pi x) \cos(\pi y) \sin(2t) \quad (44)$$

and the non-gravitational body force and heat source driving the flow are:

$$\mathbf{f}_b(x, y, t) = \begin{bmatrix} \left\{ \begin{array}{l} -2 \cos(\pi x) \sin(\pi y) (\cos(2t) + \pi^2 v \sin(2t)) \\ + g_1 \beta \cos(\pi x) \cos(\pi y) \sin(2t) \end{array} \right\} \\ \left\{ \begin{array}{l} 2 \sin(\pi x) \cos(\pi y) (\cos(2t) + \pi^2 v \sin(2t)) \\ + g_2 \beta \cos(\pi x) \cos(\pi y) \sin(2t) \end{array} \right\} \end{bmatrix} \quad (45)$$

$$f(x, y, t) = 2 \cos(\pi x) \cos(\pi y) (\cos(2t) + \pi^2 \alpha \sin(2t)) \quad (46)$$

A square 2D domain $\Omega = [-1, 1] \times [-1, 1]$ is used with Dirichlet boundary conditions $\mathbf{g}(\mathbf{x}) = [\mathbf{u}, p, \theta]_{\mathbf{x} \in \Gamma}$ applied at the domain boundary Γ for all solution fields (i.e., velocities, pressure, and relative temperature). All the material parameters are assumed to be unity and a unit gravitational force is applied in the negative y direction. A comprehensive temporal and spatial convergence study is carried out that employs different time integration schemes, element types, and polynomial order of interpolation functions. The error is measured in terms of L_2 norm and H^1 seminorm in space and L_2 norm in time as follows:

$$\|e\|_{L_2 L_2} = \left(\sum_{n=1}^N \Delta t \left\| \mathbf{u}(\mathbf{x}, t_n) - \mathbf{u}^h(\mathbf{x}, t_n) \right\|_{L_2(\Omega)}^2 \right)^{1/2} \quad (47)$$

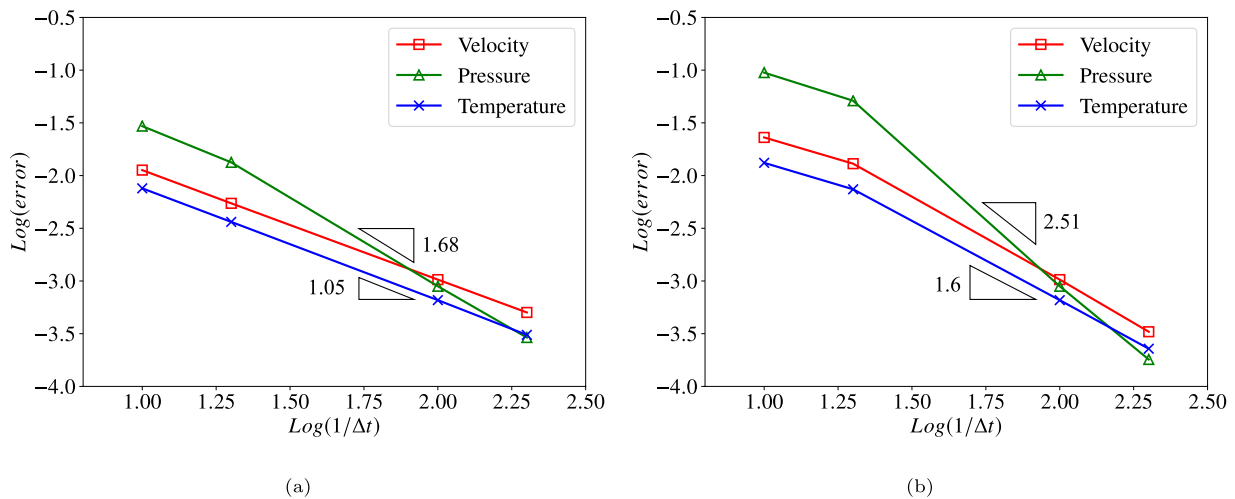


Fig. 1 Temporal convergence rate plots for **a** backward Euler, and **b** generalized alpha time integration schemes

$$\|e\|_{H^1 L_2} = \left(\sum_{n=1}^N \Delta t \left\| \mathbf{u}(\mathbf{x}, t_n) - \mathbf{u}^h(\mathbf{x}, t_n) \right\|_{H^1(\Omega)}^2 \right)^{1/2} \quad (48)$$

Figure 1a and b show convergence rates of L_2 norm of error for backward Euler and generalized- α time integration schemes, respectively. A uniform spatial grid of 100×100 is employed for the convergence study. It can be observed that all the solution fields converge optimally (> 1) for the first order backward Euler scheme. For the second order generalized alpha method, pressure converges optimally while velocity and temperature fields have an asymptotic convergence rate of 1.6.

Then a spatial convergence study with grid resolutions of 10×10 , 20×20 , 40×40 and 80×80 linear hexahedral

elements is performed. Since the problem is two-dimensional, the third dimension is constrained through periodic boundary conditions. Backward Euler time integration scheme with a time step of 0.005 is used for marching the solution to a final time level of 1.0. Figure 2a and b show the convergence rates for the standard L_2 norm and the H^1 seminorm of error at the final time level, respectively. For all the solution fields, convergence rates are optimal i.e., 2.0 for L_2 norm and 1.0 for H^1 seminorm.

We also investigated spatial convergence with other element types. A 6-node linear wedge element with the same grid resolutions as that for the hexahedral elements in the previous case is employed. Periodic boundary conditions are applied in the third dimension to run the problem in 2D. A time step of 0.005 is chosen to run the

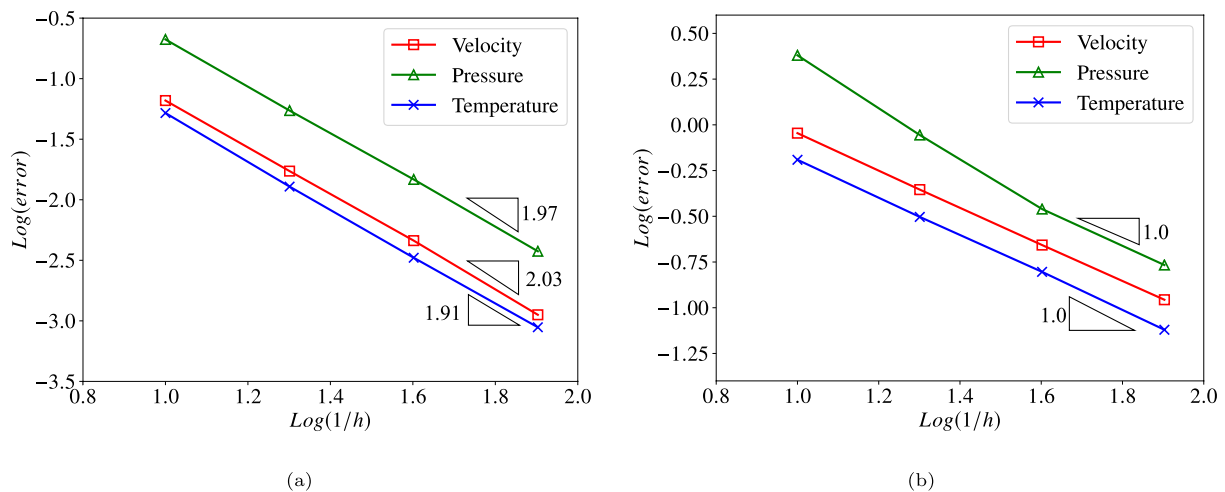


Fig. 2 Spatial convergence rate plots of **a** L_2 norm, and **b** H^1 seminorm of error for linear hexahedral elements

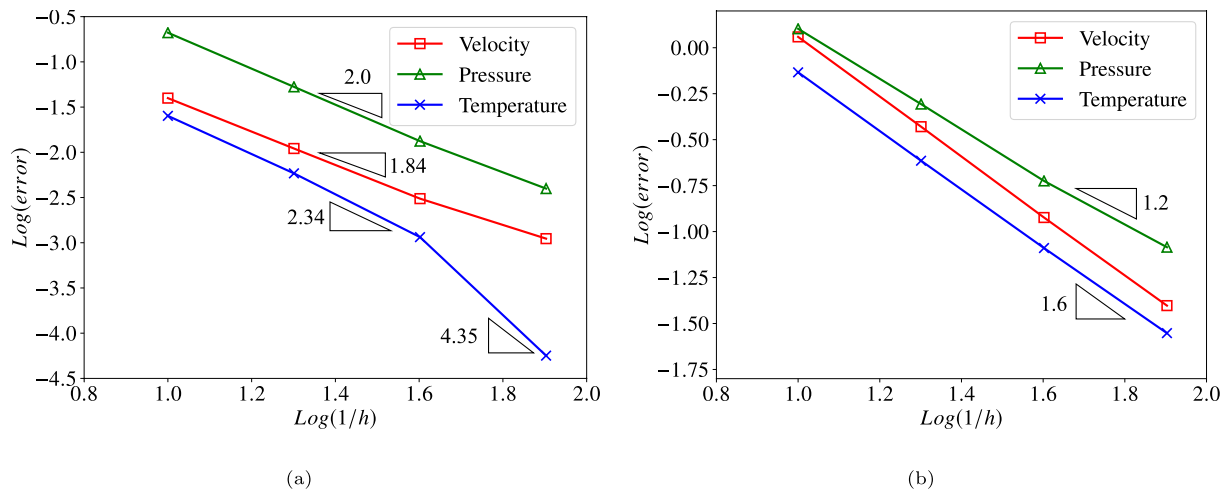


Fig. 3 Spatial convergence rate plots of **a** L_2 norm, and **b** H^1 seminorm of error for linear wedge elements

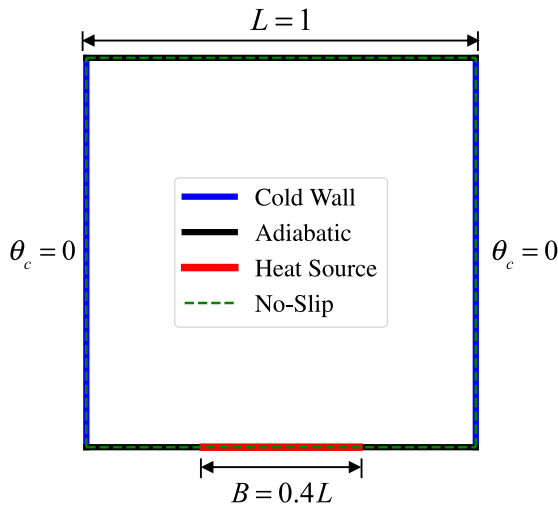


Fig. 4 Schematic diagram and boundary conditions of nanofluid filled square cavity test problem

simulation to a final time level of 1.0 and then standard error norms are computed. Figure 3a, b show optimal convergence rates (2.0 and 1.0) for L_2 norm and H^1 seminorm of error, respectively. This is a numerical verification of the variational consistency of the method that leads to mathematically predicted convergence rates across different element types and time integration schemes.

5.2 Nanofluid-filled square cavity with a heat source

This test case explores the attributes of the proposed method for unsteady natural convection in a nanofluid-filled square enclosure cavity with a heat source at the bottom. The problem domain comprises of a unit square $[0, 1] \times [0, 1]$ as shown in Fig. 4. A heat source of length $B = 0.4L$ is located at the center of the bottom wall, and a unit thermal flux is applied at the bottom of the square cavity which serves as the heat source. No-slip boundary condition for velocity is prescribed at all the walls. For the temperature field, vertical walls are kept at a relatively lower temperature while adiabatic boundary conditions are applied at the horizontal walls. The value of Prandtl number is chosen as 6.2. All the solution fields are initially set equal to zero. Based on the grid independence study shown in Table 2, a uniform grid resolution of 60×60 and a time step of 5×10^{-4} is used in the subsequent simulations. The heat transfer performance is measured in terms of mean Nusselt number which is defined as follows:

$$Nu_s(X) = \frac{1}{\theta_s(X)} \quad (49)$$

$$Nu_m = \frac{1}{B} \int_{0.5-0.5B}^{0.5+0.5B} Nu_s(X) dX \quad (50)$$

Table 2 Grid independence study (Cu-water, $\phi = 10\%$, $Ra = 10^7$)

Mesh resolution	30×30	40×40	50×50	60×60	70×70	80×80
$\theta_{\max}$	0.0813	0.0793	0.0781	0.0772	0.0765	0.0760
Nu_m	20.617	20.941	21.178	21.355	21.441	21.490

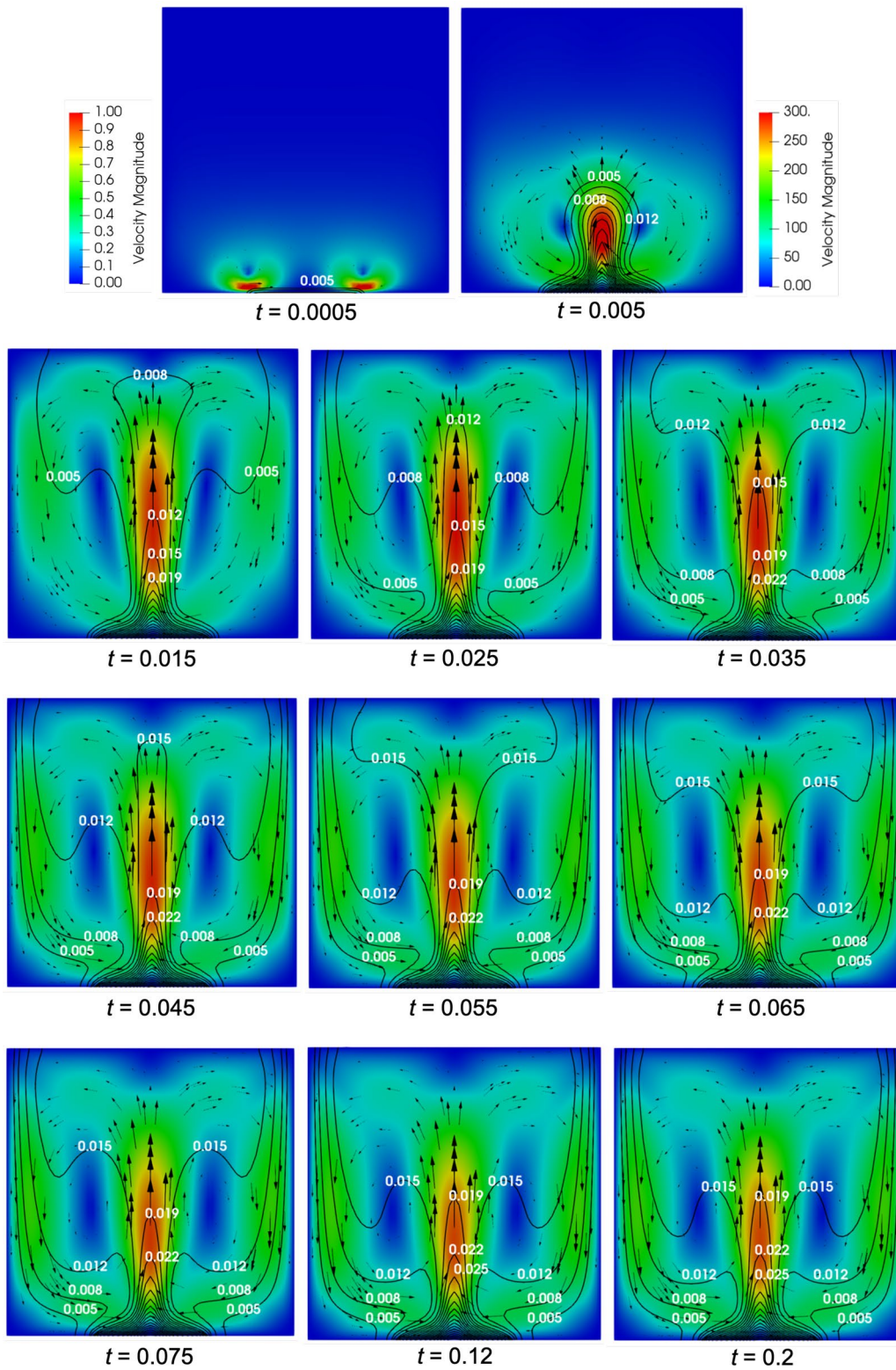


Fig. 5 Isotherms colored with velocity magnitude and local flow direction at various time levels (Cu-water, $\phi = 10\%$, $Ra = 10^7$). Time is non-dimensionalized by L^2/α as detailed at the end of Sect. 2.1

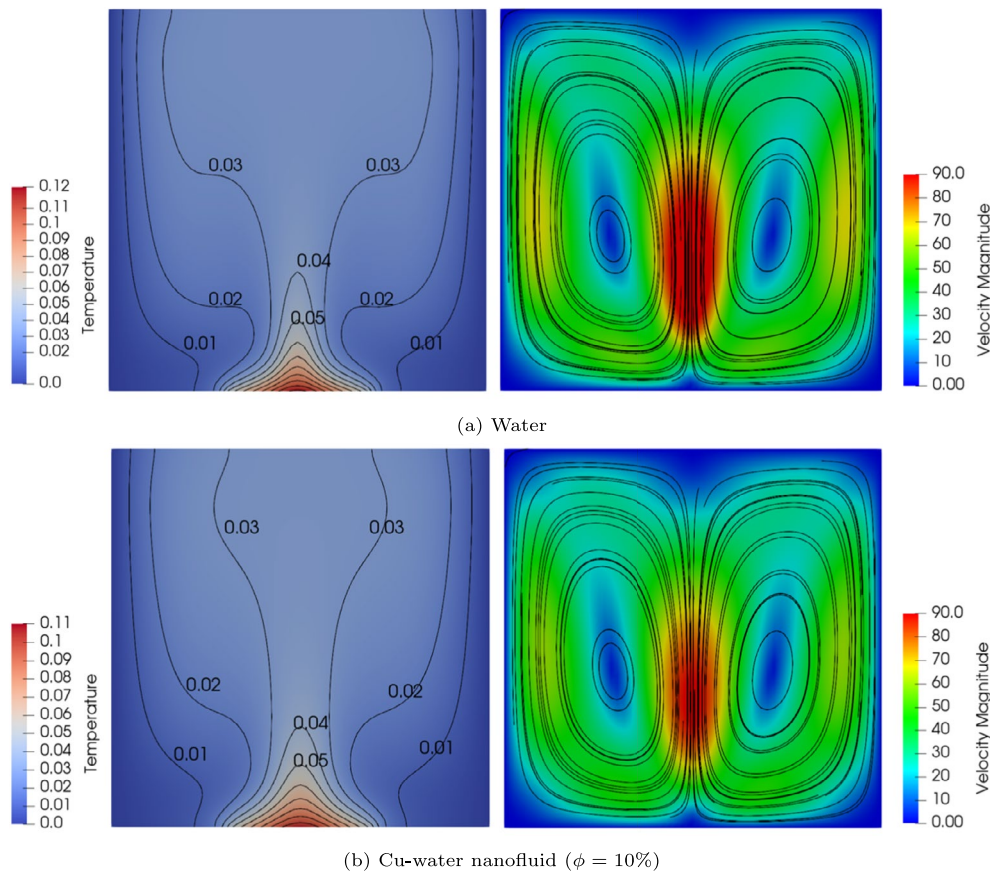


Fig. 6 Isotherms and velocity streamlines for $Ra = 10^6$ at steady state stage

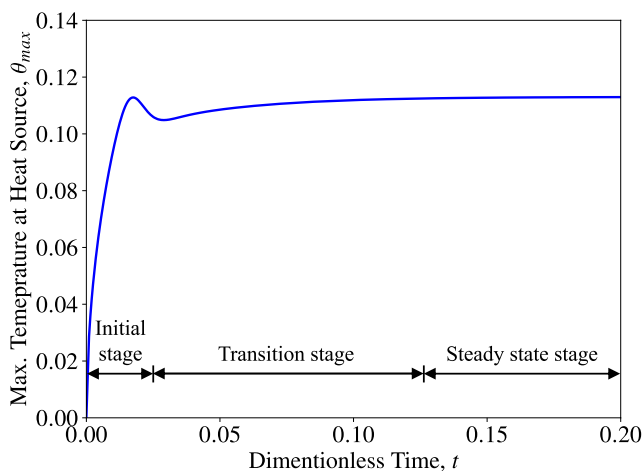


Fig. 7 Flow development of Cu-water nanofluid at $Ra = 10^6$

where θ_s is the dimensionless temperature at the heat source.

Figure 5 shows the development and transition of the thermal layer inside the cavity at a Rayleigh number 10^7 . Initially a thermal plume rises rapidly upward from the heat source ($t = 0.005$) and spreads outwards after reaching the top

adiabatic wall. As the plume descends along the cold walls it generates a recirculation flow by creating two large vortices on either side of the center line of the heat source. The recirculation results in heat transfer between hot and cold fluids. This process generates oscillations and variations in isothermal patterns until a steady state is reached ($t \geq 0.2$).

Figure 6 shows the effect of nanoparticles on flow dynamics and on heat transfer properties by presenting a qualitative comparison of isotherms and velocity streamlines of water and Cu-water nanofluid at $Ra = 10^6$ when flow reaches steady state. It is shown that as the nanoparticles are added, maximum temperature in the cavity is reduced while the isotherms are narrower and concentrated around the rising plume which is an indication of the enhanced heat transfer performance of nanofluids. This increase in thermal convection due to addition of nanoparticles reduces the strength of re-circulating flow in the nanofluid as shown in Fig. 6b.

Temporal variation of maximum relative temperature of the heat source is shown in Fig. 7. After the heat flux is applied at the bottom, the resulting flow development can be classified into three stages: an initial stage, a transition stage, and a steady stage. The initial stage is dominated by pure conduction due to the formation of a thermal boundary layer when the heat

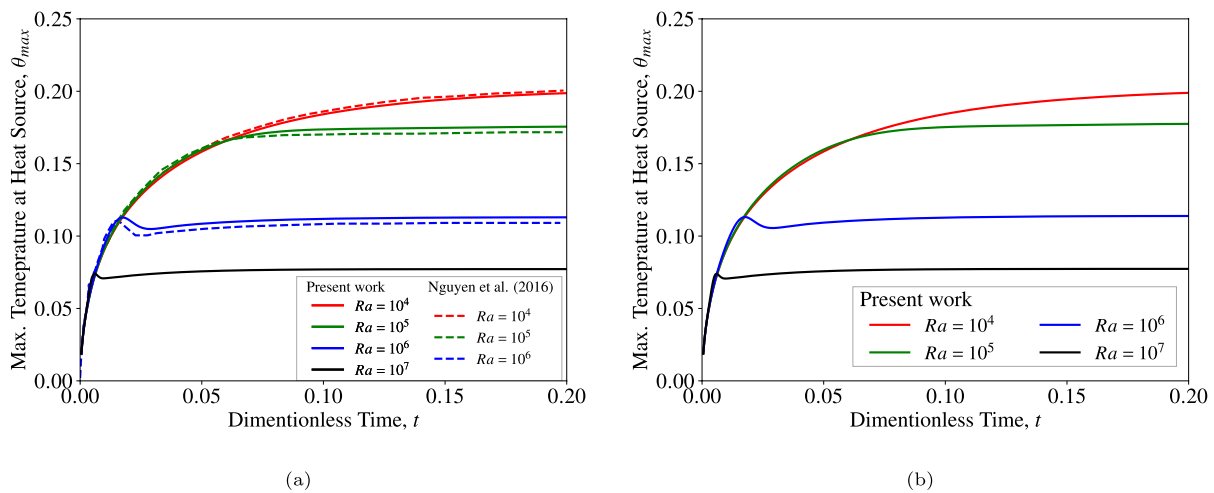


Fig. 8 Variation of maximum temperature of the heat source with dimensionless time at different Rayleigh numbers **a** Cu-water nanofluid, and **b** Al_2O_3 -water nanofluid. ($\phi = 10\%$)

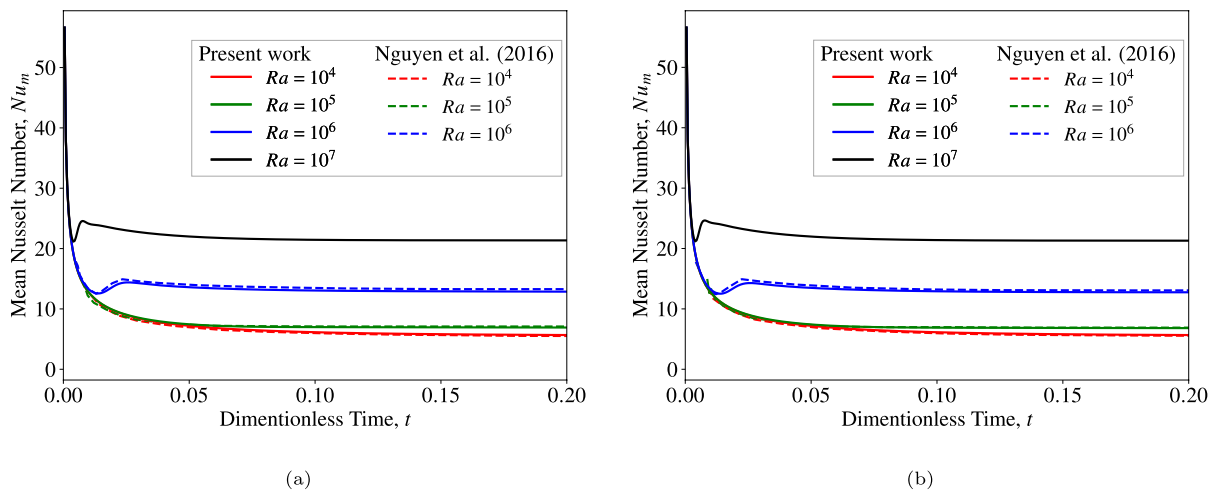


Fig. 9 Variation of mean Nusselt number with dimensionless time at different Rayleigh numbers **a** Cu-water nanofluid, and **b** Al_2O_3 -water nanofluid. ($\phi = 10\%$)

flux is applied. This boundary layer is responsible for a rapid initial rise in the temperature. The transition stage involves temperature oscillations when heat exchange takes place because of recirculating flow. A convection dominated steady stage is reached with gradual decay of oscillations. Figures 8 and 9 show temporal variation of the maximum temperature and mean Nusselt number of the two nanofluids at different Rayleigh numbers, respectively. It is observed that at lower Ra , the transition region is smooth and there is no temperature drop at the end of initial stage. This is because at lower Ra , the thermal plume from the heat source does not rise rapidly and the exchange of heat is slower. While at $Ra = 10^6$, mixing is more rapid and chaotic, and we observe a drop in temperature as well as a jump in Nusselt number at the end of the initial

stage. This vigorous mixing is also the reason for the flow to reach steady state earlier at higher Rayleigh numbers. The oscillations in the initial stage vanish as the flow recovers during the transition stage and becomes stable as it reaches steady state. We have compared our results with the published data that employed first order semi-implicit FEM [44]. The results match well at lower Ra , however, the semi-implicit scheme is found to be dissipative at higher Rayleigh number.

5.3 Differentially heated enclosure with sinusoidal side walls

To study transient natural convection in an irregular geometry, we employ a differentially heated cavity with sinusoidal

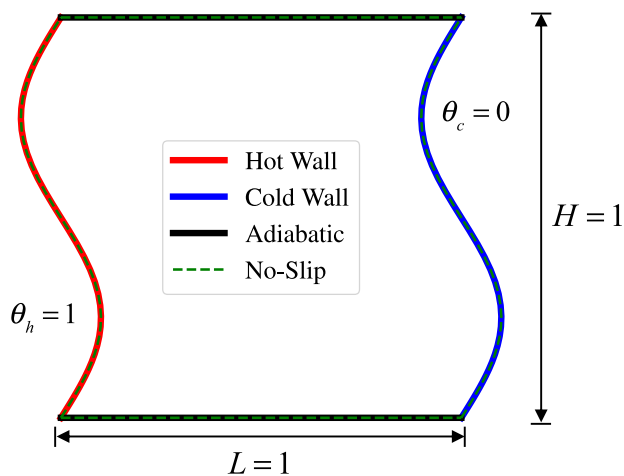


Fig. 10 Problem description and boundary conditions

side walls. The sinusoidal geometry is defined by the following equation:

$$y = a_1 \sin \left(a_2 \left(\frac{2\pi x}{L} \right) \right) \quad (51)$$

where a_1 and a_2 are geometric parameters controlling the corrugations in the sinusoidal wall. For this test case, we have chosen $a_1 = 0.1$ and $a_2 = 1.0$. The cavity is differentially heated with hot wall on the left and a cold wall on the right. Horizontal walls are considered adiabatic. A no-slip boundary condition for velocity is prescribed on all the walls. The schematic diagram as well as the prescribed boundary conditions are shown in Fig. 10. The initial conditions for the

velocity and pressure fields are set equal to zero. The initial temperature inside the cavity is set equal to the mean of the temperatures on differentially heated vertical walls.

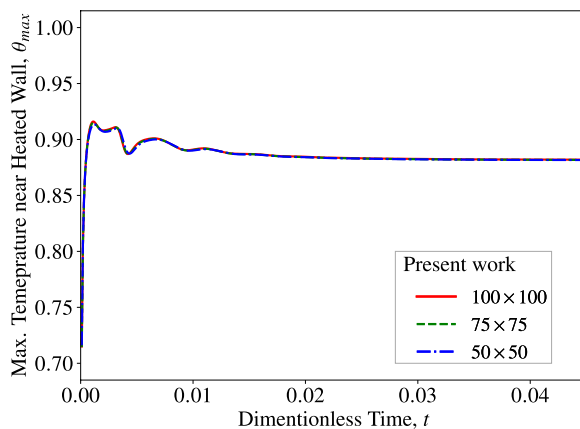
Based on the grid independence study shown in Fig. 11, the simulation is run on a mesh of 50×50 hexahedral elements. The fluid chosen for convection is air with a Prandtl number of 0.72. The heat transfer is measured in terms of mean Nusselt number which is defined as:

$$Nu_m = - \int_0^1 Y \frac{\partial \theta_s}{\partial X} \bigg|_{X=0,1} dY \quad (52)$$

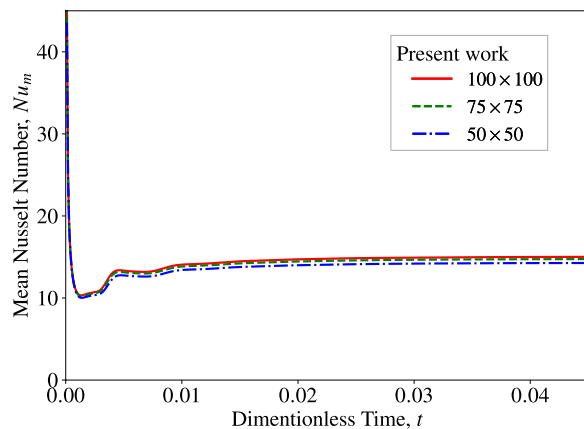
where $Y = y/L_{ref}$ is dimensionless vertical coordinate and θ_s is the dimensionless temperature on vertical walls.

Remark 4 To demonstrate the robustness and practical utility of the formulation on unstructured grids and irregular geometries common in complex industrial designs, we conducted a comparative study between the structured mesh with hexahedral elements presented here and a corresponding unstructured mesh with tetrahedral elements, as detailed in Appendix 2. The results from both the structured and unstructured meshes show close agreement.

Figure 12a shows temporal variation of dimensionless temperature at a very close point near the heated left wall (0.01, 0.5) for various Rayleigh numbers. As seen in the previous test case, the flow development can be categorized into three different stages: an initial rise stage, followed by an intermediate oscillatory or transition stage which gradually turns into a steady state stage. In the initial stage the development of the boundary layer is dominated by pure



(a)



(b)

Fig. 11 Temporal variation of **a** maximum temperature near the hot wall (0.01, 0.5), and **b** mean Nusselt number along the vertical walls at different grid resolutions ($Ra = 10^6$)

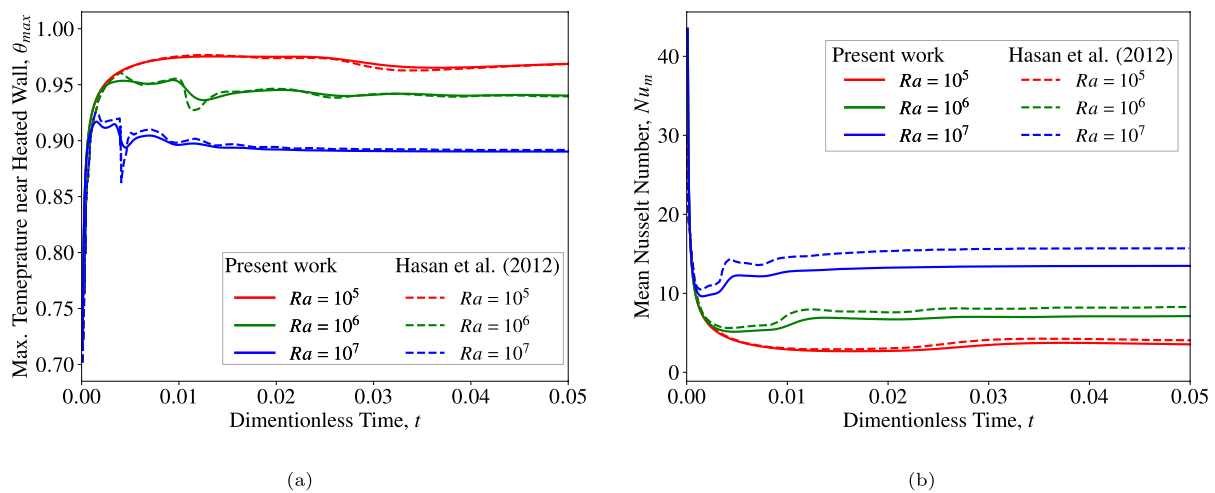


Fig. 12 Temporal variation of **a** maximum temperature near hot wall (0.01, 0.5), and **b** mean Nusselt number along vertical walls

conduction, the transition stage involves mixing of hot and cold fluid due to formation of a vortex and re-circulating flow. Finally, the steady state stage is dominated by convection. We have compared our results with the ones published in the literature [45] that used an adaptive finite element method with 7,873 elements. Our numerical results are obtained with 2,500 elements, i.e., one third the number of elements, and they match well with the reference data [45]. At $Ra = 10^7$, our temperature values are slightly lower than the reference, however, the trend for the temporal variation is the same.

The flow development is qualitatively shown in Fig. 13. Initially boundary layers are formed at each of the two differentially heated vertical walls of the cavity. These layers move inward in the opposite directions. The heated layer rises upward while the colder layer settles downward ($t = 0.01$). These oppositely traveling waves interact with the adiabatic horizontal walls and generate a re-circulating flow with a big central vortex inside the cavity ($t = 0.02$). The re-circulation alters the boundary layers and creates mixing of hot and cold layers that results in a temperature dip near the heated wall as shown in Fig. 12a. Since mean Nusselt number depends on the gradient of the temperature field at the vertical walls, an opposite behavior is observed in Figure 12b. As Ra increases, strength of the re-circulation increases, and we observe a sharper dip in the temperature field as well as more oscillations in the transition stage. A stronger re-circulation also gives rise to some secondary vortices at the corners of the cavity and helps the flow to reach steady state faster. As the flow reaches steady state, flow recirculation weakens, and it becomes horizontally stratified ($t = 0.05$). A better mixing due to stronger buoyancy at higher Ra results in more stratification of the flow as it reaches steady state. This is shown in Figure 14.

Finally, we present evolution of the residual as a function of the iteration number for three time steps in Table 3. We see that the consistently linearized coupled system (41) yields quadratic convergence in the nonlinear Newton–Raphson solver at various time levels where different physical flow conditions (parameters) are dominant. The quadratic convergence of the proposed numerical method all through the evolution of the problem shows the robustness of the formulation and the solution algorithm.

5.4 Forced convection in a square cavity under the existence of Lorentz force

In this test case, the influence of a magnetic source on the hydrothermal behavior of a ferrofluid in a cavity with a hot cylinder is investigated. In addition, effect of Rayleigh and Hartmann numbers, and volume fraction of Fe_3O_4 on heat transfer properties are also considered. Due to the external electromagnetic field acting on the ferrofluid, the forcing term $\mathbf{f}_b$ in (1) and the heat source/sink term f in (3) are taken to be of the following forms.

$$\mathbf{f}_b(x, y) = -Ha^2 \text{Pr} \left(\frac{\sigma_{nf}/\sigma_{bf}}{\rho_{nf}/\rho_{bf}} \right) \begin{bmatrix} (H_y^2 u_1 - H_x H_y u_2) \\ (H_x^2 u_2 - H_x H_y u_1) \end{bmatrix} \quad (53)$$

$$f(x, y) = Ha^2 E_c \left(\frac{\sigma_{nf}/\sigma_{bf}}{(\rho C_p)_{nf}/(\rho C_p)_{bf}} \right) (H_x u_2 - H_y u_1)^2 + E_c \left[\frac{\mu_{nf}/\mu_{bf}}{(\rho C_p)_{nf}/(\rho C_p)_{bf}} \right] \left\{ 2u_{1,x}^2 + 2u_{2,y}^2 + (u_{1,y} + u_{2,x})^2 \right\} \quad (54)$$

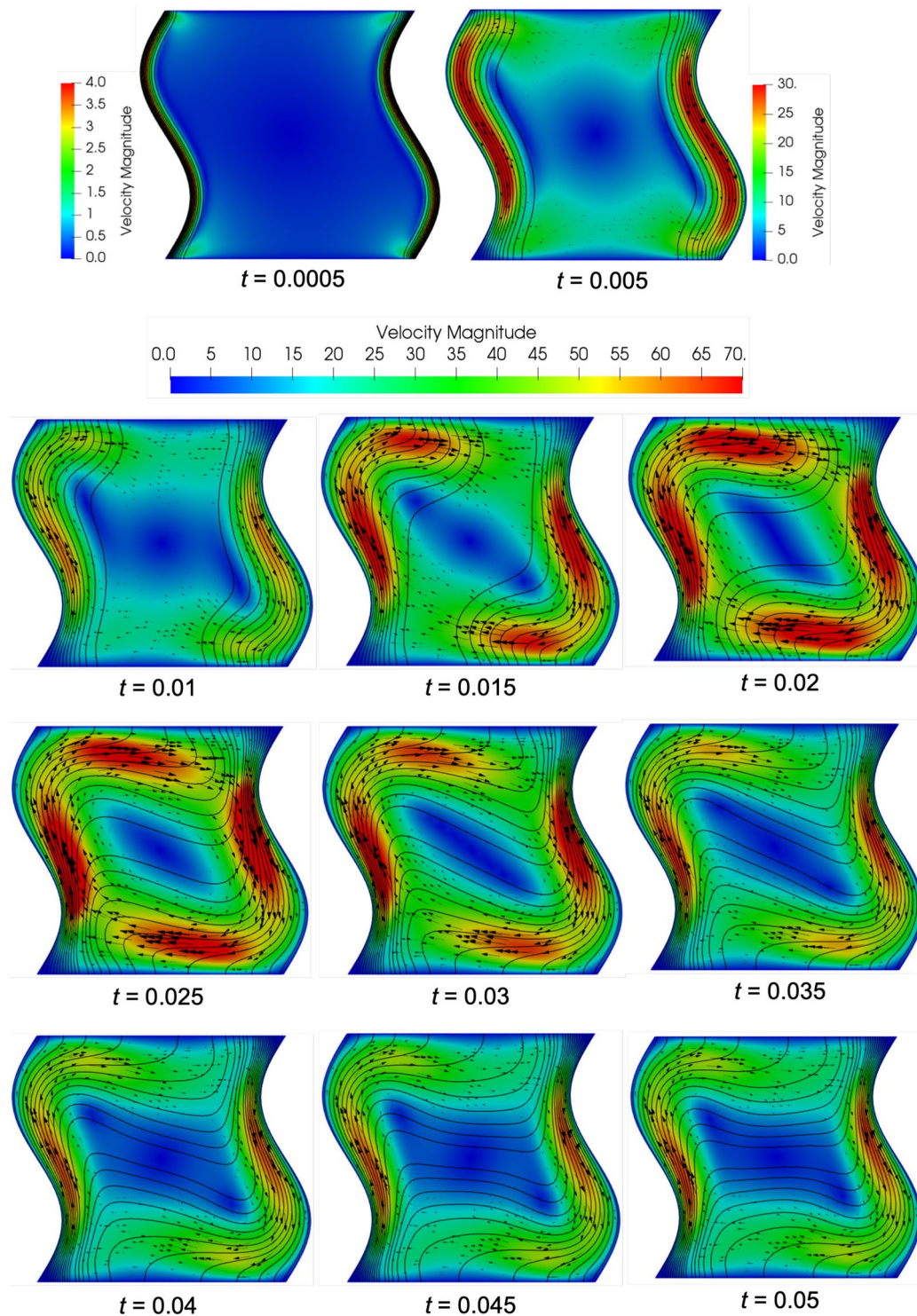


Fig. 13 Isotherms colored with velocity magnitude and local flow direction at various dimensionless times ($Ra = 10^5$). Isotherm magnitudes are 0.02, 0.06, 0.1, 0.14, 0.18, 0.22, 0.26, 0.3, 0.34,

0.38, 0.42, 0.46, 0.54, 0.58, 0.62, 0.66, 0.7, 0.74, 0.78, 0.82, 0.86, 0.9, 0.94, 0.98 from right to left

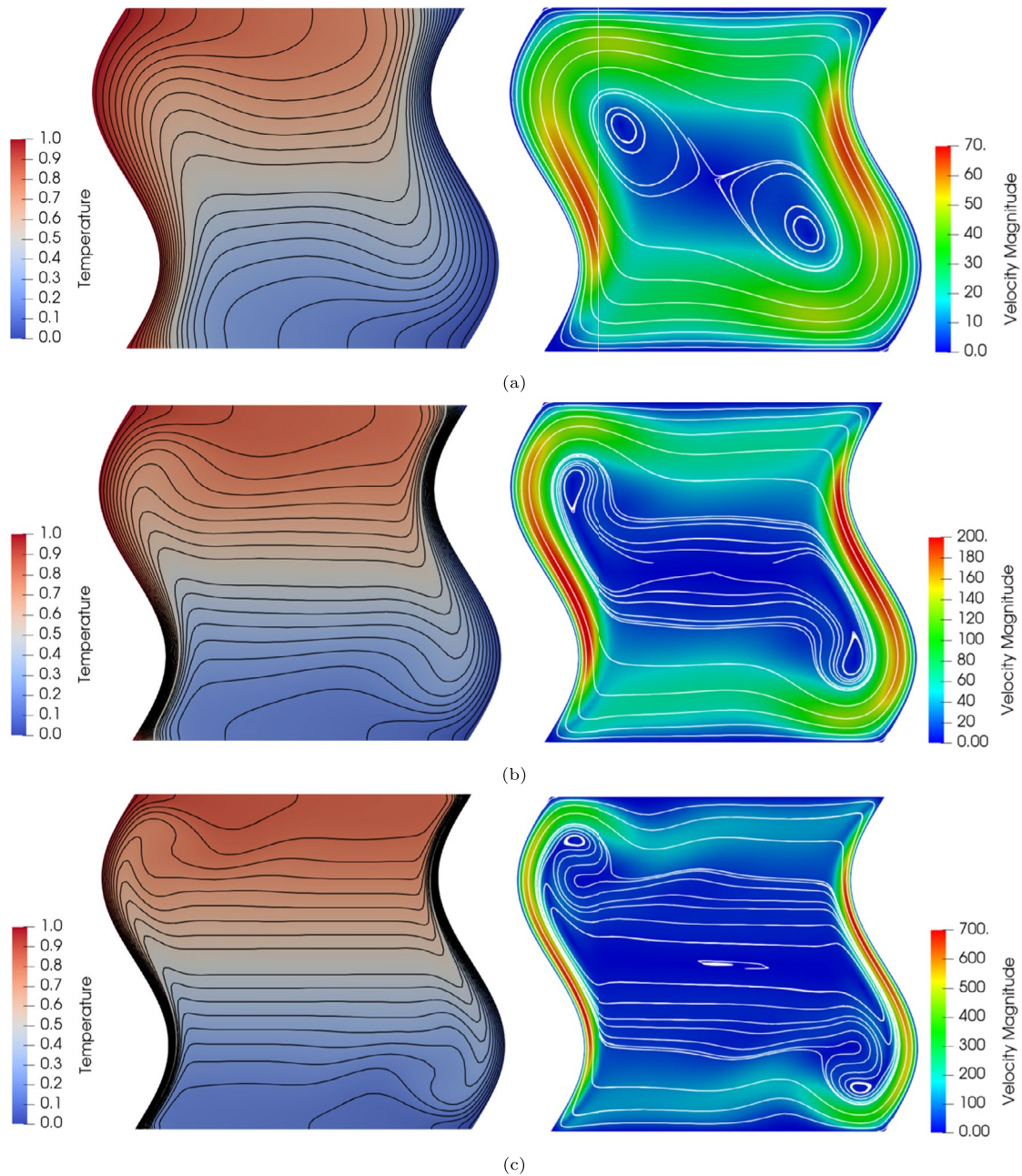


Fig. 14 Isotherms and streamlines at steady state for **a** $Ra = 10^5$, **b** $Ra = 10^6$, and **c** $Ra = 10^7$. Isotherm magnitudes are same as given in Fig. 13

Table 3 Residual reduction in Newton–Raphson scheme

Iteration number	Step 3	Step 10	Step 80
1	4.9798×10^5	1.0299×10^6	5.0191×10^5
2	2.0723	7.7461	2.2716
3	1.4851×10^{-8}	3.5623×10^{-7}	1.5917×10^{-7}
4	-2.7812×10^{-11}		

where σ is electrical conductivity, Ha and Ec are the dimensionless parameters, H_x and H_y are the components of the magnetic field. Hartmann number (Ha) is defined as the ratio of electromagnetic force to the viscous force. These parameters are defined as follows.

$$\frac{\sigma_{nf}}{\sigma_{bf}} = 1 + \frac{3\phi(\sigma_{np}/\sigma_{bf} - 1)}{(\sigma_{np}/\sigma_{bf} + 2) - \phi(\sigma_{np}/\sigma_{bf} - 1)} \quad (55)$$

$$Ec = \frac{\mu_{bf}\alpha_{bf}}{(\rho C_p)_{bf}\Delta TL^2} \quad (56)$$

$$Ha = L\mu_0 H_0 \sqrt{\frac{\sigma_{bf}}{\mu_{bf}}} \quad (57)$$

Problem domain consists of a unit square $\Omega = [-0.5, 0.5] \times [-0.5, 0.5]$ with a cavity of radius 0.2. At the outer walls, zero temperature is prescribed, while unit temperature flux $q''_o = \Delta T k_{bf}/L = 1$ is applied at the inner circular wall. No-slip velocity boundary condition is prescribed at all the walls. Lorentz force is applied through a magnetic source which is placed at a vertical distance of 0.505 away from the center of the domain and the magnetic field at any point in the domain is given by the following relations [46]:

$$H_x = \frac{(y-b)}{\sqrt{(a-x)^2 + (b-y)^2}} \frac{\gamma}{2\pi} \quad (58)$$

$$H_y = \frac{(a-x)}{\sqrt{(a-x)^2 + (b-y)^2}} \frac{\gamma}{2\pi} \quad (59)$$

$$H = \sqrt{H_x^2 + H_y^2} \quad (60)$$

where (a, b) is the location of the magnetic source, and γ is its strength. For this problem, $(a, b) = (0, 0.505)$ and $\gamma = 2.5$. The schematic diagram of the problem is shown in Fig. 15. A grid of 8,900 hexahedral elements is employed to run this simulation. Values of parameters Pr and Ec are 6.8 and 10^{-5} ,

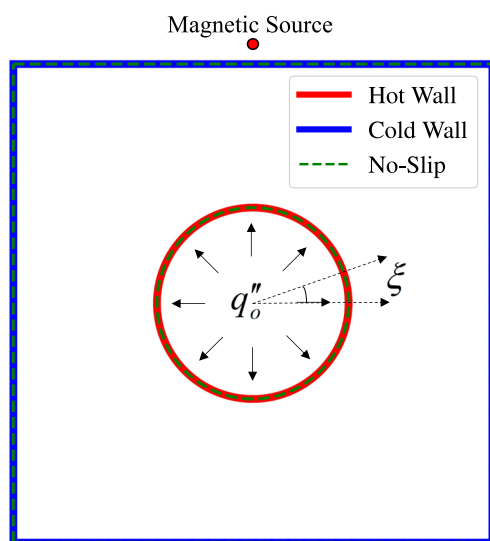


Fig. 15 Schematic diagram of the Lorentz force problem

respectively. Table 4 shows quadratic rate of convergence of the nonlinear iterative solution scheme, which confirms the consistent linearization of the formulation.

Figure 16 shows the effect of Rayleigh and Hartmann numbers on streamlines and isotherms of the nanofluid. At lower Rayleigh number ($Ra = 10^3$), conduction mode dominates heat transfer, while the convection mode governs at higher Ra due to the presence of a strong buoyancy force generating a thermal plume. Application of Lorentz force diminishes the eddies in the flow as well as the thermal plume. It also pushes the heat transfer mechanism from convection dominated to conduction dominated, thus reducing the Nusselt number.

Local and average Nusselt numbers are defined as follows:

$$Nu_{loc} = \left(\frac{k_{nf}}{k_{bf}} \right) \frac{1}{\theta_s} \quad (61)$$

$$Nu_m = \frac{1}{2\pi} \int_0^{2\pi} Nu_{loc} d\xi \quad (62)$$

Figure 17 shows the distribution of local Nusselt number (Nu_{loc}) along the hot wall as a function of Rayleigh number and Hartmann number. The extreme points in the plot are due to the thermal plume which is generated at high Rayleigh number. As the magnetic force is applied, conduction mode dominates which enhances the temperature near the circular wall and thus Nu_{loc} decreases. The effect is more pronounced at higher Rayleigh number which is shown in Fig. 18. We also show a comparison of our results with the reported data in the literature [18] in Figs. 17 and 19. The reference uses control volume FEM with 19,200 elements. Our method produces comparable accuracy with 54% less elements.

To investigate the percentage increase in effective heat transfer with the addition of ferroparticles and magnetic force, we define another parameter $\mathcal{E}$ as follows.

$$\mathcal{E} = \frac{Nu_m^{nf} - Nu_m^{bf}}{Nu_m^{bf}} \times 100 \quad (63)$$

A study is performed for water and nanofluid with 4% volume fraction of Fe_3O_4 at different Hartmann numbers as shown in Fig. 19. Results illustrate that at lower Rayleigh numbers ($Ra \leq 10^4$), percentage increase in heat transfer due to the addition of nanoparticles is independent of the Lorentz force. This is because Lorentz force pushes heat transfer mechanism from convection dominated to conduction dominated and at lower Ra the heat transfer mechanism is already conduction dominated. At $Ra = 10^5$, we see a slight enhancement in the heat transfer

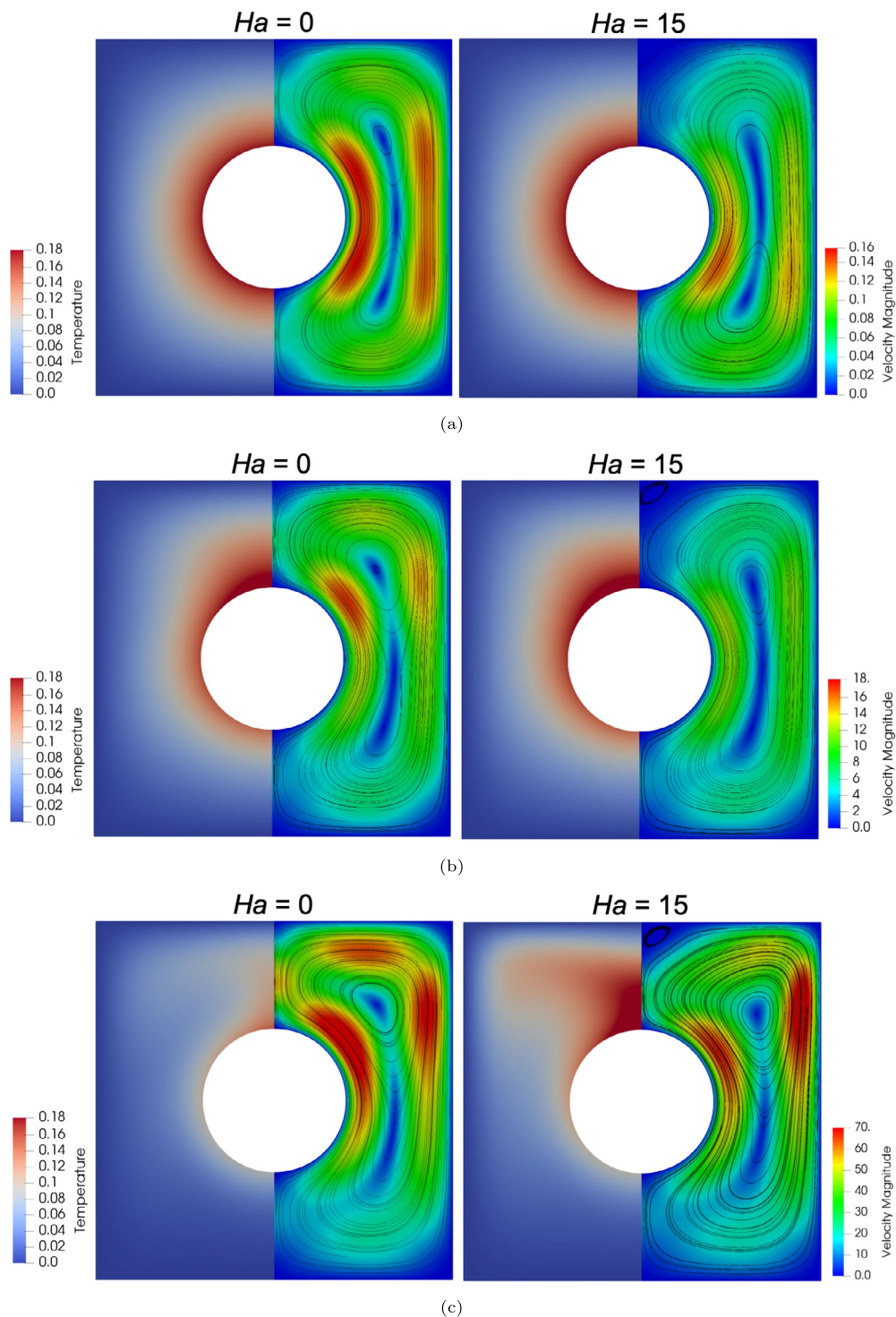


Fig. 16 Isotherms (left) and velocity streamlines (right) contours for different values of Hartmann number with $\phi = 4\%$ **a** at $Ra = 10^3$, **b** at $Ra = 10^5$, and **c** at $Ra = 10^6$

as we increase the Lorentz force. However, the behavior becomes interesting at $Ra = 10^6$. As we know that heat transfer mechanism is mainly dominated by convection at higher Ra due to a stronger buoyancy force, and since

Lorentz force pushes heat transfer mechanism from convection to conduction resulting in a lesser mean Nusselt number, we see a reduction in heat transfer performance

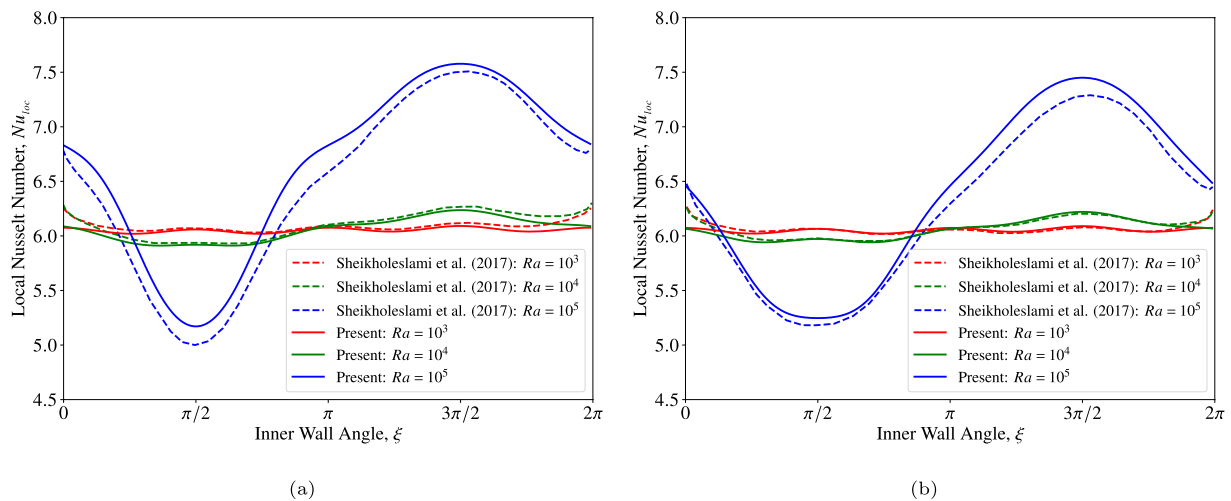


Fig. 17 Effect of Hartmann number and Rayleigh number on local Nusselt number Nu_{loc} along inner wall at **a** $Ha = 0$, and **b** $Ha = 15$ ($\phi = 4\%$)

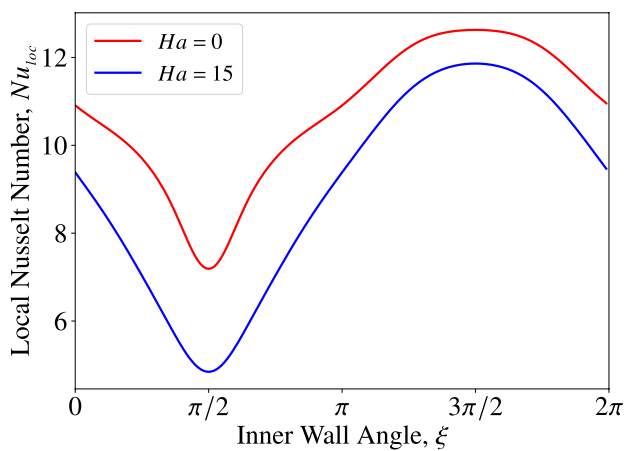


Fig. 18 Effects of Hartmann number on local Nusselt number Nu_{loc} along inner wall at $Ra = 10^6$ ($\phi = 4\%$)

Table 4 Residual reduction in the Newton–Raphson scheme for the Lorentz force problem ($\phi = 4\%$, $Ra = 10^3$, $Ha = 15$)

Iteration number	Step 3	Step 10
1	8.8670	8.8853
2	2.5077×10^{-7}	2.5210×10^{-7}
3	4.3790×10^{-19}	5.0643×10^{-18}

with an increase in Hartmann number at higher Ra . Therefore, we can conclude from these results that the addition of nanoparticles yields optimal heat transfer enhancement at lower Rayleigh number ($\leq 10^5$) and higher Hartmann number.

Finally, a transient test case for the same problem is carried out at different Hartmann numbers and a temporal variation of mean Nusselt number is shown in Fig. 20. It is observed that there is no apparent effect of Hartmann number on the mean Nusselt number for lower Rayleigh numbers. Moreover, the flow development is smooth and is not affected by the Lorentz force. At higher $Ra = 10^6$, heat transfer mechanism is dominated by convection and since Lorentz force reduces heat convection, therefore a decrease in the mean Nusselt number is observed as we increase the magnitude of Lorentz force through Hartmann number. This is consistent with the observations for steady state test case. At $Ha = 100$, a combination of increased Lorentz and buoyancy forces induces a strong re-circulation flow. This results in temperature oscillations as the heat transfer mechanism transitions from conduction to convection. The oscillations die out as the convection dominated steady state is reached.

6 Conclusion

We have presented a stabilized numerical method for simulating thermal convection in nanofluids, governed by the conservation laws of mass, momentum, and energy. Buoyancy-driven convection was modeled by introducing a buoyancy term in the momentum equation via the Boussinesq approximation, while the thermal field was governed by an advection–diffusion equation. The nanofluid was treated as a single-phase medium using standard homogenization techniques to define effective material properties. The resulting coupled system of variational equations leads to a mixed-field problem that needs to address some classical numerical difficulties, namely, the *inf-sup* condition on the mixed-field

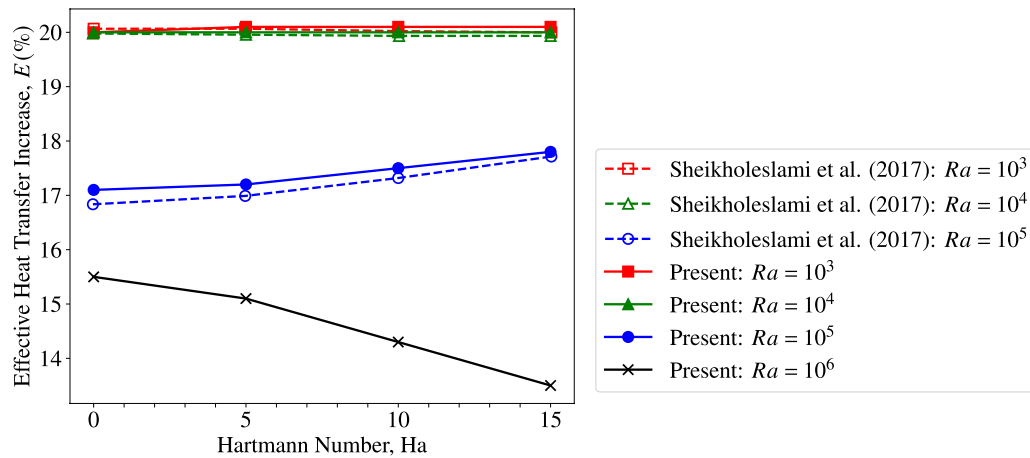


Fig. 19 Effect of the Hartmann number and Rayleigh number on the percentage heat transfer enhancement due to addition of nanoparticles

formulation, the destabilizing influence of nonlinear convective terms in both momentum and energy equations, and the

requirement for quadratic convergence rates in solving the nonlinear system.

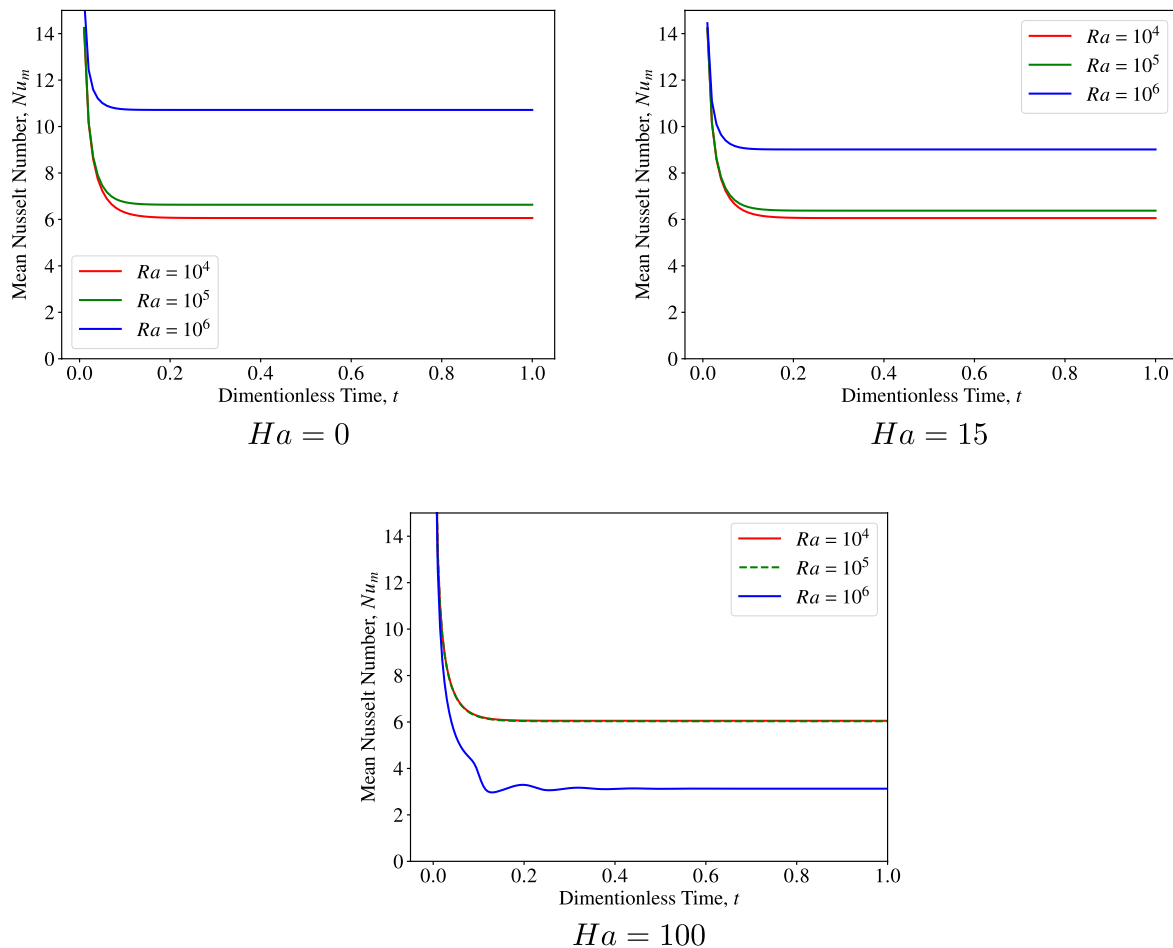


Fig. 20 Effects of the Hartmann number and Rayleigh number for porous medium on the ratio of heat transfer enhancement due to addition of nanoparticles

To address these issues, a robust stabilized formulation was derived within the Variational Multiscale (VMS) framework. The method is based on a scale separation of the unknown fields, consistent linearization of the fine-scale equations, and the preservation of thermomechanical coupling across all scales. By applying a bubble function-based approach at the fine-scale level, an explicit stabilization tensor was obtained, naturally incorporating cross-coupling terms essential for numerical stability and accuracy. These coupling terms proved particularly important in handling strongly nonlinear regimes where scale anisotropy becomes significant. The enhanced stability of the method enables a monolithic solution strategy, and the variational structure supports consistent linearization, yielding quadratic convergence in the iterative solution process.

A comprehensive convergence study was conducted to verify the variational consistency and robustness of the method. Optimal spatial convergence rates were demonstrated using linear hexahedral and wedge elements, while optimal temporal accuracy was achieved with both first-order backward Euler and second-order generalized- α time integration schemes. The method's applicability was further demonstrated through benchmark problems of engineering relevance, including natural convection in a wavy enclosure and forced convection in the presence of a Lorentz force. The results showed excellent agreement with published data and confirmed the method's capability to accurately capture transient thermal convection phenomena and the evolution of thermal boundary layers. The proposed stabilized VMS formulation provides a reliable and accurate framework for simulating complex thermofluidic problems in nanofluids, with potential applicability to a wide range of engineering and scientific domains.

Appendix 1: Explicit representation of the fully-coupled stabilization tensor

To help the interested reader in the implementation of the method, an explicit expression of the stabilization tensor τ derived in (35) is presented below for 2-node (in one dimension) and 3-node (in two dimensions) canonical elements using linear interpolation functions. The quadratic bubble function in 1-D is the mid-node of the Lagrange element. For 3-node linear triangle (in two dimensions) it is the barycenter node.

Appendix 1.1: 2-node element

See Fig. 21.

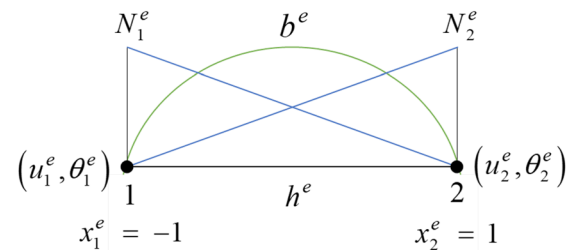


Fig. 21 Canonical 2-node element of length h_e in one-dimension. Linear nodal interpolation functions and quadratic bubble function is also shown

Interpolation functions

$$N^e = [N_1^e \ N_2^e]^T = \frac{1}{h^e} \left[(x_2^e - x) \ (x - x_1^e) \right]^T; h^e = x_2^e - x_1^e \quad (\text{A.1})$$

$$\nabla N^e = \begin{bmatrix} \partial N_1^e / \partial x \\ \partial N_2^e / \partial x \end{bmatrix} = \frac{1}{h^e} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (\text{A.2})$$

$$u = N_1^e u_1^e + N_2^e u_2^e; \quad \theta = N_1^e \theta_1^e + N_2^e \theta_2^e \quad (\text{A.3})$$

Bubble function

$$b^e = (1 - x^2); \quad b_{,x}^e = -2x \quad (\text{A.4})$$

Stabilization tensor

$$\tau = -b^e \int_{-1}^{+1} b^e dx \left\{ \frac{\mathbf{M}}{\Delta t} + \begin{bmatrix} \hat{\tau}_M & \hat{\tau}_{ME} \\ \hat{\tau}_{EM} & \hat{\tau}_E \end{bmatrix} \right\}^{-1} \quad (\text{A.5})$$

where,

$$\begin{aligned} \mathbf{M} &= \int_{-1}^{+1} b^{e2} dx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\tau}_M & \hat{\tau}_{ME} \\ \hat{\tau}_{EM} & \hat{\tau}_E \end{bmatrix} \\ &= \int_{-1}^{+1} \begin{bmatrix} \hat{b}^e u b_{,x}^e + (b^e)^2 u_{,x} + 2\nu (b^e_{,x})^2 & g\beta (b^e)^2 \\ (b^e)^2 \theta_{,x} & \hat{b}^e u b_{,x}^e + \alpha (b^e_{,x})^2 \end{bmatrix} dx \end{aligned} \quad (\text{A.6})$$

Appendix 1.2

3-node element

See Fig. 22.

Interpolation functions

$$N^e = [N_1^e \ N_2^e \ N_3^e]^T = [x \ y \ (1 - x - y)]^T \quad (\text{A.7})$$

$$\nabla \mathbf{N}^e = \begin{bmatrix} \partial N_1^e / \partial x & \partial N_1^e / \partial y \\ \partial N_2^e / \partial x & \partial N_2^e / \partial y \\ \partial N_3^e / \partial x & \partial N_3^e / \partial y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \quad (\text{A.8})$$

Bubble function

$$b^e = xy(1 - x - y); \quad b^e_{,x} = y(1 - y - 2x); \quad b^e_{,y} = x(1 - x - 2y) \quad (\text{A.9})$$

Stabilization tensor

$$\tau = -b^e \int_0^1 \int_0^1 b^e dx dy \left\{ \frac{\mathbf{M}}{\Delta t} + \begin{bmatrix} \hat{\tau}_M & \hat{\tau}_{ME} \\ \hat{\tau}_{EM} & \hat{\tau}_E \end{bmatrix} \right\}^{-1} \quad (\text{A.10})$$

where,

$$\begin{aligned} \mathbf{M} &= \int_{-1}^{+1} \int_{-1}^{+1} b^{e2} dx dy \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \hat{\tau}_M &= \int_0^1 \int_0^1 \left[\begin{array}{c} \hat{b}^e (ub^e_{,x} + vb^e_{,y}) + (b^e)^2 \\ u_{,x} + v \left(2(b^e_{,x})^2 + (b^e_{,y})^2 \right) \\ (b^e)^2 v_{,x} + vb^e_{,x} b^e_{,y} \end{array} \right] \begin{array}{c} (b^e)^2 u_{,y} + vb^e_{,y} b^e_{,x} \\ \hat{b}^e (ub^e_{,x} + vb^e_{,y}) + (b^e)^2 \\ v_{,x} + v \left((b^e_{,x})^2 + 2(b^e_{,y})^2 \right) \end{array} \right] dx dy \\ \hat{\tau}_{ME} &= \int_0^1 \int_0^1 (b^e)^2 dx dy \quad \beta \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \\ \hat{\tau}_{EM} &= \int_0^1 \int_0^1 (b^e)^2 [\theta_{,x} \quad \theta_{,y}] dx dy \\ \hat{\tau}_E &= \int_0^1 \int_0^1 \hat{b}^e (ub^e_{,x} + vb^e_{,y}) + (b^e)^2 v_{,x} + \alpha \left((b^e_{,x})^2 + (b^e_{,y})^2 \right) dx dy \end{aligned} \quad (\text{A.11})$$

where $\hat{b}^e$ is the advection bubble function. The reason for employing a different advection bubble is mentioned in Remark 1. One choice for the advection bubble can be the interpolation function of the most upwind node in the upwind direction as presented in [34]. The above integrals can be exactly evaluated using appropriate Gauss quadrature rules.

Appendix 2: Unstructured mesh for differentially heated enclosure with sinusoidal side walls

To show the robustness of the proposed stabilized method for irregular geometries and unstructured meshes, we conducted the test case in Sect. 5.3 using an unstructured mesh composed of tetrahedral (T4) elements. A comparison between the structured mesh with hexahedral (B8)

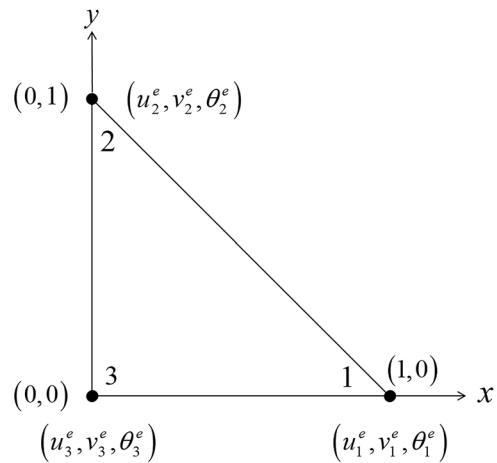


Fig. 22 Canonical 3-node triangular element in two-dimension

elements and the unstructured mesh with equal number of nodal points is presented in Fig. 23.

The simulations demonstrated close agreement between the results obtained from both the structured and unstructured meshes. Fig. 23 presents the temperature contours for each mesh (see Fig. 24). A quantitative comparison confirms that the numerical method accurately captures the temperature variations near the wall, as well as the mean Nusselt number in the case of the unstructured mesh (Fig. 25).

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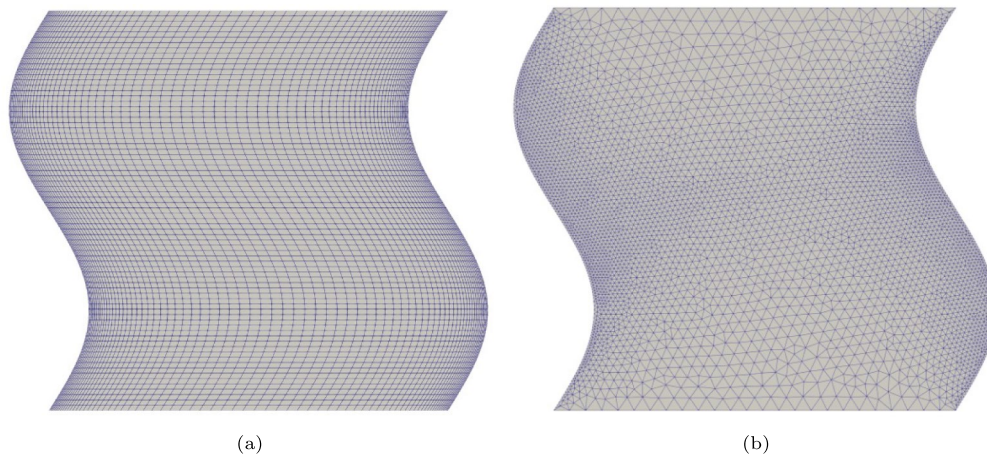


Fig. 23 Comparison of **a** structured mesh comprised of hexahedral (B8) elements, and **b** a corresponding unstructured mesh comprised of tetrahedral (T4) elements

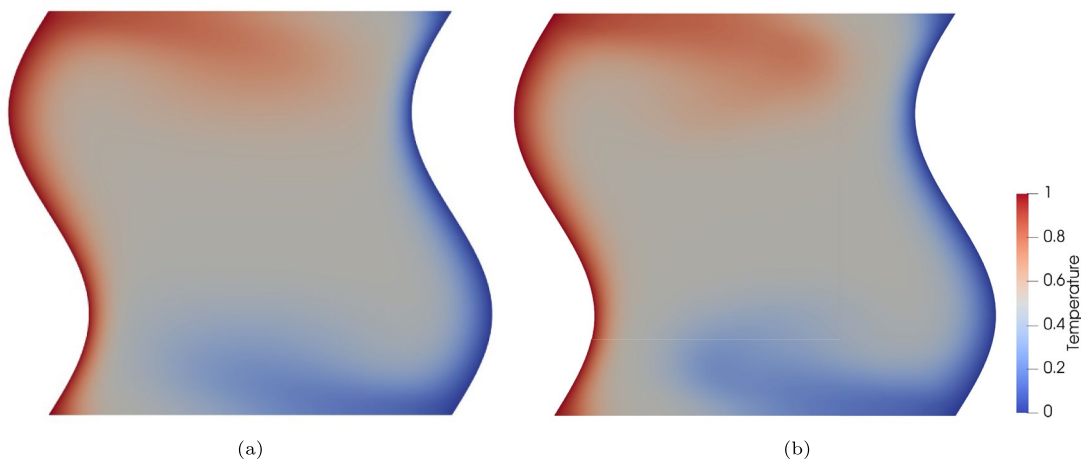


Fig. 24 Temperature field visualized at $t = 0.075$ for **a** structured mesh, and **b** unstructured mesh

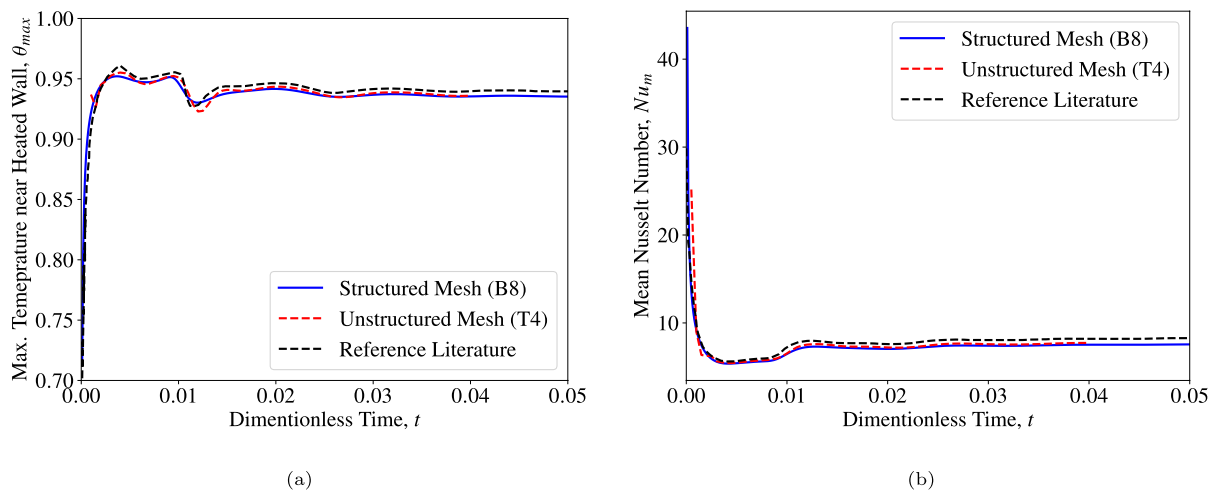


Fig. 25 Temporal variation of **a** maximum temperature near hot wall (0.01, 0.5), and **b** mean Nusselt number along vertical walls

Declarations

Conflict of interest The authors declare no potential conflict of interest.

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