

Nonlinear electromechanical coupling dynamics of a two-degree-of-freedom hybrid energy harvester*

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(Received Feb. 23, 2025 / Revised Apr. 25, 2025)

Abstract Vibration energy harvesting presents a significant opportunity for powering wireless sensor networks and internet of things (IoT) devices, offering a sustainable alternative to traditional battery-based power sources. However, environmental vibrations are predominantly low-frequency, which presents a significant challenge to the efficient conversion of such energy. To address this challenge, this paper proposes a novel two-degree-of-freedom (2-DOF) energy harvester. The first layer of the harvester incorporates a piezoelectric composite beam (PCB) paired with permanent magnets to form a negative stiffness mechanism (NSM), which counteracts the stiffness of linear springs, thereby achieving quasi-zero stiffness (QZS) or bistable characteristics. The second layer integrates piezoelectric transduction units with triboelectric nanogenerator (TENG) units to further enhance the efficiency of low-frequency vibration energy conversion. By considering the modal characteristics of the PCB, this paper establishes the electromechanical coupling equations of the harvester from an energy perspective. The mechanical responses of the masses in both layers, as well as the electrical outputs of the PCB, are analytically solved. Furthermore, the effects of the system parameters on the efficiency of low-frequency vibration energy harvesting are thoroughly analyzed. This work provides a theoretical foundation for the development of self-powered IoT sensor nodes, enabling efficient energy harvesting from ambient low-frequency vibrations.

Key words energy harvesting, quasi-zero stiffness (QZS), low frequency, piezoelectric beam, triboelectric nanogenerator (TENG)

Chinese Library Classification O322

2010 Mathematics Subject Classification 34C15, 74H45

* Citation: CHEN, T. T., WANG, K., CHEN, S. C., XU, Z. Y., LI, Z., and ZHOU, J. X. Nonlinear electromechanical coupling dynamics of a two-degree-of-freedom hybrid energy harvester. *Applied Mathematics and Mechanics (English Edition)*, **46**(6), 989–1010 (2025) <https://doi.org/10.1007/s10483-025-3264-7>

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Project supported by the National Key R&D Program of China (No. 2024YFB3408700), the National Natural Science Foundation of China (Nos. 12272129 and 12122206), the Natural Science Foundation of Hunan Province (Nos. 2024JJ4004 and 2024JJ3003), and the Postgraduate Scientific Research Innovation Project of Hunan Province (No. CX20240444)

1 Introduction

Vibrations are commonly regarded as a significant trouble in industrial settings, often leading to structural fatigue, noise pollution, and reduced operational efficiency. These adverse effects can result in premature machinery wear, increased maintenance costs, and even catastrophic failures in extreme cases^[1–4]. However, this negative perception overlooks the potential opportunities that vibrations present. In fact, vibrations can be harnessed for a variety of beneficial applications that contribute to societal progress^[5–6]. For example, vibrating screens represent a key technology in mining and construction industries for efficient material separation^[7]. Notably, mechanical vibration also plays a crucial role in biological systems, where it induces cellular mechanotransduction and modulates cell behavior by activating mechanosensitive pathways and receptors. In engineering applications, vibration-assisted drilling has demonstrated significant advantages in enhancing manufacturing precision and process efficiency^[8]. Beyond these applications, one of the most promising advantages of vibrations lies in their ability to be captured and converted into usable energy^[9–10]. This capability not only mitigates the detrimental effects of vibrations but also enables the development of self-powered systems for real-time monitoring and diagnostics, offering a sustainable solution for industrial and environmental applications.

The capture and utilization of vibration energy have attracted considerable attention in recent years, leading to the development of various energy harvesting technologies. Among these, piezoelectric^[11–14], electromagnetic^[15–17], electrostatic^[18], and triboelectric^[19–20] mechanisms are the most widely studied and implemented. Piezoelectric energy harvesting relies on the direct conversion of mechanical strain into electrical energy through piezoelectric materials, offering high energy density and simplicity in design^[21]. However, its performance is often limited by the brittleness of piezoelectric materials and their sensitivity to environmental factors such as temperature and humidity^[22]. Electromagnetic energy harvesting, on the other hand, utilizes the relative motion between a magnet and a coil to generate electrical energy via electromagnetic induction^[23]. This method is advantageous for its robustness and ability to harvest energy from low-frequency vibrations, but it typically requires larger device sizes and complex configurations. Triboelectric energy harvesting, a relatively newer approach, exploits the contact electrification and electrostatic induction between two dissimilar materials to generate power^[24]. While certain implementation challenges, including material degradation, scalability limitations, and environmental sensitivity, may constrain potential applications, triboelectric nanogenerators (TENGs) remain a highly promising technology for efficient mechanical-to-electrical energy conversion. These challenges primarily affect long-term reliability and large-scale deployment, but ongoing advances in material engineering and device architecture continue to address these limitations. The fundamental merits of TENGs, including their high instantaneous power density, material versatility, and ability to harvest low-frequency mechanical energy, position them as a compelling solution for distributed energy harvesting applications^[9]. Each of these methods has its unique strengths and limitations, making them suitable for different applications depending on the specific requirements of the vibration environment and the desired energy output.

Piezoelectric energy harvesters have evolved through several design generations, each addressing specific operational requirements and environmental conditions. The foundational configurations, including cantilever beams^[25], stacked layers^[26], and cymbal or bridge-type designs^[27], have demonstrated effectiveness in conventional vibration energy harvesting scenarios. However, their applications in low-frequency vibration environments, characteristic of human motion, ocean waves, and machinery operations, revealed significant performance limitations, prompting the development of advanced piezoelectric structures specifically engineered for low-frequency energy conversion^[27]. This paradigm shift has driven innovative energy harvesting advancements, particularly through multi-stable structures leveraging nonlinear dynamics for enhanced energy conversion^[28–31], geometrically optimized piezoelectric compos-

ite beams (PCBs) to reduce resonant frequencies^[32–33], frequency-up conversion mechanisms enabling efficient low-to-high frequency energy transfer^[34], ferrite nanoparticle-doped nanocomposites for improved dielectric and piezoelectric properties^[35], and quasi-zero stiffness (QZS) harvesters tailored for low-frequency vibration environments^[36–37]. While these advancements have expanded the capabilities of piezoelectric energy harvesters, critical limitations persist, particularly in terms of constrained operational bandwidth and suboptimal energy conversion efficiency in low-frequency domains^[38]. These limitations not only highlight the complexity of low-frequency vibration energy harvesting but also underscore the need for continued innovation in structural design and energy conversion mechanisms.

Recent advancements in vibration energy harvesting have demonstrated promising approaches to address key technical challenges. The integration of PCBs with varying resonant frequencies within a single structure^[25,39], coupled with the strategic utilization of nonlinearity^[40], has effectively expanded the operational bandwidth for energy harvesting. Furthermore, the synergistic combination of different energy conversion mechanisms has shown significant potential in enhancing overall energy conversion efficiency^[41–45]. Despite these developments, the field continues to face a fundamental challenge: the realization of a compatible system that simultaneously fulfills the critical requirements of low-frequency operation, high efficiency, broad bandwidth, and compact structural design. This multifaceted optimization problem represents a significant barrier in the advancement of vibration energy harvesting technologies.

To address these challenges, this study proposes an innovative two-degree-of-freedom (2-DOF) energy harvester with a dual-layer configuration. The first layer incorporates a PCB integrated with permanent magnets, forming a negative stiffness mechanism (NSM) specifically designed to counteract the stiffness of linear springs. This configuration enables the system to achieve either QZS characteristic or bistable behavior, depending on operational requirements. The secondary layer represents a significant advancement through the integration of piezoelectric transduction units with TENG units, creating a hybrid energy conversion system optimized for enhanced low-frequency vibration energy conversion efficiency. From a theoretical perspective, this research establishes a comprehensive electromechanical coupling model based on energy principles, taking into account the modal characteristics of the PCB. The model enables the systematic analysis of mechanical responses and electrical outputs for both the mass-spring components and the PCB across both layers. Furthermore, this study conducts a detailed parametric analysis to investigate the effects of key system parameters on low-frequency vibration energy harvesting efficiency, providing valuable insights for system optimizations and performance enhancements.

2 Design and operating principle

The fundamental configuration and operational principles of the proposed energy harvester are illustrated in Fig. 1. The core concept, depicted in Fig. 1(a), involves achieving low stiffness and energy harvesting through the integration of piezoelectric beams. Specifically, a piezoelectric film is bonded to an elastic substrate beam to form a PCB. A magnet is then affixed to the midpoint of the PCB via a bracket, maintaining a specific clearance between the lower surface of the magnet and the PCB to prevent interference during bending and ensure unimpeded deformation.

The designed structure, as shown in Fig. 1(b), consists of two layers, providing the energy harvester with 2-DOF. The first layer includes two PCBs positioned on the front and rear sides, with corresponding magnets installed on the sliding rail. These magnets maintain a specific gap with the magnets on the PCBs and share the same polarity, generating a repulsive force that establishes an NSM. The computational model of the first layer energy harvester is detailed in Fig. 1(c). Mass 1 represents the total mass of the second layer energy harvester, which comprises Mass 2, two pairs of magnets, two PCBs on the left and right sides, and one slice TENGs film

positioned on the bottom of Mass 2. Linear springs are installed on the left and right sides of the linear sliding guide, pre-compressed at the initial position to ensure continuous contact during horizontal motion. These springs provide a combined stiffness equivalent to four times the individual spring stiffness ($k/4$) in the horizontal direction. It should be noted that in the computational model, while the two springs on each side have been reduced to a single spring, its stiffness is set to be twice that of an individual spring. This means such simplification does not compromise the accuracy of the system analysis. The negative stiffness value of the magnet-composite beam structure can be less than, equal to, or greater than the stiffness provided by the springs, allowing Mass 1 to exhibit resultant positive stiffness, QZS, or negative stiffness. This configuration enables versatile stiffness tuning and enhances energy harvesting performance. The second layer energy harvester, illustrated in Fig. 1(e), differs from the first layer in that the PCBs and magnet pairs are positioned on the left and right sides of the mass block rather than on its front and rear surfaces. On the bottom of Mass 2, a glass plate is securely fixed, with its surface bonded to metal electrodes (copper film, 0.01 mm) and a dielectric layer (fluorinated ethylene propylene (FEP) film, 28 mm long and 0.22 mm thick). One additional glass plate with metal electrodes is affixed to the internal surface of the linear sliding guide, forming a sliding-mode TENG.

Figure 1(d) illustrates the working principle of the PCBs on the first layer. When the energy harvester is in its static equilibrium position, as shown in Subplot I, the magnet mounted on the mass block and the magnet installed on the PCB have the maximum gap. In other words, at this moment, the PCB exhibits the largest mid-span deflection. As the mass block moves to one side (upward in the figure), the distance between the two magnets decreases, resulting in a reduction of the force acting at the midpoint of the PCB. Consequently, the mid-span deflection of the composite beam decreases. As the displacement of the mass block further increases, the force exerted on the beam rapidly decreases, leading to a sharp reduction in the deflection, which eventually approaches zero. As the mass block returns from its farthest position to the static equilibrium position, the mid-span deflection of the PCB gradually increases, reaching its maximum value when the mass block is at the static equilibrium position. As the mass block continues to move to the opposite side (downward in the figure), the mid-span deflection of the PCB decreases again, reaching its minimum value when the mass block reaches its farthest position. When the mass block returns to the static equilibrium position, the PCB once again exhibits its maximum mid-span deflection. It can thus be observed that during one complete oscillation cycle of the mass block, the PCB undergoes two full cycles of deflection from maximum to minimum. In other words, the proposed system exhibits a frequency up-conversion effect.

Figure 1(e) illustrates the working principle of the PCBs on the second layer. When the mass undergoes horizontal motion, the magnet-composite beams on both sides generate restoring forces. These forces act in opposite directions: one side provides positive stiffness, resisting the motion of the mass, while the other side offers negative stiffness, facilitating its movement. When Mass 2 is at the static equilibrium position, the forces exerted by the left and right magnet-composite beam mechanisms are equal in magnitude but opposite in direction. When Mass 2 moves to the right (see Subplot I in Fig. 1(e)), the magnetic force from the left magnet assists motion, while the force from the right magnet opposes it. However, as the distance between the left magnet pair increases, the magnetic force decreases, whereas the distance between the right magnet pair decreases, leading to an increase in magnetic force. This results in a nonlinear increase in the restoring force of the second layer energy harvester with displacement, indicating an increase in stiffness. Conversely, when Mass 2 moves to the left (see Subplot II in Fig. 1(d)), the scenario is reversed: the right magnet assists the motion (negative stiffness), while the left magnet opposes it (positive stiffness). Similar to the first layer energy harvester, the overall stiffness of the second layer system can be adjusted by regulating the gap d_2 between the magnet fixed on the mass and that attached to the PCB. This adjustability allows for precise

control over the dynamic response of the energy harvester and enhances its energy harvesting performance.

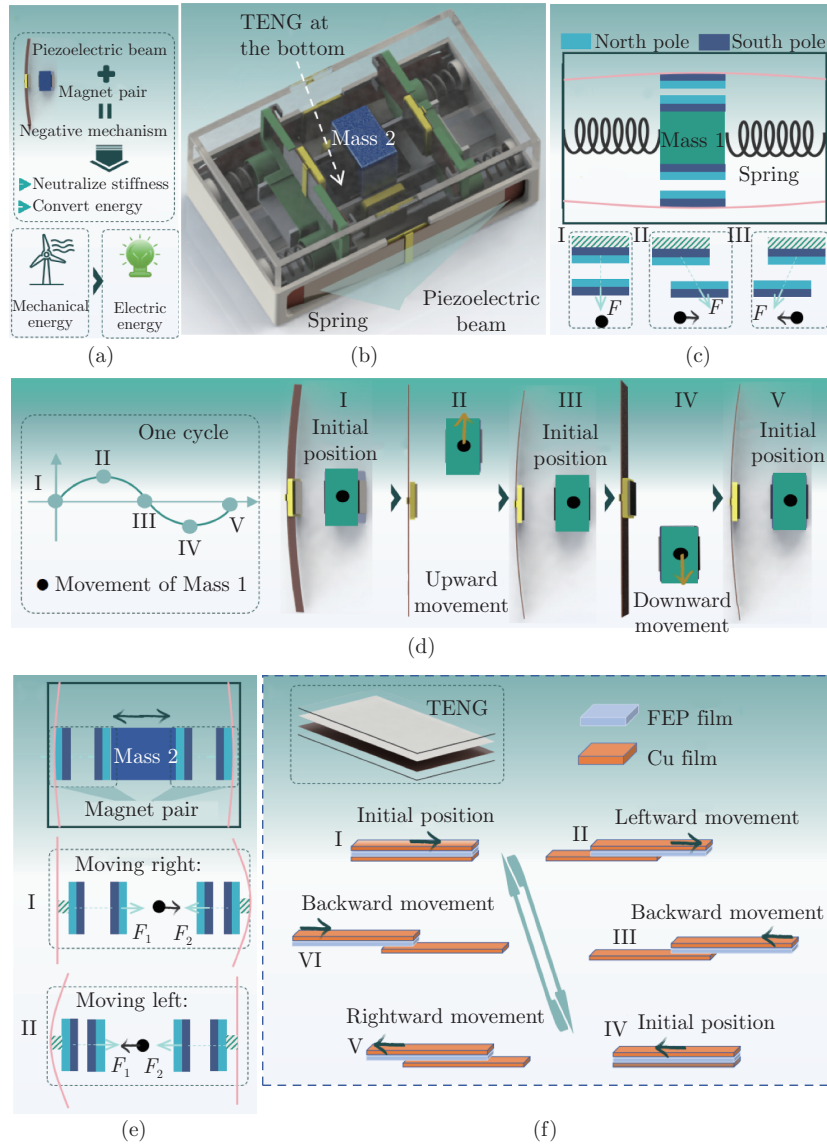


Fig. 1 Fundamental configuration and operational principles of the energy harvester: (a) low stiffness and energy harvesting concept using PCBs; (b) three-dimensional (3D) model of the energy harvester; (c) the computational model of Mass 1; (d) the motion sequence of the piezoelectric energy conversion unit over one operational cycle; (e) the computational model of Mass 2; (f) the motion mechanism of the TENG unit (color online)

The operational principle of the TENG throughout a complete motion cycle is depicted in Fig. 1(f), with brown and silver elements representing the copper (Cu) film and fluorinated ethylene propylene (FEP) film, respectively. The cycle initiates with the mass block positioned at the static equilibrium point (see Subplot I in Fig. 1(f)), where the FEP film establishes full contact with the Cu film. Owing to the disparity in electron affinities between the two materials, electrons migrate from the Cu film to the FEP film, resulting in the accumulation of

positive and negative charges on their respective surfaces, while the potential difference between them remains negligible. As the mass block deviates from the equilibrium position, the Cu film (or the FEP film in the case of a fixed Cu film) undergoes a displacement of distance y_2 , disrupting the charge distribution and generating an electric field that induces current flow, thereby neutralizing the triboelectric potential. At the point of maximum displacement, the relative velocity between the mass block and the linear sliding guide reaches a minimum, while the open-circuit (OC) voltage attains its peak value. Subsequently, the Cu film (or FEP film) reverses its direction of motion (see Subplot III in Fig. 1(f)), returning to the equilibrium position (see Subplot IV in Fig. 1(f)) and achieving the maximum displacement in the opposite direction (see Subplot VI in Fig. 1(f)). The cycle culminates as the mass block restores the Cu film to its equilibrium state (see Subplot I in Fig. 1(f)). This cyclic process elucidates the energy conversion mechanism inherent in the TENG, wherein mechanical energy is transformed into electrical energy through the combined effects of triboelectrification and electrostatic induction, thereby facilitating continuous energy harvesting.

3 Analysis

To investigate the dynamic response and electrical output characteristics of the proposed energy harvester, it is essential to derive the electromechanical coupling equations that describe its behavior. The computational schematics of the energy harvester are presented in Fig. 2. Specifically, Fig. 2(a) provides a simplified schematic representation of the energy harvester, where the variables y_1 and y_2 represent the displacements of Mass 1 and Mass 2, respectively. Figure 2(b) illustrates the schematic of the neutral axis within the PCB. Furthermore, Figs. 2(c) and 2(d) illustrate the computational schematics of the magnets located in the first and second layers of the energy harvester, respectively.

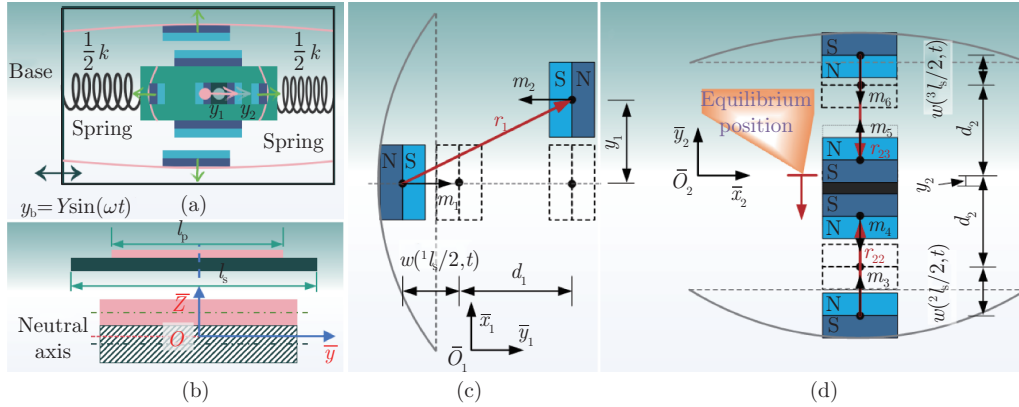


Fig. 2 (a) Computational model of the energy harvester; (b) schematic representation of the PCB, including its neutral axis; (c) and (d) schematic diagrams of the magnetic pair configurations in the first and second layers of the energy harvester (color online)

Following the energy method framework, the governing variational equation characterizing the electromechanically coupled system can be expressed as follows:

$$\delta \int_{t_1}^{t_2} (T - U) dt + \int_{t_1}^{t_2} \delta W dt = 0, \quad (1)$$

where δ is the infinitesimal variation, and T denotes the total kinetic energy of the energy harvester. For the proposed 2-DOF energy harvester incorporating magnet pairs, the total kinetic energy comprises four distinct components: the kinetic energy of the proof mass, the

linear sliding guide, the moving magnet, and the PCB. The potential energy, denoted by U , represents the cumulative potential energy of the entire energy harvester. The term W accounts for the work performed by non-conservative forces within the energy harvester. To establish the theoretical framework, the mass per unit length of the PCB is defined first as follows:

$$m_b = \rho_s b_s h_s + \rho_p b_p h_p \left(H\left(\bar{x} - \frac{l_s - l_p}{2}\right) - H\left(\bar{x} - \frac{l_s + l_p}{2}\right) \right). \quad (2)$$

The material and geometric parameters of the PCB are defined as follows: ρ_s and ρ_p denote the density of the substrate beam and piezoelectric film, respectively; b_s and b_p represent their respective widths; h_s and h_p indicate their thicknesses; l_s and l_p correspond to their lengths. The function $H(a)$ stands for the Heaviside function, where a is a variable.

The kinetic energy of Mass 1 within the first layer energy harvester can be mathematically expressed as follows:

$$T_M = \frac{1}{2} M (\dot{y}_b(t) + \dot{y}_1(t))^2, \quad (3)$$

where $\dot{y}_b$ is the derivative of the base excitation with respect to time t ; $\dot{y}_1(t)$ denotes the relative velocity response of Mass 1. Similarly, the kinetic energy associated with Mass 2 in the second layer energy harvester can be written as

$$T_m = \frac{1}{2} m (\dot{y}_b(t) + \dot{y}_1(t) + \dot{y}_2(t))^2, \quad (4)$$

where $\dot{y}_2(t)$ represents the velocity of Mass 2 relative to the base.

The kinetic energy of the PCB within the energy harvester can be given by

$$T_{b1} = \frac{1}{2} \int_0^{l_s} m_b \dot{w}(\bar{x}, t)^2 d\bar{x}. \quad (5)$$

Given that the magnet is rigidly mounted at the geometric center of the PCB, its displacement is identical to the mid-span deflection of the PCB. Consequently, the kinetic energy associated with the magnet can be evaluated by

$$T_m = \frac{1}{2} m_m \left(\frac{\partial w(\bar{x}, t)}{\partial t} \Big|_{\bar{x}=l_s/2} \right)^2. \quad (6)$$

Here, m_m denotes the mass of the magnet mounted on the substrate beam, and l_s stands for the length of the substrate beam.

The elastic potential energy stored in four linear springs, each with a stiffness coefficient of $k/4$, can be mathematically expressed as

$$U_{\text{spring}} = 4 \times \frac{1}{2} \left(\frac{1}{4} k \right) y_1(t)^2. \quad (7)$$

Upon obtaining the magnetic potential energy, the subsequent step involves the derivation of the elastic potential energy of the PCB. For a rectangular cross-sectional beam, the moment of inertia of the substructure beam is given by $I_s = b_s h_s^3/12$, while that of the piezoelectric film is expressed as $I_p = b_p h_p^3/12$. Taking the bottom of the substructure beam as the reference point and defining the coordinate system, the neutral axis (see Fig. 2(b)) of the substructure beam is positioned at the center of its cross-section, located at a distance of $z_s = h_s/2$ from the reference position. Similarly, the neutral axis of the piezoelectric film is situated at a distance of $z_p = (h_s + h_p)/2$ from the reference position. Applying the formula for the neutral axis of the PCB, the following relationship can be established:

$$\hat{z} = \frac{E_s A_s z_s + E_p A_p z_p}{E_s A_s + E_p A_p}. \quad (8)$$

Here, $A_s = b_s \times h_s$ and $A_p = b_p \times h_p$ are the cross-sectional areas of the substructure beam and piezoelectric film, respectively. The variables z_s and z_p represent the positions of the neutral axes for each layer within the PCB.

The elastic potential energy of the substrate beam, in the absence of an adhered piezoelectric film, can be expressed as follows:

$$U_{s1} = \frac{1}{2} \left(\int_0^{\frac{l_s - l_p}{2}} E_s I_{s1} \kappa(\bar{x}, t)^2 d\bar{x} + \int_{\frac{l_p + l_s}{2}}^{l_s} E_s I_{s1} \kappa(\bar{x}, t)^2 d\bar{x} \right). \quad (9)$$

Here,

$$I_{s1} = \int_0^{b_s} \int_{-h_s/2}^{h_s/2} \bar{z}^2 d\bar{y} d\bar{z} = \frac{1}{12} b_s h_s^3, \quad (10)$$

and $\kappa(\bar{x}, t)$ denotes the curvature of the PCB, with its expression given by $\kappa(\bar{x}, t) = \frac{\partial^2 w(\bar{x}, t)}{\partial \bar{x}^2}$.

The elastic potential energy of the PCB in the region $(l_s - l_p)/2 \leq \bar{x} \leq (l_s + l_p)/2$, which is covered with a piezoelectric film, can be expressed as

$$U_{s2} = \frac{1}{2} \int_{\frac{l_s - l_p}{2}}^{\frac{l_s + l_p}{2}} E_s I_{s2} \kappa(\bar{x}, t)^2 d\bar{x}, \quad (11)$$

where

$$I_{s2} = \int_0^{b_s} \int_{-\hat{z}}^{h_s - \hat{z}} \bar{z}^2 d\bar{y} d\bar{z} = \frac{1}{3} b_s ((h_s - \hat{z})^3 - (-\hat{z})^3). \quad (12)$$

The electromechanical behavior of the piezoelectric film is governed by the following constitutive equation, which describes the relationship between the stress component and the electromechanical field,

$$T_{xx}(\bar{x}, \bar{z}, t) = \bar{c}_{11}^E S_1 - \bar{e}_{31} E_3 = \bar{c}_{11}^E S_{xx}(\bar{x}, \bar{z}, t) + \bar{e}_{31} \frac{v(t)}{h_p}. \quad (13)$$

Here, the electric field intensity, E_3 , is defined as a function of the voltage output across the electrode through the relation $E_3(t) = -v(t)/h_p$, where $v(t)$ represents the instantaneous voltage, and h_p is the thickness of the piezoelectric film. The material properties are characterized by the following constitutive parameters: the reduced elastic modulus ($\bar{c}_{11}^E = 1/s_{11}^E$), the piezoelectric stress coefficient ($\bar{e}_{31} = d_{31}/s_{11}^E$), and the permittivity constant ($\bar{\epsilon}_{33}^S = \epsilon_{33}^T - d_{31}^2/s_{11}^E$). Based on these defined material parameters and in conjunction with the governing equation (see Eq. (13)), the total potential energy stored in the piezoelectric film can be given by

$$U_e = \frac{1}{2} \int_{V_p} \left(\bar{c}_{11}^E \left(-\bar{z} \frac{\partial^2 w(\bar{x}, t)}{\partial \bar{x}^2} \right)^2 + \bar{e}_{31} \frac{v(t)}{h_p} \left(-\bar{z} \frac{\partial^2 w(\bar{x}, t)}{\partial \bar{x}^2} \right) \right) dV_p. \quad (14)$$

The internal electrical energy density in the piezoelectric film is governed by the interaction between the electric field and the dielectric properties of the material, which can be expressed as

$$W_e = \frac{1}{2} \int_{V_p} \mathbf{E}^t \mathbf{D} dV_p = \frac{1}{2} \int_{V_p} \mathbf{E}^t (\mathbf{eS} + \epsilon \mathbf{E}) dV_p. \quad (15)$$

Here, $\mathbf{E}$ represents the vector of electric field components, and $\mathbf{D}$ is the vector of electric displacement components. Substituting the non-zero electric field and the electric displacement

components into Eq. (15), the internal electrical energy in the piezoelectric film can be rewritten as

$$W_e = \frac{1}{2} \int_{V_p} -\frac{v(t)}{h_p} \left(-\bar{e}_{31} z \frac{\partial^2 w(\bar{x}, t)}{\partial^2 \bar{x}} - \bar{\varepsilon}_{33}^S \frac{v(t)}{h_p} \right) dV_p. \quad (16)$$

The preceding derivation of kinetic and potential energies for the subsystem adopts a generalized formulation that applies universally across different PCB configurations, without dependence on specific PCB properties. By inserting relevant PCB parameters into these general formulations, one can derive corresponding energy expressions, where a left superscript j identifies each specific case. Note that in the first layer energy harvester, two PCBs exhibit identical structural and functional characteristics, including matching geometry and loading conditions. This symmetry permits analysis to concentrate on a single representative beam while preserving complete generality ($j = 1$). In contrast, the second layer harvester integrates a pair of mechanically distinct PCBs, each tailored with specific properties to meet different performance requirements ($j = 2$ and $j = 3$).

In addition to the potential energy of the PCB, the magnetic potential energy constitutes a significant component of the total potential energy in the 2-DOF energy harvester. The schematic representation of the magnetic pair configuration is illustrated in Figs. 2(c) and 2(d), where two distinct subplots delineate the magnetic pairs within the first layer and the second layer energy harvesters, respectively. The elastic potential energy arising from the first layer magnetic field is given by $U_m = -\mathbf{B} \cdot \mathbf{m}_1$, where $\mathbf{m}_1$ denotes the magnetic moment vector of the magnet, and $\mathbf{B}$ represents the magnetic field generated by other magnets^[46]. The expression of the magnetic field can be formulated as follows:

$$\mathbf{B} = -\frac{\mu_0}{4\pi} \nabla \frac{\mathbf{m}_2 \cdot \mathbf{r}}{\|\mathbf{r}\|_2^3}, \quad (17)$$

where μ_0 represents the vacuum permeability, and $\mathbf{m}_2$ denotes the magnetic moment vector of the magnet. The vector gradient operator and the L_2 -norm are symbolized by ∇ and $\|\cdot\|_2$, respectively. The relative position vector between the two magnets is expressed as $\mathbf{r}$.

Following the above equations, the elastic potential energy induced by the magnetic field can be reformulated as

$$U_m = \frac{\mu_0}{4\pi} \left(\frac{1}{r^3} (\mathbf{m}_1 \cdot \mathbf{m}_2) - \frac{3}{r^5} (\mathbf{m}_1 \cdot \mathbf{r})(\mathbf{m}_2 \cdot \mathbf{r}) \right). \quad (18)$$

Here, r denotes the L_2 -norm of the vector $\mathbf{r}$.

As illustrated in Fig. 2(c), the position vector of the two magnets in the first layer energy harvester can be expressed as follows:

$$\mathbf{r} = -\left(d_1 + w\left(\frac{l_s}{2}, t\right)\right) \bar{\mathbf{x}}_1 + y_1(t) \bar{\mathbf{y}}_1. \quad (19)$$

Here, d_1 represents the distance between the magnet pair mounted on PCB 1. $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{y}}_1$ denote the unit vectors in two orthogonal directions. The transverse deflection at the geometric center ($\bar{x} = l_s/2$) of PCB 1 is denoted as $w(l_s/2, t)$. Then, the L_2 -norm of the first position vector $\mathbf{r}$ can be expressed as

$$r = \sqrt{\left(d_1 + w\left(\frac{l_s}{2}, t\right)\right)^2 + y_1(t)^2}. \quad (20)$$

In the magnetic field, magnetic moment vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ can be defined based on the magnetization magnitude and the volume of the respective magnets. Specifically, they are expressed as follows:

$$\mathbf{m}_1 = M_1 V_1 \bar{\mathbf{x}}_1, \quad \mathbf{m}_2 = -M_2 V_2 \bar{\mathbf{y}}_1. \quad (21)$$

Here, M_1 and M_2 denote the magnetization magnitude of the magnets attached to the PCB and that fixed on the second layer frame, respectively. Based on the residual flux B_r , the magnetization magnitude of the magnets can be expressed as $M_1 = M_2 = B_r/\mu_0$. The variables V_1 and V_2 represent the volumes of the magnets mounted on the PCB and the second layer frame, respectively.

By substituting Eqs. (19), (20), and (21) into Eq. (18), and utilizing the orthogonality of the unit vectors, the elastic potential energy induced by the first layer magnetic field can be reformulated as

$${}^1U_m = \frac{\mu_0 M_1 V_1 M_2 V_2 \left(2 \left(d_1 + w \left(\frac{1l_s}{2}, 2 \right) \right)^2 - y_1(t)^2 \right)}{4\pi \left(\left(d_1 + w \left(\frac{1l_s}{2}, 2 \right) \right)^2 + y_1(t)^2 \right)^{5/2}}. \quad (22)$$

For the second and third magnet pairs depicted in Fig. 2(d), both magnets exhibit magnetic dipole moments aligned along the vertical direction. Consequently, the position vector lacks any component in the horizontal direction. Thus, the relative displacement vector between the two magnets can be expressed as

$$\mathbf{r}_{22} = \left(d_2 - y_2 + w \left(\frac{2l_s}{2}, t \right) \right) \bar{\mathbf{y}}_2, \quad (23)$$

$$\mathbf{r}_{23} = - \left(d_2 + y_2 + w \left(\frac{3l_s}{2}, t \right) \right) \bar{\mathbf{y}}_2. \quad (24)$$

Therefore, the L_2 -norm of the position vector is given by

$$r_{22} = d_2 - y_2(t) + w \left(\frac{2l_s}{2}, t \right), \quad r_{23} = d_2 + y_2(t) + w \left(\frac{3l_s}{2}, t \right). \quad (25)$$

The magnetic moment vector of the magnet in the second layer energy harvester can be expressed as follows:

$$\mathbf{m}_i = (-1)^i M_i V_i \bar{\mathbf{y}}_2, \quad i = 3, 4, 5, 6. \quad (26)$$

By substituting Eqs. (23), (24), and (26) into Eq. (18), the elastic potential energy produced by the magnets within the second layer energy harvester can be derived as

$${}^2U_m = \frac{2\mu_0 M_3 V_3 M_4 V_4}{4\pi \left(d_2 - y_2(t) + w \left(\frac{2l_s}{2}, t \right) \right)^3}, \quad (27)$$

$${}^3U_m = \frac{2\mu_0 M_5 V_5 M_6 V_6}{4\pi \left(d_2 + y_2(t) + w \left(\frac{3l_s}{2}, t \right) \right)^3}. \quad (28)$$

Assuming that the simply-supported beam follows the Euler-Bernoulli beam theory with uniform cross-sectional properties, the transverse vibration displacement corresponding to the mode shape can be expressed as

$$w(\bar{x}, t) = \sum_{i=1}^n \phi_i(\bar{x}) \varphi_i(t), \quad (29)$$

where $\varphi_i(t)$ represents the temporal modal coordinate, and $\phi_i(x)$ is the mode shape function. The variable n is the number of the mode shape. For a simply supported beam satisfying the classical boundary conditions, the mode shape function can be expressed as $\phi_i(\bar{x}) = \sin(i\pi\bar{x}/l_s)$,

where l_s represents the length of the beam. Given that the operating frequency of the designed energy harvester is significantly lower than the fundamental natural frequency of the simply supported beam, the dynamic response can be adequately characterized by considering only the first vibration mode. Consequently, the transverse vibration displacement of the substrate beams can be rewritten as $w(\bar{x}, t) = \sin(\pi\bar{x}/l_s)\varphi_1(t)$.

The mid-span deflection of the substrate beam, corresponding to the midpoint of the beam span, can be derived as $w(l_s/2, t) = \varphi(t)$. Following the above equations, the total potential energy of the j th piezoelectric film can be expressed as

$${}^jU_e = \left(\frac{\pi}{jl_s}\right)^4 \bar{c}_{11}^E {}^jI_p^j \varphi(t)^2 \left(2^j l_p - \frac{j l_s}{\pi} \sin\left(\frac{2\pi^j l_p}{jl_s}\right)\right) - \frac{{}^jJ_p {}^jv(t) {}^j\varphi(t) \pi}{2^j l_s} \left(\cos\left(\frac{\pi^j l_p}{jl_s}\right) - 1\right), \quad j = 1, 2, 3. \quad (30)$$

Here,

$${}^jI_p = \int_0^j b_p \int_{j h_s - j \hat{z}}^{j h_s + j h_p - j \hat{z}} (j \bar{z})^2 d\bar{y} d\bar{z} = \frac{1}{3} {}^j b_p ((j h_s + j h_p - j \hat{z})^3 - (j h_s - j \hat{z})^3), \quad j = 1, 2, 3, \quad (31)$$

$${}^jJ_p = \int_0^j b_p \int_{j h_s - j \hat{z}}^{j h_s + j h_p - j \hat{z}} \frac{\bar{e}_{31}}{j h_p} {}^j \bar{z} d\bar{y} d\bar{z} = \frac{1}{2} \frac{{}^j b_p \bar{e}_{31}}{j h_p} ((j h_s + j h_p - j \hat{z})^2 - (j h_s - j \hat{z})^2), \quad j = 1, 2, 3. \quad (32)$$

Based on the obtained kinetic and potential energy expressions, the Lagrange function for this 2-DOF energy harvester can be expressed as

$$L = T + \sum_{j=1}^3 ({}^jW_e - {}^jU_{s_1} - {}^jU_{s_2} - {}^jU_e) - U_{\text{spring}} - U_m, \quad j = 1, 2, 3, \quad (33)$$

where T denotes the kinetic energy of the whole system, whose expression can be given by

$$T = T_M + \sum_{j=1}^2 ({}^jT_m + {}^jT_b), \quad (34)$$

and U_m denotes the magnetostatic potential energy arising from the magnetic field, i.e.,

$$U_m = \sum_{j=1}^3 {}^jU_m. \quad (35)$$

Here, the left superscript j ($j = 1, 2, 3$) identifies the magnetic field configurations: $j = 1$ corresponds to the first layer energy harvester, while $j = 2$ and $j = 3$ represent the two distinct magnetic dipole pairs in the second layer energy harvester.

The virtual work done by the nonconservative forces acting on the system can be expressed as

$$\delta W = -2 \int_0^t {}^jC_1 {}^j\dot{w}(\bar{x}, t) \delta {}^jw(\bar{x}, t) d\bar{x} - C_{21} \dot{y}_1(t) \delta y_1(t) - C_{22} \dot{y}_2(t) \delta y_2(t) + 2^1 q(t) \delta^1 v(t) + 2^2 q(t) \delta^2 v(t) + 3^3 q(t) \delta^3 v(t), \quad j = 1, 2, 3. \quad (36)$$

Here, ${}^jC_1$ represents the viscous damping coefficient characterizing the energy dissipation mechanism in the j th substructure beam. The viscous damping coefficients C_{21} and C_{22} characterize the energy dissipation arising from the mechanical friction components. Furthermore, ${}^jq(t)$ denotes the charge generated by the j th piezoelectric film through the direct piezoelectric effect in response to the mechanical strain.

Based on the extended Hamiltonian principle, which incorporates both mechanical and electrical energy components, the coupled electromechanical governing equations of the energy harvester can be derived as follows:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\bar{y}}_i(t)} \right) - \frac{\partial L}{\partial \bar{y}_i(t)} = \frac{\delta W}{\delta \bar{y}_i(t)}, \quad i = 1, 2, \quad (37)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}^j(t)} \right) - \frac{\partial L}{\partial \varphi^j(t)} = \frac{\delta W}{\delta \varphi^j(t)}, \quad j = 1, 2, 3, \quad (38)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{v}(t)} \right) - \frac{\partial L}{\partial v(t)} = \frac{\delta W}{\delta v(t)}, \quad j = 1, 2, 3. \quad (39)$$

By substituting the aforementioned equations into Eq. (37), the equations of motion for both the second layer frame and the mass element within the second layer energy harvester can be derived as follows:

$$M(\ddot{y}_b(t) + \ddot{y}_1(t)) + C_{21}\dot{y}_1(t) + F_1(y_1, {}^1\varphi) - F_2(y_2, {}^2\varphi, {}^3\varphi) = 0, \quad (40)$$

$$m(\ddot{y}_b(t) + \ddot{y}_1(t) + \ddot{y}_2(t)) + C_{22}\dot{y}_2(t) + F_2(y_2, {}^2\varphi, {}^3\varphi) = 0. \quad (41)$$

Here, $F_1(\bar{y}_1, {}^1\varphi)$ and $F_2(\bar{y}_2, {}^2\varphi, {}^3\varphi)$ represent the restoring force components generated by the mechanical systems (magnetic pair and linear springs), which can be mathematically expressed as

$$F_1(y_1, {}^1\varphi) = ky_1(t) + 2 \times \frac{3\mu_0 M_1 V_1 M_2 V_2 y_1(t)(y_1(t)^2 - 4(d_1 + {}^1\varphi(t))^2)}{4\pi((d_1 + {}^1\varphi(t))^2 + y_1(t)^2)^{7/2}}, \quad (42)$$

$$F_2(y_2, {}^2\varphi, {}^3\varphi) = \frac{3\mu_0 M_3 V_3 M_4 V_4}{2\pi(d_2 + {}^2\varphi(t) - y_2(t))^4} - \frac{3\mu_0 M_5 V_5 M_6 V_6}{2\pi(d_2 + {}^3\varphi(t) + y_2(t))^4}. \quad (43)$$

The equation of the motion of the PCB in the energy harvester can be written as

$$({}^j m_{\text{eff-b}} + {}^j m_m) {}^j \ddot{\varphi}_1 + {}^j c_{\text{eff}} {}^j \dot{\varphi}_1 + {}^j k_{\text{eff-b}} {}^j \varphi_1 + {}^j \theta {}^j v(t) + F_3({}^j \varphi) = 0, \quad j = 1, 2, 3. \quad (44)$$

Here, the equivalent damping ${}^j c_{\text{eff}} = C_j {}^j l_s$, and the equivalent mass of the PCB can be given by

$${}^j m_{\text{eff-b}} = \frac{1}{2\pi} \left({}^j b_p {}^j h_p {}^j l_s {}^j \rho_p \sin \left(\frac{\pi {}^j l_p}{{}^j l_s} \right) + ({}^j b_p {}^j h_p {}^j \rho_p {}^j l_p + {}^j b_s {}^j h_s {}^j l_s {}^j \rho_s \pi) \right), \quad (45)$$

and the equivalent stiffness of the PCB can be written as

$$\begin{aligned} {}^j k_{\text{eff-b}} = & \frac{1}{2} {}^j E_s {}^j I_s \left(\frac{\pi}{{}^j l_s} \right)^4 \left(({}^j l_s - {}^j l_p) - \frac{{}^j l_s}{\pi} \sin \left(\frac{\pi {}^j l_p}{{}^j l_s} \right) \right) + {}^j E_s {}^j I_{s2} \left(\frac{{}^j l_p}{2} + \frac{{}^j l_s}{2\pi} \left(\sin \frac{{}^j l_p \pi}{{}^j l_s} \right) \right) \left(\frac{\pi}{{}^j l_s} \right)^4 \\ & + \frac{1}{4} \bar{c}_{11}^E I_p \left(\frac{\pi}{{}^j l_s} \right)^4 \left(2{}^j l_p - \frac{{}^j l_s}{\pi} \sin \left(\frac{2\pi {}^j l_p}{{}^j l_s} \right) \right). \end{aligned} \quad (46)$$

The variable ${}^j \theta$ represents the electromechanical coupling coefficient, which quantifies the energy conversion efficiency between the mechanical and electrical domains. This fundamental parameter can be expressed as

$${}^j \theta = \frac{{}^j J_p \pi}{{}^j l_s} \left(1 - \cos \left(\frac{\pi {}^j l_p}{{}^j l_s} \right) \right). \quad (47)$$

$F_3(j\varphi)$ stands for the repulsive force arising from the interaction between two permanent magnets, which can be quantitatively described by the following expression:

$$F_3(j\varphi) = \begin{cases} -\frac{3\mu_0 M_1 V_1 M_2 V_2 (d_1 + {}^1\varphi(t))(2(d_1 + {}^1\varphi(t))^2 - 3y_1(t)^2)}{4\pi((d_1 + {}^1\varphi(t))^2 + y_1(t)^2)^{7/2}}, & j = 1, \\ -\frac{3\mu_0 M_{2j-1} V_{2j-1} M_{2j} V_{2j}}{2\pi(d_2 + (-1)^{j+1}y_2(t) + {}^j\varphi)^4}, & j = 2, 3. \end{cases} \quad (48)$$

Based on Eq. (39), the electrical domain equation describing the electromechanical behavior of the PCB can be expressed as

$${}^j\theta {}^j\dot{\varphi}(t) - {}^jC_p {}^j\dot{v}(t) = {}^j\dot{q}(t), \quad j = 1, 2, 3, \quad (49)$$

where ${}^jC_p$ denotes the internal capacitance of the j th piezoelectric film, which is defined as

$${}^jC_p = \bar{\varepsilon}_{33}^S \frac{{}^jA_p}{{}^jh_p}, \quad j = 1, 2, 3. \quad (50)$$

Differentiating both sides of the aforementioned equation with respect to the temporal variable t yields the following expression:

$${}^j\theta {}^j\dot{\varphi}(t) - {}^jC_p {}^j\dot{v}(t) = {}^j\dot{q}(t) = \frac{{}^j\dot{v}(t)}{{}^jR}, \quad j = 1, 2, 3. \quad (51)$$

Consequently, the governing electrical equation for the piezoelectric film, incorporating the effect of an external resistive load, can be reformulated as

$${}^jC_p {}^j\dot{v}(t) + \frac{{}^j\dot{v}(t)}{{}^jR} - {}^j\theta {}^j\dot{\varphi}(t) = 0, \quad j = 1, 2, 3. \quad (52)$$

By employing the given parameters and solving Eq. (52), the output voltage of the PCB can be determined. For the TENG integrated in the second layer energy harvester, when only its specific operating conditions, namely, the OC voltage and short-circuit (SC) current, are taken into account, its electrical output can be described by the following expressions^[47]:

$$|V_{OC}| = \frac{\sigma y_2}{\varepsilon_0(L - y_2)} \left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right), \quad (53)$$

$$I_{SC} = \sigma w \frac{dy_2}{dt} = \sigma w \dot{y}_2. \quad (54)$$

Here, σ and ε_0 represent the charge density and the permittivity of free space, respectively. The variables L and w denote the length and width of the dielectric material, respectively, while d_1 , d_2 , and ε_1 , ε_2 correspond to the thicknesses and relative permittivities of Dielectric 1 and Dielectric 2, respectively. It is evident that once the displacement and velocity responses of Mass 2 are determined through the general solution to Eqs. (40) and (41), the electrical output of the TENG can be subsequently predicted using Eqs. (53) and (54).

4 Results and discussion

By substituting the following parameters into Eq. (42): substrate beam length ${}^1l_s = 69$ mm, piezoelectric film length ${}^1l_p = 56$ mm, substrate beam width ${}^1b_s = 10$ mm, piezoelectric film width ${}^1b_p = 10$ mm, substrate beam thickness ${}^1h_s = 0.3$ mm, piezoelectric film thickness ${}^1h_p = 0.3$ mm, substrate beam Young's modulus $E_s = 119$ GPa, piezoelectric film Young's modulus $E_p = 56$ GPa, residual flux density $B_r = 0.84$, magnet volumes $V_1 = V_2 = 10 \text{ mm} \times 10 \text{ mm}$

$\times 1.8$ mm, and vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$, the static mechanical properties of Mass 1 in the first layer energy harvester can be gained, which are provided in the electronic supplementary material (see <https://doi.org/10.1007/s10483-025-3264-7>). Notably, such an analysis describes F_1 as a function of displacement and mid-span deflection, but the figure itself primarily lists displacement amplitudes and corresponding mid-span deflection values to compute the restoring forces and stiffnesses. This approach effectively serves as a qualitative analysis of the static characteristics of Mass 1. However, in the context of the energy harvester proposed in this paper, the restoring forces and stiffnesses of both Mass 1 and Mass 2 are intrinsically linked to their motion displacements and mid-span deflections. The PCB, coupled to the main system (Mass 1 or Mass 2) via permanent magnet pairs, ensures that the mid-span deflection of the PCB is rigorously correlated with the motion displacement of the mass. Therefore, a quantitative evaluation of the stiffness characteristics can be achieved by solving the dynamic equations of the energy harvester to determine the motion displacement of the mass and the mid-span deflection of the PCB. These derived values are then substituted into the expressions for F_1 and F_2 to yield the restoring force and stiffness curves of the mass. This methodological approach, as illustrated in Figs. 3(a) and 3(b), facilitates a comprehensive understanding of the dynamic behavior and the implications for energy harvesting efficiency of the energy harvester.

Theoretically, the stiffness curve of the presented energy harvester should exhibit symmetry about the static equilibrium position. However, the curve shown in Fig. 3(a) is shifted to the left, and the curves on both sides are not strictly symmetrical. This discrepancy arises primarily because the restoring force and stiffness curve in Fig. 3 are derived from the displacement and mid-span deflection relationships obtained by solving the dynamic equations. The energy harvester proposed in this paper is a typical nonlinear system, and its dynamic response exhibits inherent nonlinear behaviors, leading to deviations between the obtained restoring force, stiffness curves, and theoretical predictions. Nevertheless, the curves depicted in the figure reveal that, for Mass 1, the introduction of the NSM formed by the PCB and permanent magnet pairs can significantly counteract the stiffness of the positive stiffness mechanism at the static equilibrium position. This results in a system with low stiffness characteristics, enabling large-amplitude responses at lower frequencies and thereby enhancing vibration energy harvesting efficiency. For Mass 2, its stiffness value approaches zero at the static equilibrium position and increases with displacement. It should be noted that although the stiffness curves in Fig. 3 do not exhibit QZS characteristics, this is primarily due to the specific parameter combination used to generate the plot, which prevents the NSM from fully counteracting the positive stiffness. By appropriately adjusting the system parameters, such as the dimensions of the PCB or the gap between the permanent magnet pairs, the proposed energy harvester can indeed achieve QZS or even bistable mechanical characteristics.

Figures 3(d) and 3(e) illustrate the time-domain curves of the displacement response of the energy harvester and the mid-span deflection response of the PCBs. Notably, in the first layer of the energy harvester, PCB 1 and PCB 2 are symmetrically distributed on both sides of Mass 1, resulting in identical mid-span deflection responses. Therefore, the deflection response curve of PCB 2 is not separately provided. Additionally, in the second layer of the energy harvester, PCB 3 and PCB 4 are positioned along the front and rear motion directions of Mass 2. The deflection responses of these two beams exhibit equal amplitudes with a phase difference of 90° . Consequently, only the time-domain deflection response curve of PCB 3 is presented. As shown in the figures, for the first layer of the energy harvester, when the motion displacement of Mass 1 reaches its maximum, the mid-span deflection response of the PCB attains its minimum. This phenomenon is primarily determined by the structure of the energy harvester. Specifically, at the static equilibrium position, the PCB exhibits its maximum mid-span deflection. As Mass 1 moves away from the static equilibrium position, the distance between the two magnets increases, leading to a reduction in the repulsive force and, consequently, a decrease in the mid-span deflection amplitude of the composite beam. For the second layer of the energy harvester,

one of the two PCBs always exhibits a deflection response that aligns with the displacement response of Mass 2, meaning that the mid-span deflection increases as the displacement of Mass 2 increases. The phase diagrams of Mass 1 and Mass 2 are shown in Figs. 3(c) and 3(f), respectively. Clearly, neither the phase diagram of Mass 1 nor that of Mass 2 exhibits a regular circle or ellipse. Furthermore, their Poincare sections are not concentrated at a single point but are instead distributed across a region. Therefore, under this parameter combination, the dynamic responses of both the first and second layers of the energy harvester exhibit chaotic characteristics.

The amplitude-frequency response curves of Mass 1, Mass 2, PCB 1, PCB 3, and PCB 4 are depicted in Figs. 3(g)–3(i). In these figures, the left Y -axis represents the response of the primary system (Mass 1 or Mass 2), while the right Y -axis denotes the amplitude of the mid-span deflection response of the PCBs. Evidently, both the displacement response amplitude and the deflection response amplitude exhibit pronounced resonance phenomena within the frequency ranges around 3.5 Hz and 15.5 Hz. This behavior is entirely distinct from that of a single-degree-of-freedom system, indicating that the proposed 2-DOF energy harvester is capable of generating significant dynamic responses across two frequency ranges. This characteristic provides an opportunity for efficient vibration energy harvesting over a broader frequency band.

Figure 4 illustrates the electric output of the proposed energy harvester. As shown in Fig. 4(a), the output voltage amplitude of PCB 1 exhibits significant variations with excitation frequency. At 2 Hz, the amplitude remains relatively small at approximately 1 V. When the frequency increases to 3.5 Hz, the voltage output rises sharply to exceed 2 V, though the time-domain curve displays pronounced nonlinear characteristics. As the excitation frequency reaches 12 Hz, the voltage amplitude suddenly decreases to levels compared with 3.5 Hz excitation. With further increases in frequency, the output voltage shows another rise at 15.5 Hz before dropping again to relatively low values at 18 Hz. Notably, the voltage output exhibits two distinct resonance peaks within the 3.5 Hz to 15.5 Hz frequency range. This dual-peak characteristic represents a significant advantage over conventional single-degree-of-freedom energy harvesters, as it enables effective voltage generation across two different frequency bands, significantly broadening the operational bandwidth of the energy harvesting system.

To further analyze the voltage output characteristics of the PCB, Fig. 4(b) presents the statistical properties of the output voltage when the excitation frequencies are 3.5 Hz, 6 Hz, and 12 Hz. In this plot, the box represents the data ranging from the lower quartile (Q1) to the upper quartile (Q3) of the output voltage time series, and the horizontal line and the hollow box inside the box indicate the median value and mean value (MV) of the output voltage, respectively. The lines outside the box denote the lower and upper whiskers, and the solid diamond points represent output voltage values that significantly deviate from the overall data distribution, primarily caused by the nonlinearity. Evidently, compared with the case at 6 Hz, the box range is larger at 3.5 Hz, indicating a more stable output voltage at this frequency. For the case at 12 Hz, the box height is small and outliers are present, suggesting a small-amplitude nonlinear response. To more clearly observe the variation of the output voltage with frequency, Fig. 4(c) provides the amplitude-frequency response curves of the output voltage peak and root mean square (RMS) values. Both the RMS output voltage and the peak voltage exhibit resonance peaks near the two frequency intervals of 3.5 Hz and 15.5 Hz. Additionally, the RMS value of the output voltage differs significantly from the peak value in the resonance region, primarily due to the pronounced nonlinear characteristics of the time series in this region.

Figures 4(d)–4(f) illustrate the voltage output of the PCB within the second layer energy harvester. Specifically, Fig. 4(d) depicts the time series of the output voltage at different excitation frequencies, Fig. 4(e) compares the peak and RMS values of the output voltage, and Fig. 4(f) presents comparative time series of the output voltages of PCB 3 and PCB 4. Similar to the first layer energy harvester, the second layer energy harvester also exhibits two resonance regions, located approximately at 3.5 Hz and 15.5 Hz, which align with the resonance

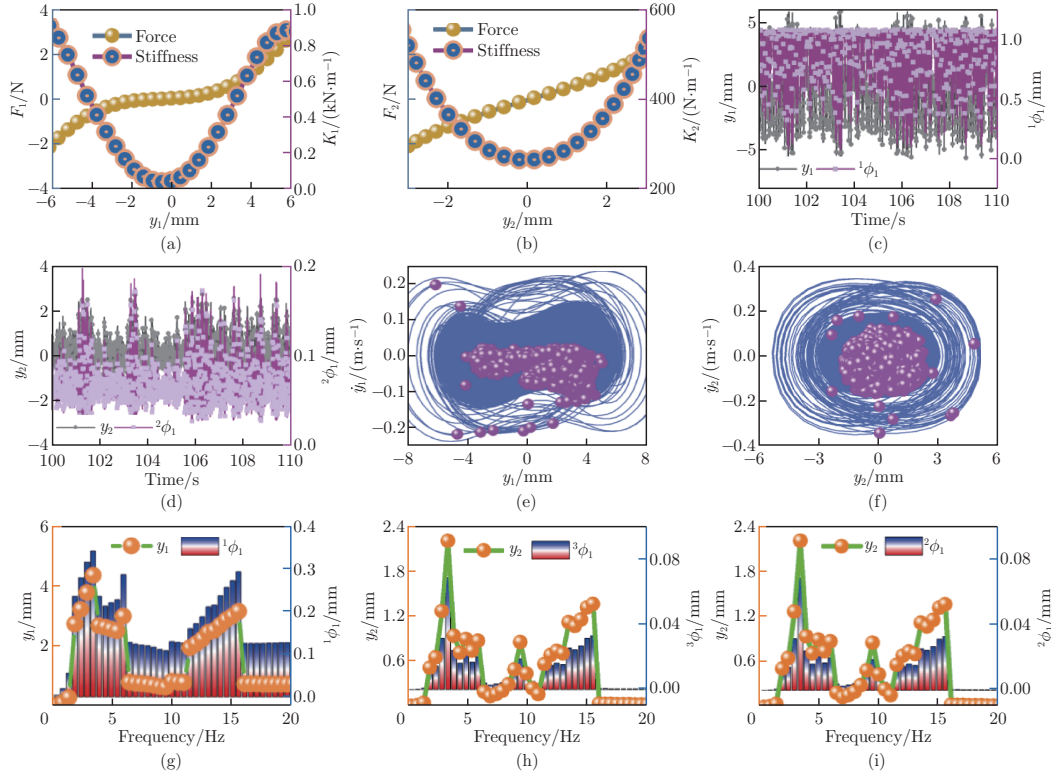


Fig. 3 The restoring force and stiffness of (a) the first layer and (b) the second layer energy harvesters calculated by the dynamic data; (c) the phase portraits and Poincaré sections corresponding to the motion of Mass 1; (d) the comparison of the time series depicting the motion displacement of Mass 1 and the mid-span deflection response of the PCB; (e) the comparison of the time series depicting the motion displacement of Mass 2 and the mid-span deflection response of the PCB; (f) the phase portraits and Poincaré sections corresponding to the motion of Mass 2; (g)–(i) the amplitude-frequency response curves for the mass and mid-span deflection of the PCBs (color online)

characteristics of the output voltage for the first layer energy harvester. However, compared with the output voltage of PCB 1, the amplitude of the output voltage of PCB 3 is slightly smaller. Similarly, the peak output voltage of PCB 3 is also greater than its RMS value, with a more significant difference observed in the first resonance region. As previously mentioned, the dynamic responses of PCB 3 and 4 share the same amplitude but exhibit a phase difference of 90° . Since the output voltage of the PCB is strictly correlated with its dynamic response, it can be predicted that the output voltage will exhibit the same characteristics as the dynamic response. To validate this prediction, Fig. 4(f) provides the time series of the output voltages of PCB 3 and PCB 4. Clearly, consistent with the prediction, the two beams exhibit identical amplitudes with a phase difference of 90° . Therefore, this paper focuses solely on the analysis of PCB 3, while a detailed discussion of PCB 4 is omitted.

By substituting the parameters of the TENG, namely, $\sigma = 80 \mu\text{C}/\text{m}^2$, $w = 11 \text{ mm}$, $L = 20 \text{ mm}$, $d_1 = d_2 = 2.2 \times 10^{-4} \text{ m}$, $\varepsilon_1 = 4$, and $\varepsilon_2 = 2$, into Eqs. (53) and (54), the electrical output of the TENG can be obtained, as illustrated in Figs. 4(g)–4(i). It should be noted that the parameters of the TENG are partially derived from Ref. [19], while the remaining parameters are selected based on the geometric configuration of the energy harvester. As shown in Fig. 4(g), the OC voltage of the TENG is strictly correlated with the displacement response of Mass 2,

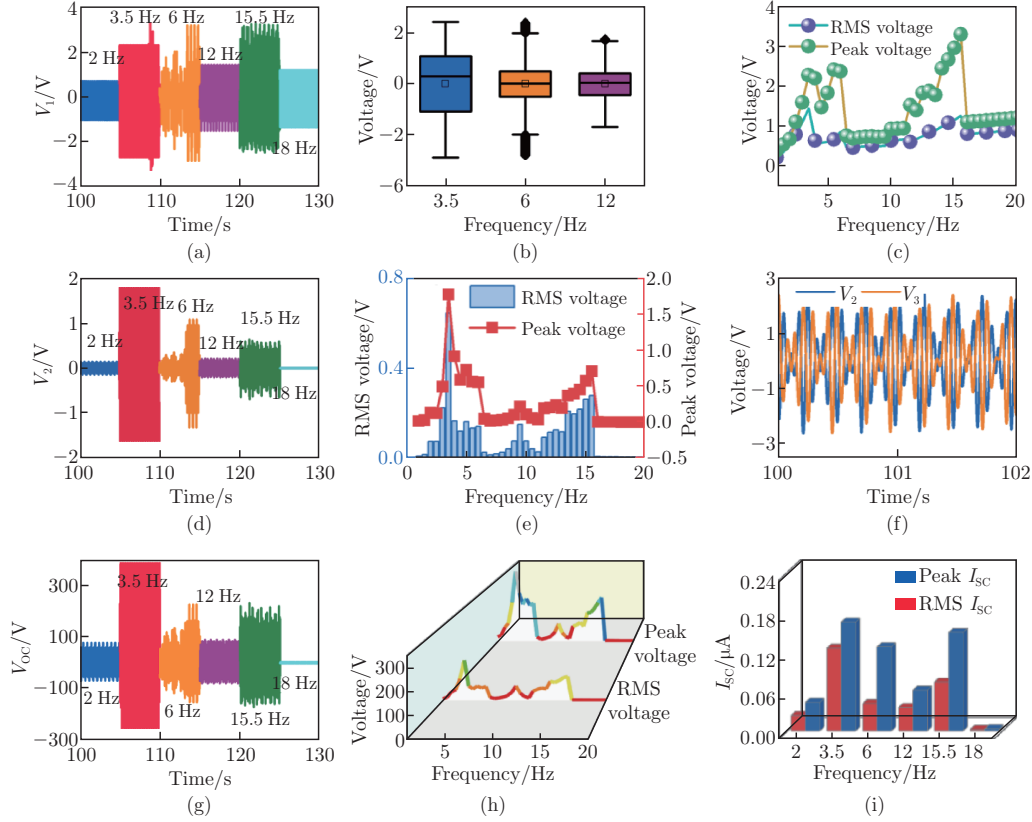


Fig. 4 (a) Time series of voltage responses for PCB 1 under varying excitation frequencies; (b) data distribution of the output voltage for the PCB in the first layer; (c) comparison of the RMS and peak voltage of the PCB in the first layer; (d) time series of voltage responses for the PCB 3 under varying excitation frequencies; (e) comparison of the RMS and peak voltages of the PCB in the second layer; (f) comparison of the output voltage for the PCB 3 and PCB 4; (g) time series of voltage responses for the TENG under varying excitation frequencies; (h) RMS and peak output voltages of the TENG; (i) RMS and peak SC current of the TENG under different excitation frequencies (color online)

resulting in time series of electrical output that exhibits a similar pattern and envelope to the voltage output time series of PCB 3. Specifically, within the frequency range of 3.5 Hz to 15.5 Hz, two resonance peaks are observed. However, compared with the voltage output of the PCB, the amplitude of the OC voltage is two orders of magnitude larger, reaching a maximum of over 200 V. Figure 4(h) depicts the frequency response curves of the RMS and peak values of the OC voltage. Clearly, consistent with the amplitude-frequency response curves of the electrical output of PCB, there is a significant discrepancy between the RMS and peak values of the OC voltage, with the difference being more pronounced in the resonance regions where nonlinear characteristics are more evident. In other words, while the TENG may exhibit larger peaks in the resonance regions, its RMS value could be relatively smaller. Although the OC voltage of the TENG is significantly higher than the voltage output of the PCB, does this imply that the TENG has a higher output power? In fact, this is not the case, as shown in Fig. 4(i), where the SC current of the TENG is only on the order of microamperes, which is much smaller than the output current of the piezoelectric material^[21].

To provide a comprehensive guidance for the design optimization of the 2-DOF energy harvester, Fig. 5 systematically presents the electrical output characteristics under various param-

eter configurations. Figures 5(a)–5(c) illustrate the electrical outputs of distinct transduction units as a function of the total stiffness of the four linear springs, ranging from 400 N/m to 1 000 N/m. Notably, PCB 1 maintains substantial electrical output across two frequency bands throughout the stiffness variation. Particularly, when the stiffness is below 600 N/m, the low-frequency range exhibiting significant electrical output from the PCB expands considerably. This phenomenon can be attributed to the negative stiffness of the energy harvester surpassing the stiffness of the linear springs, thereby inducing negative stiffness characteristics. Under such conditions, the energy harvester can overcome the potential barrier across a broader low-frequency spectrum, resulting in enhanced dynamic responses and consequently large electrical outputs.

The effects of spring stiffness on the electrical output of PCB 3 are illustrated in Fig. 5(b). While the stiffness of the linear springs does not directly affect the mechanical properties of the second layer energy harvester, the inherent coupling between the first and second layers impacts the electrical output of PCB 3. As depicted, increasing stiffness to 1 000 N/m enhances the output voltage of PCB 3, primarily due to the amplified dynamic response amplitude of the first layer harvester, which induces larger relative displacements of Mass 2. Figure 5(c) presents the stiffness effect on the electrical output characteristics of the TENG, where gray bars represent the frequency range with OC voltage exceeding 100 V, and colored bars represent the peak output voltage. The results indicate that extreme stiffness values optimize electrical output performance, yielding higher peak voltages and broader efficient energy harvesting bandwidths.

Figures 5(d)–5(f) illustrate the effects of the initial magnet gap d_1 in the first layer energy harvester on energy harvesting performance. As d_1 increases from 7 mm to 10 mm, the peak electrical output of PCB 1 significantly diminishes, primarily due to the reduced negative stiffness from the magnets, causing a transition from bistable to high-stiffness characteristics. This transition reduces dynamic response amplitudes, consequently reducing energy conversion efficiency. Figure 5(e) demonstrates the effect of the gap d_1 on the output voltage of PCB 3, showing opposite trends with Beam 1: increasing d_1 maintains nearly constant average output voltage, but widens confidence intervals (CIs). For the SC current of the TENG, increasing d_1 shifts the larger current frequency range toward lower frequencies while decreasing the amplitude.

The effects of the gap d_2 between the two magnets in the second layer energy harvester on the output voltage of PCB 3 and the OC voltage of the TENG are illustrated in Figs. 5(g)–5(h). Increasing d_2 from 10 mm to 12 mm reduces both median value and MV of the RMS output voltage amplitude-frequency response curve for PCB 3 within 1 Hz–20 Hz, significantly narrowing the RMS voltage range. The maximum value of the amplitude-frequency response curve also decreases from approximately 1.8 V to around 0.7 V, indicating the detrimental effect of the increasing of d_2 on vibration energy conversion. Further increasing d_2 to 14 mm exacerbates these effects, reducing peak, median, and mean output voltages due to weakened magnet interactions and consequent decreased PCB deformation and transduction efficiency. Conversely, the effect of d_2 on the OC voltage of the TENG opposes that on PCB 3, with smaller gaps yielding higher OC voltages. This inverse relationship stems from reduced magnetic repulsion on Mass 2 at smaller gaps, creating larger low-stiffness regions near static equilibrium and enabling broader mass displacement under base excitation.

Furthermore, the electrical output characteristics of the energy harvester are significantly influenced by external load resistance. Figure 5(i) presents the output voltage of the PCB as a function of varying resistance. It is evident that within a lower resistance range, the RMS output voltage of the PCB exhibits a rapid increase with increasing resistance. However, when the resistance reaches 109 M Ω , further increments in resistance do not significantly affect the output voltage of the PCB. This behavior suggests that the output voltage asymptotically approaches the OC voltage under these conditions, indicating a saturation point in the voltage-resistance curve.

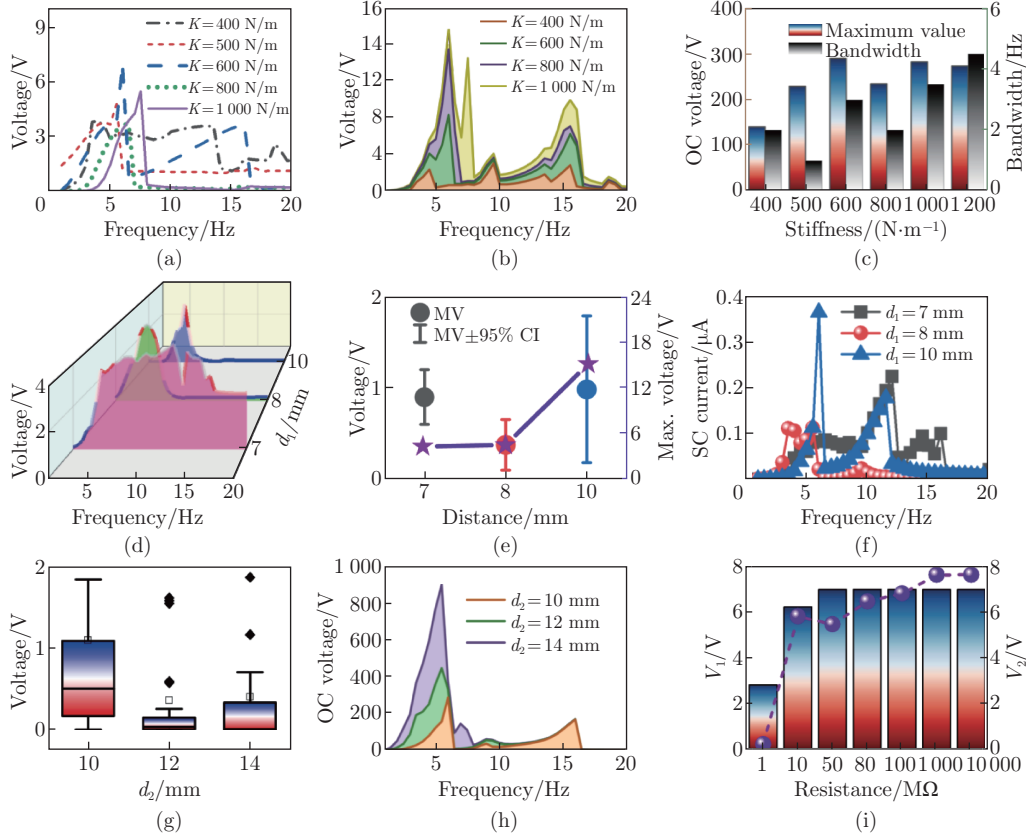


Fig. 5 Effects of system parameters on electrical outputs: (a) output voltage of PCB 1; (b) PCB 3 as a function of the stiffness; (c) OC voltage of the TENG as a function of the stiffness; (d)–(f) electrical outputs of the energy harvester as a function of the distance between the first magnet pair; (g)–(h) electrical outputs as a function of the distance between the second layer magnet pair; (i) output voltage of the first and second layer PCBs as a function of external resistance (color online)

5 Conclusions

This study introduces an innovative 2-DOF hybrid energy harvester specifically developed to overcome challenges in low-frequency vibration energy harvesting. The proposed design integrates a PCB combined with magnet pairs in the first layer to create an NSM, while the second layer effectively combines piezoelectric transduction units with a TENG. The deflection characteristics of the piezoelectric film depend not only on its inherent material properties but also on the dynamic response of the complete energy harvesting system, enabling efficient conversion of vibration energy in low-frequency ranges. A principal achievement of this research lies in establishing a comprehensive nonlinear electromechanical coupling dynamic model that offers a robust analytical framework for understanding the system behavior. Analytical results reveal that the designed structure substantially expands the effective frequency range for energy conversion, particularly within low-frequency domains. The strategic integration of piezoelectric and triboelectric conversion mechanisms in the second layer significantly boosts overall energy conversion efficiency. These advancements demonstrate the potential of this method for practical implementation in powering wireless sensor networks and internet of things (IoT) devices, providing a sustainable energy solution. Subsequent investigations will concentrate on

further optimization of the harvester design and experimental verification to evaluate real-world performance characteristics.

Conflict of interest The authors declare no conflict of interest.

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Supporting information Supporting information is available online <https://doi.org/10.1007/s10483-025-3264-7>.

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