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Jon Sporring
Lars Schjøth
Kenny Erleben

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Spatial and Temporal Ray Differentials

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Jon Sporring, Lars Schjøth, and Kenny Erleben
eScience Center, Department of Computer Science, University of Copenhagen
Universitetsparken 1, DK-2100 Copenhagen, Denmark
Contact email: sporring@diku.dk

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Abstract

We consider ray bundles emanating from a source such as a camera or light source. We derive the complete spatial and temporal structure to first order of the intersection of ray bundles with scene geometry where scene geometry is given as any implicit function. Further, we present the complete details of two often used geometrical representations. The first order structure may be used as the linear approximation of the change of photon shape as the camera, objects, and light source change as function of space and time. Our work generalizes previous work on ray differentials [Igehy, 1999] and photon differentials [Schjøth et al., 2007].

1 Ray differential

In this work we consider reflection and refraction of light rays off and through surfaces as illustrated in Figure 1. We will derive the spatial-temporal first order structure of these processes without any simplifying assumptions. Consider two points in 3-space, $\boldsymbol{P}, \boldsymbol{x} \in \mathbb{R}^3$. We define the line $\boldsymbol{Q}(s)$ parametrized by $s \in \mathbb{R}$ passing through $\boldsymbol{x}$ and $\boldsymbol{V}$ as,

$$\boldsymbol{Q} = \boldsymbol{P} + s\boldsymbol{V} \tag{1}$$

where

$$\boldsymbol{V} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_2}, \tag{2a}$$

$$\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{P}. \tag{2b}$$

The line is henceforth called the ray, and we use column vectors, such that $\|\boldsymbol{v}\|_2 = \sqrt{\boldsymbol{v}^T\boldsymbol{v}}$. This work extends [Igehy, 1999, Schjøth et al., 2007]: We develop the complete first order structure of $\boldsymbol{Q}$ in terms of its intersection with scene geometry and the following directions of reflection and refraction.

We use the notation of differential matrix calculus [Magnus and Neudecker, 1988]. Differentials are rooted in Taylor series, i.e., consider an analytical function $f : \mathbb{R} \to \mathbb{R}$, and write its Taylor series as, $f(y + \Delta y) = f(y) + f'(y)\Delta y + \mathcal{O}\left(\Delta y^2\right)$, where $\mathcal{O}$ is the remainder in Landau notation, and f' is the first order derivative of f. We define the differential of f as the limit, $f(y + \Delta y) - f(y) \to \mathrm{d}f$ as $\Delta y \to \mathrm{d}y$, and write

$$\mathrm{d}f = f'(y)\, dy. \tag{3}$$

The extension to vector and matrix equations is straight forward, since their Taylor series are element-wise Taylor series. We use the same notation and note that derivative are now Jacobian matrix, e.g., for vector equations such as $\boldsymbol{V} \in \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{x} \in \mathbb{R}^n$, the Jacobian of $\boldsymbol{V}$ w.r.t. the variable $\boldsymbol{x}$ is the matrix $\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}$ who's ij'th entry is the partial derivative $\frac{\partial \boldsymbol{V}_i}{\partial \boldsymbol{x}_j}$. Hence, the j'th column is the change vector of $\boldsymbol{V}$ when only considering the coordinate direction $\boldsymbol{x}_j$. The Jacobian w.r.t. the complete space of parameters is often just written as $\partial \boldsymbol{V}$ for convenience. The differential embodies the complete first order structure of a function, and a first order estimate of the change of, e.g., $\Delta\boldsymbol{Q}$ is obtained by replacing the infinitesimals with finite values, i.e., $d\boldsymbol{x}$ with $\Delta\boldsymbol{x}$ in the expression of the differential of $\mathrm{d}\boldsymbol{Q}$.

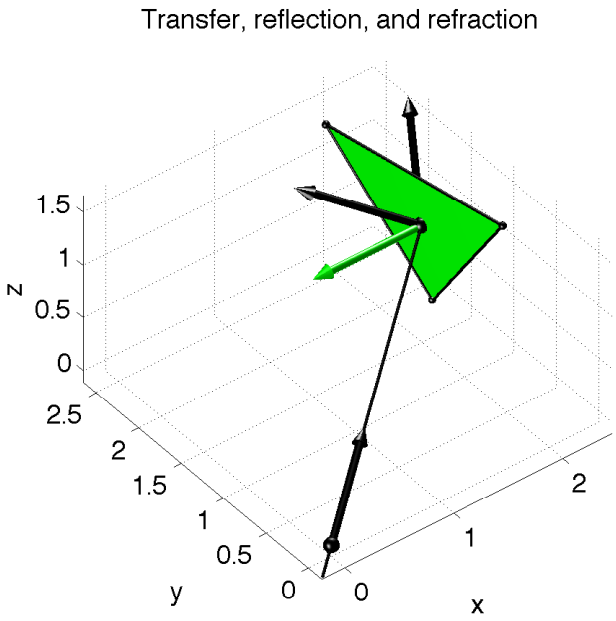


Figure 1: Transfer, Reflection, and Refraction by a flat surface. Black arrows are ray directions from origin, through reflection and refraction, green arrow is the normal for the green surface patch.

The differentials are useful, since by the usual rules of differentiation, such as the chain rule and the derivative of inverse functions, they allow for a step-by-step derivation of the Jacobian of a function w.r.t. its parameters much like the peels of an onion. As an example, consider $\boldsymbol{V}$, which is a composition of functions $\boldsymbol{v}$, $\boldsymbol{x}$, and $\boldsymbol{P}$. To find the Jacobian of $\boldsymbol{V}(\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{P}))$ we first find the Jacobian of $\boldsymbol{V}(\boldsymbol{v})$ by evaluating the differential of (2a) to d$\boldsymbol{v}$ as follows.

$$\mathrm{d}\boldsymbol{V} = \frac{(\mathrm{d}\boldsymbol{v})\left(\boldsymbol{v}^T\boldsymbol{v}\right)^{1/2} - \boldsymbol{v}\left(\boldsymbol{v}^T\boldsymbol{v}\right)^{-1/2}\boldsymbol{v}^T\mathrm{d}\boldsymbol{v}}{\boldsymbol{v}^T\boldsymbol{v}} \tag{4a}$$

$$= \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{\left(\boldsymbol{v}^T\boldsymbol{v}\right)^{3/2}}\,\mathrm{d}\boldsymbol{v} \tag{4b}$$

where $\boldsymbol{I}_3$ is the 3×3 identity matrix. Thus, if $\boldsymbol{v}$ is the free parameter, then the above expression is the final form of the Jacobian. However, as in our case, when $\boldsymbol{v}$ is a function of other variables, we repeat the exercise until we have an expression solely in terms of the differentials of the free parameters. I.e., the Jacobian of $\boldsymbol{v}$ w.r.t. parameters $\boldsymbol{x}$ and $\boldsymbol{P}$ is found to be,

$$\mathrm{d}\boldsymbol{v} = \left(\mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{P}\right), \tag{5}$$

and hence, we conclude that the Jacobian of $\boldsymbol{V}$ w.r.t. parameters $\boldsymbol{x}$ and $\boldsymbol{P}$ is

$$\mathrm{d}\boldsymbol{V} = \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{\left(\boldsymbol{v}^T\boldsymbol{v}\right)^{3/2}}\left(\mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{P}\right), \tag{6}$$

where it is to be understood that all variables that are not differentials, are evaluated at the particular point of interest as terms in Taylor series usually are. This is the complete first order structure of $\boldsymbol{V}$ in terms of $\boldsymbol{x}$ and $\boldsymbol{P}$, and we identify the Jacobians as

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} = \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{\left(\boldsymbol{v}^T\boldsymbol{v}\right)^{3/2}}, \qquad \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{P}} = -\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}. \tag{7}$$

By replacing d$\boldsymbol{x}$ and d$\boldsymbol{P}$ with finite steps $\Delta\boldsymbol{x}$ and $\Delta\boldsymbol{P}$ we may evaluate the linear approximation to the changes of $\boldsymbol{V}$. Geometrically speaking, this approximation is a point in the tangent space at $\boldsymbol{V}$ w.r.t. $\boldsymbol{x}$ and $\boldsymbol{P}$ and parametrized by $\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\,\Delta\boldsymbol{x}$ and $\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{P}}\,\Delta\boldsymbol{P}$ where $\Delta\boldsymbol{x}$ and $\Delta\boldsymbol{P}$ are the free vector parameters of the approximation. Finally, note that if

one parameter is constant, then it's differential is evaluated to be zero, e.g., consider that $\boldsymbol{P} = \text{const.}$ in which case $\mathrm{d}\boldsymbol{P} = 0$, and the differential of $\boldsymbol{V}$ reduces to $\mathrm{d}\boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}$.

Often, $\boldsymbol{x}$ is restricted to be a point on a 2-dimensional surface such as an image plane or a sphere, and as a consequence we must further evaluate the Jacobian of the parametrization of $\boldsymbol{x}$. E.g., a plane may be parametrized as, $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$ for some constant and orthogonal coordinate axes $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, and we evaluate $\mathrm{d}\boldsymbol{x} = \begin{bmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2\end{bmatrix}\begin{bmatrix}\mathrm{d}x_1 & \mathrm{d}x_2\end{bmatrix}^T$ and $\frac{\partial \boldsymbol{V}}{\partial [x_1\ x_2]^T} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\begin{bmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2\end{bmatrix}$. Likewise, when $\boldsymbol{x}$ restricted to the surface of a unite sphere, centered at $\boldsymbol{P}$, then we may use the spherical parametrization,

$$\boldsymbol{x} = \begin{bmatrix}\cos\phi(t)\sin\theta(t) & \sin\phi(t)\sin\theta(t) & \cos\theta(t)\end{bmatrix}^T, \tag{8a}$$

$$\boldsymbol{P} = \begin{bmatrix}\boldsymbol{P}_1(t) & \boldsymbol{P}_2(t) & \boldsymbol{P}_3(t)\end{bmatrix}^T, \tag{8b}$$

such that

$$\mathrm{d}\boldsymbol{x} = \begin{bmatrix}-\sin\phi\sin\theta & \cos\phi\cos\theta & \frac{\partial \boldsymbol{x}_1}{\partial t} \\ \cos\phi\sin\theta & \sin\phi\cos\theta & \frac{\partial \boldsymbol{x}_2}{\partial t} \\ 0 & -\sin\theta & \frac{\partial \boldsymbol{x}_3}{\partial t}\end{bmatrix}\begin{bmatrix}\mathrm{d}\phi \\ \mathrm{d}\theta \\ \mathrm{d}t\end{bmatrix}, \tag{9a}$$

$$\mathrm{d}\boldsymbol{P} = \begin{bmatrix}\frac{\partial \boldsymbol{P}_1}{\partial t} & \frac{\partial \boldsymbol{P}_2}{\partial t} & \frac{\partial \boldsymbol{P}_3}{\partial t}\end{bmatrix}^T \mathrm{d}t, \tag{9b}$$

where $\frac{\partial \boldsymbol{x}}{\partial t}$ and $\frac{\partial \boldsymbol{P}}{\partial t}$ are the linear part of the time dependence of $\boldsymbol{x}$ and $\boldsymbol{P}$. Hence, the differential of $\boldsymbol{V}$ w.r.t. the θ, ϕ, and t parameters is,

$$\mathrm{d}\boldsymbol{V} = \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{\left(\boldsymbol{v}^T\boldsymbol{v}\right)^{3/2}}\left(\begin{bmatrix}-\sin\phi\sin\theta & \cos\phi\cos\theta & \frac{\partial \boldsymbol{x}_1}{\partial t} \\ \cos\phi\sin\theta & \sin\phi\cos\theta & \frac{\partial \boldsymbol{x}_2}{\partial t} \\ 0 & -\sin\theta & \frac{\partial \boldsymbol{x}_3}{\partial t}\end{bmatrix}\begin{bmatrix}\mathrm{d}\phi \\ \mathrm{d}\theta \\ \mathrm{d}t\end{bmatrix} - \begin{bmatrix}\frac{\partial \boldsymbol{P}_1}{\partial t} \\ \frac{\partial \boldsymbol{P}_2}{\partial t} \\ \frac{\partial \boldsymbol{P}_3}{\partial t}\end{bmatrix}\mathrm{d}t\right) \tag{10a}$$

$$= \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{\left(\boldsymbol{v}^T\boldsymbol{v}\right)^{3/2}}\begin{bmatrix}-\sin\phi\sin\theta & \cos\phi\cos\theta & \frac{\partial \boldsymbol{x}_1}{\partial t} - \frac{\partial \boldsymbol{P}_1}{\partial t} \\ \cos\phi\sin\theta & \sin\phi\cos\theta & \frac{\partial \boldsymbol{x}_2}{\partial t} - \frac{\partial \boldsymbol{P}_2}{\partial t} \\ 0 & -\sin\theta & \frac{\partial \boldsymbol{x}_3}{\partial t} - \frac{\partial \boldsymbol{P}_3}{\partial t}\end{bmatrix}\begin{bmatrix}\mathrm{d}\phi \\ \mathrm{d}\theta \\ \mathrm{d}t\end{bmatrix}. \tag{10b}$$

Hence, we identify the Jacobian,

$$\frac{\partial \boldsymbol{V}}{\partial [\phi, \theta, t]^T} = \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{\left(\boldsymbol{v}^T\boldsymbol{v}\right)^{3/2}}\begin{bmatrix}-\sin\phi\sin\theta & \cos\phi\cos\theta & \frac{\partial \boldsymbol{x}_1}{\partial t} - \frac{\partial \boldsymbol{P}_1}{\partial t} \\ \cos\phi\sin\theta & \sin\phi\cos\theta & \frac{\partial \boldsymbol{x}_2}{\partial t} - \frac{\partial \boldsymbol{P}_2}{\partial t} \\ 0 & -\sin\theta & \frac{\partial \boldsymbol{x}_3}{\partial t} - \frac{\partial \boldsymbol{P}_3}{\partial t}\end{bmatrix}, \tag{11}$$

which is evaluated at a specific value of the parameters θ_0, ϕ_0, and t_0. For static scenes, all terms involving $\mathrm{d}t$ is zero, i.e., the right most column of the Jacobian is zero.

The above illustrates the advantage of matrix differential algebra: When the parametrization of $\boldsymbol{v}$ is changed, then $\mathrm{d}\boldsymbol{v}$ changes, but $\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{v}}$ is remains the same, all of which matrix differential algebra allows us to express in terms of matrices.

2 Transfer, Reflection, and Refraction

In the following we will investigate light's interaction with dielectric material, i.e., reflection and refraction. We consider a ray from a source at point $\boldsymbol{P}$ with direction $\boldsymbol{V}$, which intersects a surface at position $\boldsymbol{Q}$ and is reflected and refracted in directions $\boldsymbol{W}_{\text{reflect}}$ and $\boldsymbol{W}_{\text{refract}}$ respectively. To avoid discussion on any particular parametrization of $\boldsymbol{V}$, we will in the following evaluate expressions no further than $\mathrm{d}\boldsymbol{V}$. The normal at $\boldsymbol{Q}$ will be denoted $\boldsymbol{N}$, and the ratio of refraction at $\boldsymbol{Q}$ will be denoted η. Our main goal will be to calculate the differentials, $\mathrm{d}\boldsymbol{Q}$, $\mathrm{d}\boldsymbol{W}_{\text{reflect}}$, and $\mathrm{d}\boldsymbol{W}_{\text{refract}}$ as a function of relevant parametrizations, e.g.,

$$\mathrm{d}\boldsymbol{Q} = \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{V}}\,\mathrm{d}\boldsymbol{V} + \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{P}}\,\mathrm{d}\boldsymbol{P} + \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{N}}\,\mathrm{d}\boldsymbol{N}, \tag{12a}$$

$$\mathrm{d}\boldsymbol{W}_{\text{reflect}} = \frac{\partial \boldsymbol{W}_{\text{reflect}}}{\partial \boldsymbol{V}}\,\mathrm{d}\boldsymbol{V} + \frac{\partial \boldsymbol{W}_{\text{reflect}}}{\partial \boldsymbol{N}}\,\mathrm{d}\boldsymbol{N}, \tag{12b}$$

$$\mathrm{d}\boldsymbol{W}_{\text{refract}} = \frac{\partial \boldsymbol{W}_{\text{refract}}}{\partial \boldsymbol{V}}\,\mathrm{d}\boldsymbol{V} + \frac{\partial \boldsymbol{W}_{\text{refract}}}{\partial \boldsymbol{N}}\,\mathrm{d}\boldsymbol{N} + \frac{\partial \boldsymbol{W}_{\text{refract}}}{\partial \eta}\,\mathrm{d}\eta, \tag{12c}$$

and subsequently identify the respective Jacobians.

Following [Igehy, 1999] we sketch an iterative process, where in each iteration: 1) The ray is transfered to the point of intersecting geometry, $\boldsymbol{Q}$, 2) the directions of reflection and refraction,$\boldsymbol{W}_{\text{reflect}}$ and $\boldsymbol{W}_{\text{refract}}$, are calculated simultaneously. The pairs $(\boldsymbol{Q}, \boldsymbol{W}_{\text{reflect}})$ and $(\boldsymbol{Q}, \boldsymbol{W}_{\text{refract}})$ are used as two new source points and directions for following iterations.

2.1 Transfer

The transfer of a ray onto a surface at distance s is,

$$\boldsymbol{Q} = \boldsymbol{P} + s\boldsymbol{V}. \tag{13}$$

We will assume that the smooth surface $\boldsymbol{Q} \in \Omega$ embedded in $\mathbb{R}^3$ is given implicitly as the zero level-set of a scalar function of M parameters, $F : \mathbb{R}^M \times \mathbb{R}^3 \to \mathbb{R}$, as $0 = F_{\boldsymbol{\alpha}}(\boldsymbol{Q})$, where $\boldsymbol{\alpha} \in \mathbb{R}^M$ is a vector of parameters that specify the shape of F, and we will assume that there exists a method for solving for the smallest $s^* > 0$, such that $0 = F_{\boldsymbol{\alpha}}(\boldsymbol{P} + s^*\boldsymbol{V})$. Finally, we require that the surface unit normal, $\boldsymbol{N}$, exists at $\boldsymbol{Q}$.

The differential is found to be

$$\mathrm{d}\boldsymbol{Q} = \mathrm{d}\boldsymbol{P} + s^*\,\mathrm{d}\boldsymbol{V} + \boldsymbol{V}\,\mathrm{d}s. \tag{14}$$

Had $\boldsymbol{P}$, $\boldsymbol{V}$, and s been independent of each other, then this would be the final form and Jacobians w.r.t. the 3 variables. However, $\mathrm{d}s$ depends on $\boldsymbol{P}$, $\boldsymbol{V}$, and $\boldsymbol{\alpha}$ through F, which is why we expand $\mathrm{d}s$ as, $\mathrm{d}s = \frac{\partial s^*}{\partial \boldsymbol{P}}\mathrm{d}\boldsymbol{P} + \frac{\partial s^*}{\partial \boldsymbol{V}}\mathrm{d}\boldsymbol{V} + \frac{\partial s^*}{\partial \boldsymbol{\alpha}}\mathrm{d}\boldsymbol{\alpha}$, and get

$$\mathrm{d}\boldsymbol{Q} = \mathrm{d}\boldsymbol{P} + s^*\,\mathrm{d}\boldsymbol{V} + \boldsymbol{V}\left(\frac{\partial s^*}{\partial \boldsymbol{P}}\mathrm{d}\boldsymbol{P} + \frac{\partial s^*}{\partial \boldsymbol{V}}\mathrm{d}\boldsymbol{V} + \frac{\partial s^*}{\partial \boldsymbol{\alpha}}\mathrm{d}\boldsymbol{\alpha}\right) \tag{15a}$$

$$= \left(\boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s^*}{\partial \boldsymbol{P}}\right)\mathrm{d}\boldsymbol{P} + \left(s^*\boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s^*}{\partial \boldsymbol{V}}\right)\mathrm{d}\boldsymbol{V} + \boldsymbol{V}\frac{\partial s^*}{\partial \boldsymbol{\alpha}}\mathrm{d}\boldsymbol{\alpha}, \tag{15b}$$

where $\boldsymbol{I}_3$ is the 3×3 identity matrix. The Jacobians of $\boldsymbol{Q}$ are now identifiable as, $\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{P}} = \boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s^*}{\partial \boldsymbol{P}}$, $\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{V}} = s^*\boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s^*}{\partial \boldsymbol{V}}$, and $\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{\alpha}} = \boldsymbol{V}\frac{\partial s^*}{\partial \boldsymbol{\alpha}}$. In the examples given later, we shall evaluate these terms for flat and pseudo-flat geometry F. Finally note that when the differential $\mathrm{d}s$ depends on $\mathrm{d}\boldsymbol{N}$, then $\mathrm{d}s$ will depend on the curvature of the surface at s^*. For readability we will in the remainder of this article use the symbol s to denote s^*.

2.2 Reflection

Given a ray transferred to a surface, specified by its normal vector $\boldsymbol{N}$, reflection is given by

$$\boldsymbol{W}_{\text{reflect}} = \boldsymbol{V} - 2\left(\boldsymbol{V}^T\boldsymbol{N}\right)\boldsymbol{N}. \tag{16}$$

Hence,

$$\mathrm{d}\boldsymbol{W}_{\text{reflect}} = \mathrm{d}\boldsymbol{V} - 2\left(\left(\mathrm{d}\boldsymbol{V}^T\boldsymbol{N} + \boldsymbol{V}^T\mathrm{d}\boldsymbol{N}\right)\boldsymbol{N} + \left(\boldsymbol{V}^T\boldsymbol{N}\right)\mathrm{d}\boldsymbol{N}\right) \tag{17a}$$

$$= \left(\boldsymbol{I}_3 - 2\boldsymbol{N}\boldsymbol{N}^T\right)\mathrm{d}\boldsymbol{V} - 2\left(\boldsymbol{V}^T\boldsymbol{N}\boldsymbol{I}_3 + \boldsymbol{N}\boldsymbol{V}^T\right)\mathrm{d}\boldsymbol{N}. \tag{17b}$$

Hence, we identify the Jacobians as $\frac{\partial \boldsymbol{W}_{\text{reflect}}}{\partial \boldsymbol{V}} = \left(\boldsymbol{I}_3 - 2\boldsymbol{N}\boldsymbol{N}^T\right)$, and $\frac{\partial \boldsymbol{W}_{\text{reflect}}}{\partial \boldsymbol{N}} = -2\left(\boldsymbol{V}^T\boldsymbol{N}\boldsymbol{I}_3 + \boldsymbol{N}\boldsymbol{V}^T\right)$.

2.3 Refraction

Given a ray transferred to a surface, specified by its normal vector $\boldsymbol{N}$ and its ratio of refraction indices η, refraction is given by Snell's law [Watt and Watt, 1992, Igehy, 1999],

$$\boldsymbol{W}_{\text{refract}} = \eta\boldsymbol{V} - \mu\boldsymbol{N}, \tag{18}$$

where

$$\mu = \eta \boldsymbol{V}^T \boldsymbol{N} + \sqrt{\xi}, \tag{19a}$$

$$\xi = 1 - \eta^2 \left(1 - \left(\boldsymbol{V}^T \boldsymbol{N}\right)^2\right). \tag{19b}$$

The refraction ratio between water and air is typically $\eta = 1.33$, and an often use approximation is $\eta \simeq 1$ in which case $\xi \simeq \left(\boldsymbol{V}^T \boldsymbol{N}\right)^2$, nevertheless, we will derive the complete structure to facilitate a greater range of η values as well as allow for η to vary. The differential is found to be,

$$\mathrm{d}\boldsymbol{W}_{\text{refract}} = \mathrm{d}\eta\, \boldsymbol{V} + \eta \mathrm{d}\boldsymbol{V} - \mathrm{d}\mu\, \boldsymbol{N} - \mu \mathrm{d}\boldsymbol{N} \tag{20a}$$

$$= \eta \mathrm{d}\boldsymbol{V} - \mu \mathrm{d}\boldsymbol{N} + \boldsymbol{V}\mathrm{d}\eta - \boldsymbol{N}\left(\frac{\partial \mu}{\partial \boldsymbol{V}}\mathrm{d}\boldsymbol{V} + \frac{\partial \mu}{\partial \boldsymbol{N}}\mathrm{d}\boldsymbol{N} + \frac{\partial \mu}{\partial \eta}\mathrm{d}\eta\right) \tag{20b}$$

$$= \left(\eta \boldsymbol{I}_3 - \boldsymbol{N}\frac{\partial \mu}{\partial \boldsymbol{V}}\right)\mathrm{d}\boldsymbol{V} - \left(\mu \boldsymbol{I}_3 + \boldsymbol{N}\frac{\partial \mu}{\partial \boldsymbol{N}}\right)\mathrm{d}\boldsymbol{N} + \left(\boldsymbol{V} - \boldsymbol{N}\frac{\partial \mu}{\partial \eta}\right)\mathrm{d}\eta. \tag{20c}$$

Thus, in order to identify the Jacobians of $\boldsymbol{W}_{\text{refract}}$, we must identify those of μ. Using $\mathrm{d}\xi = -2\eta\left(1 - \left(\boldsymbol{V}^T\boldsymbol{N}\right)^2\right)\mathrm{d}\eta + 2\eta^2\left(\boldsymbol{V}^T\boldsymbol{N}\right)\left(\mathrm{d}\boldsymbol{V}^T\,\boldsymbol{N} + \boldsymbol{V}^T\mathrm{d}\boldsymbol{N}\right)$, we see that

$$\mathrm{d}\mu = \boldsymbol{V}^T\boldsymbol{N}\mathrm{d}\eta + \eta\left(\mathrm{d}\boldsymbol{V}^T\boldsymbol{N} + \boldsymbol{V}^T\mathrm{d}\boldsymbol{N}\right) + \frac{\mathrm{d}\xi}{2\sqrt{\xi}} \tag{21a}$$

$$= \boldsymbol{V}^T\boldsymbol{N}\mathrm{d}\eta + \eta\left(\mathrm{d}\boldsymbol{V}^T\boldsymbol{N} + \boldsymbol{V}^T\mathrm{d}\boldsymbol{N}\right) + \frac{-\eta\left(1 - \left(\boldsymbol{V}^T\boldsymbol{N}\right)^2\right)\mathrm{d}\eta + \eta^2\boldsymbol{V}^T\boldsymbol{N}\left(\mathrm{d}\boldsymbol{V}^T\boldsymbol{N} + \boldsymbol{V}^T\mathrm{d}\boldsymbol{N}\right)}{\sqrt{\xi}} \tag{21b}$$

$$= \left(\boldsymbol{V}^T\boldsymbol{N} - \frac{\eta\left(1 - \left(\boldsymbol{V}^T\boldsymbol{N}\right)^2\right)}{\sqrt{\xi}}\right)\mathrm{d}\eta + \eta\left(\boldsymbol{N}^T\mathrm{d}\boldsymbol{V} + \boldsymbol{V}^T\mathrm{d}\boldsymbol{N}\right) + \frac{\eta^2\boldsymbol{V}^T\boldsymbol{N}\left(\boldsymbol{N}^T\mathrm{d}\boldsymbol{V} + \boldsymbol{V}^T\mathrm{d}\boldsymbol{N}\right)}{\sqrt{\xi}} \tag{21c}$$

$$= \left(\boldsymbol{V}^T\boldsymbol{N} - \frac{1-\xi}{\eta\sqrt{\xi}}\right)\mathrm{d}\eta + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}^T\mathrm{d}\boldsymbol{V} + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{V}^T\mathrm{d}\boldsymbol{N}. \tag{21d}$$

Gathering terms we find that

$$\begin{aligned}\mathrm{d}\boldsymbol{W}_{\text{refract}} = &\left(\eta\boldsymbol{I}_3 - \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{N}^T\right)\mathrm{d}\boldsymbol{V} \\ &- \left(\mu\boldsymbol{I}_3 + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{V}^T\right)\mathrm{d}\boldsymbol{N} \\ &+ \left(\boldsymbol{V} - \left(\boldsymbol{V}^T\boldsymbol{N} - \frac{1-\xi}{\eta\sqrt{\xi}}\right)\boldsymbol{N}\right)\mathrm{d}\eta,\end{aligned} \tag{22}$$

and we find the Jacobians as, $\frac{\partial \boldsymbol{W}_{\text{refract}}}{\partial \boldsymbol{V}} = \left(\eta\boldsymbol{I}_3 - \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{N}^T\right)$, $\frac{\partial \boldsymbol{W}_{\text{refract}}}{\partial \boldsymbol{N}} = -\left(\mu\boldsymbol{I}_3 + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{V}^T\right)$, and $\frac{\partial \boldsymbol{W}_{\text{refract}}}{\partial \eta} = \left(\boldsymbol{V} - \left(\boldsymbol{V}^T\boldsymbol{N} - \frac{1-\xi}{\eta\sqrt{\xi}}\right)\boldsymbol{N}\right)$.

3 Examples: Triangular Surface Models

A number of differentials described above depend on the surface of intersection. We will now evaluate the differentials to complete depth for two popular and practical surface models based on triangles. The implicit function of the interior of a triangle is identical to that of a plane, and the implicit function of a plane with normal $\boldsymbol{N}_{\text{flat}}$, and where $\boldsymbol{Q}_0$ is a point in the plane, is given as

$$F_{\boldsymbol{Q}_0, \boldsymbol{N}_{\text{flat}}}(\boldsymbol{Q}) = \left(\boldsymbol{Q}_0 - \boldsymbol{Q}\right)^T \boldsymbol{N}_{\text{flat}}. \tag{23}$$

The models we will investigate originates from flat and Phong shading. The cumbersome "flat" subscript is used to distinguish the geometry normal from the interpolated normal in Phong shading. Further, we will assume that the media is homogeneous, i.e., $\mathrm{d}\eta = 0$.

3.1 Flat Surface

Assuming that we have identified a triangle intersected by the ray, where $\boldsymbol{N}^T\boldsymbol{V} \neq 0$, then we combine (13) and (23) and seek the zero point,

$$0 = \left(\boldsymbol{Q}_0 - \boldsymbol{P} - s\boldsymbol{V}\right)^T \boldsymbol{N}_{\text{flat}}, \tag{24}$$

which implies that

$$s = \frac{\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T \boldsymbol{N}_{\text{flat}}}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}. \tag{25}$$

The complete differential of s is found as follows,

$$\mathrm{d}s = \frac{\left(\mathrm{d}\left(\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T \boldsymbol{N}_{\text{flat}}\right)\right)\left(\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}\right) - \left(\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T \boldsymbol{N}_{\text{flat}}\right)\mathrm{d}\left(\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}\right)}{\left(\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}\right)^2} \tag{26a}$$

$$= \frac{\left(\left(\mathrm{d}\boldsymbol{Q}_0 - \mathrm{d}\boldsymbol{P}\right)^T \boldsymbol{N}_{\text{flat}} + \left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T \mathrm{d}\boldsymbol{N}_{\text{flat}}\right) - s\left(\left(\mathrm{d}\boldsymbol{V}^T\right)\boldsymbol{N}_{\text{flat}} + \boldsymbol{V}^T\mathrm{d}\boldsymbol{N}_{\text{flat}}\right)}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}} \tag{26b}$$

$$= \frac{\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}\mathrm{d}\boldsymbol{Q}_0 - \frac{\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}\mathrm{d}\boldsymbol{P} + \frac{\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T - s\boldsymbol{V}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}\mathrm{d}\boldsymbol{N}_{\text{flat}} - \frac{s\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}\mathrm{d}\boldsymbol{V}, \tag{26c}$$

and thus we identify, $\frac{\partial s}{\partial \boldsymbol{Q}_0} = \frac{\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}$, $\frac{\partial s}{\partial \boldsymbol{P}} = -\frac{\partial s}{\partial \boldsymbol{Q}_0}$, $\frac{\partial s}{\partial \boldsymbol{N}_{\text{flat}}} = \frac{\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T - s\boldsymbol{V}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}$, and $\frac{\partial s}{\partial \boldsymbol{V}} = -s\frac{\partial s}{\partial \boldsymbol{Q}_0}$. Combining (15a), (17), (22), and (26) we find,

$$\begin{aligned}\mathrm{d}\boldsymbol{Q}^{\text{flat}} &= \left(\boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s}{\partial \boldsymbol{P}}\right)\mathrm{d}\boldsymbol{P} + \left(s\boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s}{\partial \boldsymbol{V}}\right)\mathrm{d}\boldsymbol{V} + \boldsymbol{V}\frac{\partial s}{\partial \boldsymbol{Q}_0}\mathrm{d}\boldsymbol{Q}_0 + \boldsymbol{V}\frac{\partial s}{\partial \boldsymbol{N}_{\text{flat}}}\mathrm{d}\boldsymbol{N}_{\text{flat}}, \\ &= \boldsymbol{K}\mathrm{d}\boldsymbol{P} + s\boldsymbol{K}\mathrm{d}\boldsymbol{V} + \left(\boldsymbol{I}_3 - \boldsymbol{K}\right)\mathrm{d}\boldsymbol{Q}_0 + \boldsymbol{L}\mathrm{d}\boldsymbol{N}_{\text{flat}}, \end{aligned} \tag{27a}$$

$$\mathrm{d}\boldsymbol{W}_{\text{reflect}}^{\text{flat}} = \frac{\partial \boldsymbol{W}_{\text{reflect}}^{\text{flat}}}{\partial \boldsymbol{V}}\mathrm{d}\boldsymbol{V} + \frac{\partial \boldsymbol{W}_{\text{reflect}}^{\text{flat}}}{\partial \boldsymbol{N}}\mathrm{d}\boldsymbol{N}_{\text{flat}} \tag{27b}$$

$$\mathrm{d}\boldsymbol{W}_{\text{refract}}^{\text{flat}} = \frac{\partial \boldsymbol{W}_{\text{refract}}^{\text{flat}}}{\partial \boldsymbol{V}}\mathrm{d}\boldsymbol{V} + \frac{\partial \boldsymbol{W}_{\text{refract}}^{\text{flat}}}{\partial \boldsymbol{N}}\mathrm{d}\boldsymbol{N}_{\text{flat}} + \frac{\partial \boldsymbol{W}_{\text{refract}}^{\text{flat}}}{\partial \eta}\mathrm{d}\eta \tag{27c}$$

where superscript on the Jacobians of the $\boldsymbol{W}$'s implies that the Jacobians have the same form as (17) and (22) but with use of $\boldsymbol{N}_{\text{flat}}$ as the normal, and where

$$\boldsymbol{K} = \boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s}{\partial \boldsymbol{P}} = \boldsymbol{I}_3 - \frac{\boldsymbol{V}\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}, \tag{28a}$$

$$\boldsymbol{L} = \boldsymbol{V}\frac{\partial s}{\partial \boldsymbol{N}_{\text{flat}}} = \frac{\boldsymbol{V}\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T - s\boldsymbol{V}\boldsymbol{V}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}. \tag{28b}$$

Although the surface is flat, and the spatial part of $\mathrm{d}\boldsymbol{N}_{\text{flat}}$ is zero, we cannot disregard terms involving $\mathrm{d}\boldsymbol{N}_{\text{flat}}$, since the temporal part need not be zero.

For static geometry, only terms of $\mathrm{d}\boldsymbol{P}$ and $\mathrm{d}\boldsymbol{V}$ are non-zero. Their Jacobians, $\boldsymbol{K}$ and $s\boldsymbol{K}$, describe the change in shape of the ray by intersection with the geometry. Their column vectors lie in the plane of the intersecting triangle, which can be proven as follows: First we show that $\boldsymbol{K}$ column vectors lie in the intersecting geometry, i.e., $\boldsymbol{0} = \boldsymbol{N}_{\text{flat}}^T\boldsymbol{K}$. By transpose and expansion we find,

$$\begin{aligned}\left(\boldsymbol{N}_{\text{flat}}^T\boldsymbol{K}\right)^T &= \boldsymbol{K}^T\boldsymbol{N}_{\text{flat}} \\ &= \left(\boldsymbol{I}_3 - \frac{\boldsymbol{N}_{\text{flat}}\boldsymbol{V}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}\right)\boldsymbol{N}_{\text{flat}} \\ &= \boldsymbol{N}_{\text{flat}} - \frac{\boldsymbol{N}_{\text{flat}}\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}} \\ &= \boldsymbol{0}. \end{aligned} \tag{29}$$

Since $\frac{\partial \boldsymbol{Q}^{\text{flat}}}{\partial \boldsymbol{P}}$ and $\frac{\partial \boldsymbol{Q}^{\text{flat}}}{\partial \boldsymbol{V}}$ depends linearly on $\boldsymbol{K}$ we conclude that the column vectors of both lie in the plane of the triangle regardless of whatever parameterization is chosen for $\boldsymbol{P}$ and $\boldsymbol{V}$. Further, in the limiting case, where $\boldsymbol{V}^T\boldsymbol{N} \to 0$, the

shape of the intersecting ray bundles diverges in the direction of $\boldsymbol{V}$. To show this, consider the limit of the in plane coordinate system, $\boldsymbol{e}_1 = \boldsymbol{V} - (\boldsymbol{N}^T\boldsymbol{V})\boldsymbol{N} \to \boldsymbol{V}$ and $\boldsymbol{e}_2 = \boldsymbol{e}_1 \times \boldsymbol{N} \to \boldsymbol{V} \times \boldsymbol{N}$. In this coordinate system we find the following limits,

$$\begin{aligned}
\boldsymbol{e}_1^T\boldsymbol{K} \to \boldsymbol{V}^T\boldsymbol{K} &= \boldsymbol{V}^T - \frac{\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}} = \boldsymbol{V}^T - \frac{\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}} = \boldsymbol{\infty} \\
\boldsymbol{e}_2^T\boldsymbol{K} \to (\boldsymbol{V}\times\boldsymbol{N})^T\boldsymbol{K} &= (\boldsymbol{V}\times\boldsymbol{N})^T - \frac{(\boldsymbol{V}\times\boldsymbol{N})^T\boldsymbol{V}\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}} = (\boldsymbol{V}\times\boldsymbol{N})^T
\end{aligned} \tag{30}$$

Thus, as should be expected, the shape of intersecting ray bundles becomes infinitely long in the ray direction, while constant in perpendicular direction. This is a consequence of modeling rays as lines, not an error in the first order approximation. But in practice, almost all triangles are finite, hence care must be taken in actual implementations.

Typically, a triangle will be parametrized by its vertices, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, and $\boldsymbol{Q}_2$, and a more natural parametrization of changes is in terms of the vertices. Such a parametrization allows us to further develop $\mathrm{d}\boldsymbol{N}_{\text{flat}}$. Assume that,

$$\boldsymbol{n}_{\text{flat}} = (\boldsymbol{Q}_2 - \boldsymbol{Q}_0)\times(\boldsymbol{Q}_1 - \boldsymbol{Q}_0), \tag{31}$$

To be consistent w.r.t. models for reflection and refraction, we will assume that $\boldsymbol{n}_{\text{flat}}^T\boldsymbol{V} < 0$, otherwise we will interchange $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$. Evaluating the differentials, we find that

$$\boldsymbol{N}_{\text{flat}} = \frac{\boldsymbol{n}_{\text{flat}}}{\|\boldsymbol{n}_{\text{flat}}\|_2}, \tag{32a}$$

$$\mathrm{d}\boldsymbol{N}_{\text{flat}} = \frac{\boldsymbol{n}_{\text{flat}}^T\boldsymbol{n}_{\text{flat}}\boldsymbol{I}_3 - \boldsymbol{n}_{\text{flat}}\boldsymbol{n}_{\text{flat}}^T}{\left(\boldsymbol{n}_{\text{flat}}^T\boldsymbol{n}_{\text{flat}}\right)^{3/2}}\mathrm{d}\boldsymbol{n}_{\text{flat}}, \tag{32b}$$

$$\mathrm{d}\boldsymbol{n}_{\text{flat}} = (\boldsymbol{Q}_2 - \boldsymbol{Q}_0)\times(\mathrm{d}\boldsymbol{Q}_1 - \mathrm{d}\boldsymbol{Q}_0) - (\boldsymbol{Q}_1 - \boldsymbol{Q}_0)\times(\mathrm{d}\boldsymbol{Q}_2 - \mathrm{d}\boldsymbol{Q}_0) \tag{32c}$$

$$= (\boldsymbol{Q}_1 - \boldsymbol{Q}_2)\times\mathrm{d}\boldsymbol{Q}_0 + (\boldsymbol{Q}_2 - \boldsymbol{Q}_0)\times\mathrm{d}\boldsymbol{Q}_1 + (\boldsymbol{Q}_0 - \boldsymbol{Q}_1)\times\mathrm{d}\boldsymbol{Q}_2. \tag{32d}$$

However, in order to isolate the differentials on the right hand side of each term, we instead used the matrix form of the cross product. Consider two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. Their cross-product may be written as, $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{a}_\times\boldsymbol{b} = \boldsymbol{b}_\times^T\boldsymbol{a} = -\boldsymbol{b}_\times\boldsymbol{a}$, where

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow \boldsymbol{a}_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \tag{33}$$

and likewise for $\boldsymbol{b}_\times$. Thus, we write, $\mathrm{d}\boldsymbol{n}_{\text{flat}} = (\boldsymbol{Q}_{1\times} - \boldsymbol{Q}_{2\times})\,\mathrm{d}\boldsymbol{Q}_0 + (\boldsymbol{Q}_{2\times} - \boldsymbol{Q}_{0\times})\,\mathrm{d}\boldsymbol{Q}_1 + (\boldsymbol{Q}_{0\times} - \boldsymbol{Q}_{1\times})\,\mathrm{d}\boldsymbol{Q}_2$. Further, Using

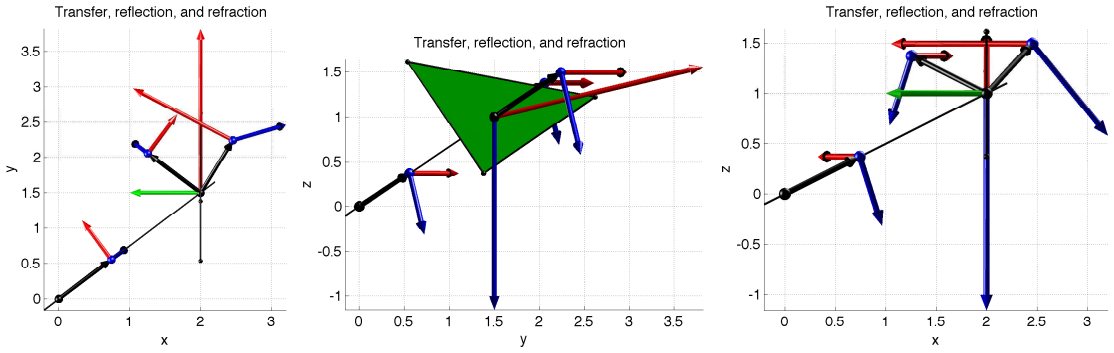
$$\boldsymbol{J} = \frac{\boldsymbol{n}_{\text{flat}}^T\boldsymbol{n}_{\text{flat}}\boldsymbol{I}_3 - \boldsymbol{n}_{\text{flat}}\boldsymbol{n}_{\text{flat}}^T}{\left(\boldsymbol{n}_{\text{flat}}^T\boldsymbol{n}_{\text{flat}}\right)^{3/2}}, \tag{34}$$

we identify $\frac{\partial\boldsymbol{N}_{\text{flat}}}{\partial\boldsymbol{Q}_0} = \boldsymbol{J}\,(\boldsymbol{Q}_{1\times} - \boldsymbol{Q}_{2\times})$, $\frac{\partial\boldsymbol{N}_{\text{flat}}}{\partial\boldsymbol{Q}_1} = \boldsymbol{J}\,(\boldsymbol{Q}_{2\times} - \boldsymbol{Q}_{0\times})$, and $\frac{\partial\boldsymbol{N}_{\text{flat}}}{\partial\boldsymbol{Q}_2} = \boldsymbol{J}\,(\boldsymbol{Q}_{0\times} - \boldsymbol{Q}_{1\times})$. Hence, we enter $\mathrm{d}\boldsymbol{N}_{\text{flat}}$ into $\mathrm{d}\boldsymbol{Q}$ from (27) and find that

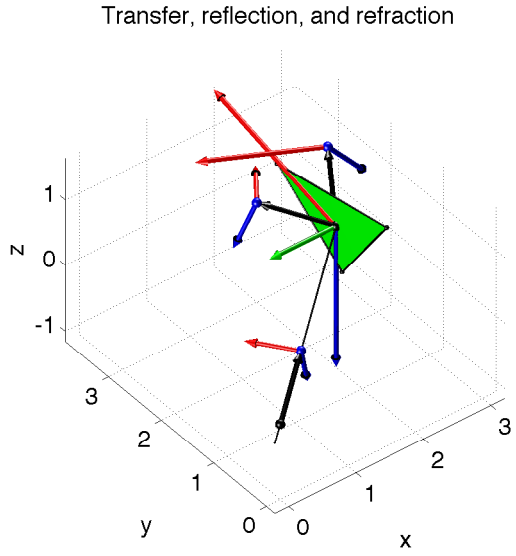
$$\begin{aligned}
\mathrm{d}\boldsymbol{Q}^{\text{flat}} &= \left(\boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s}{\partial\boldsymbol{P}}\right)\mathrm{d}\boldsymbol{P} + \left(s\boldsymbol{I}_3 + \boldsymbol{V}\frac{\partial s}{\partial\boldsymbol{V}}\right)\mathrm{d}\boldsymbol{V} \\
&\quad + \boldsymbol{V}\left(\frac{\partial s}{\partial\boldsymbol{Q}_0} + \frac{\partial s}{\partial\boldsymbol{N}_{\text{flat}}}\frac{\partial\boldsymbol{N}_{\text{flat}}}{\partial\boldsymbol{Q}_0}\right)\mathrm{d}\boldsymbol{Q}_0 + \boldsymbol{V}\frac{\partial s}{\partial\boldsymbol{N}_{\text{flat}}}\frac{\partial\boldsymbol{N}_{\text{flat}}}{\partial\boldsymbol{Q}_1}\mathrm{d}\boldsymbol{Q}_1 + \boldsymbol{V}\frac{\partial s}{\partial\boldsymbol{N}_{\text{flat}}}\frac{\partial\boldsymbol{N}_{\text{flat}}}{\partial\boldsymbol{Q}_2}\mathrm{d}\boldsymbol{Q}_2, \\
&= \boldsymbol{K}\mathrm{d}\boldsymbol{P} + s\boldsymbol{K}\mathrm{d}\boldsymbol{V} + (\boldsymbol{I}_3 - \boldsymbol{K} + \boldsymbol{L}\boldsymbol{J}\,(\boldsymbol{Q}_{1\times} - \boldsymbol{Q}_{2\times}))\,\mathrm{d}\boldsymbol{Q}_0 + \boldsymbol{L}\boldsymbol{J}\,(\boldsymbol{Q}_{2\times} - \boldsymbol{Q}_{0\times})\,\mathrm{d}\boldsymbol{Q}_1 + \boldsymbol{L}\boldsymbol{J}\,(\boldsymbol{Q}_{0\times} - \boldsymbol{Q}_{1\times})\,\mathrm{d}\boldsymbol{Q}_2.
\end{aligned} \tag{35}$$

We have now identified $\frac{\partial\boldsymbol{Q}^{\text{flat}}}{\partial\boldsymbol{P}} = \boldsymbol{K}$, $\frac{\partial\boldsymbol{Q}^{\text{flat}}}{\partial\boldsymbol{V}} = s\boldsymbol{K}$, $\frac{\partial\boldsymbol{Q}^{\text{flat}}}{\partial\boldsymbol{Q}_0} = \boldsymbol{I}_3 - \boldsymbol{K} + \boldsymbol{L}\boldsymbol{J}\,(\boldsymbol{Q}_{1\times} - \boldsymbol{Q}_{2\times})$, $\frac{\partial\boldsymbol{Q}^{\text{flat}}}{\partial\boldsymbol{Q}_1} = \boldsymbol{L}\boldsymbol{J}\,(\boldsymbol{Q}_{2\times} - \boldsymbol{Q}_{0\times})$, and $\frac{\partial\boldsymbol{Q}^{\text{flat}}}{\partial\boldsymbol{Q}_2} = \boldsymbol{L}\boldsymbol{J}\,(\boldsymbol{Q}_{0\times} - \boldsymbol{Q}_{1\times})$.

The rays and spatial differentials are illustrated in Figure 2. In Figure 3 are examples of time differentials shown. The yellow arrows denote velocity vectors, and in Figures 3(a)-(c) it should be noted that a velocity of $\boldsymbol{P}$ in different directions implies a velocity of $\boldsymbol{Q}^{\text{flat}}$ in the plane of the triangle. In Figures 3(d)-(e) we see that a rotational velocity of $\boldsymbol{V}$ implies both a velocity of $\boldsymbol{Q}^{\text{flat}}$ in the plane of the triangle as well as a rotation of $\boldsymbol{W}_{\text{reflect}}^{\text{flat}}$ and $\boldsymbol{W}_{\text{refract}}^{\text{flat}}$. Finally but not shown, motion of the triangle tangent to the triangle does not imply any velocity on any parameters, and a rotation of the triangle normal implies a velocity on $\boldsymbol{Q}^{\text{flat}}$ along the ray.



(a) (b) (c)



(d)

Figure 2: Transfer, Reflection, and Refraction for Flat surfaces. Black arrows are ray directions, green is triangle normal, blue and red arrow illustrate the row vectors of $\frac{\partial \boldsymbol{P}}{\partial[\phi,\theta]^T}$, $\frac{\partial \boldsymbol{Q}}{\partial[\phi,\theta]^T}$, $\frac{\partial \boldsymbol{V}}{\partial[\phi,\theta]^T}$, and $\frac{\partial \boldsymbol{W}}{\partial[\phi,\theta]^T}$ as relevant. Subfigures (a)-(c) show orthographic projections of (d).

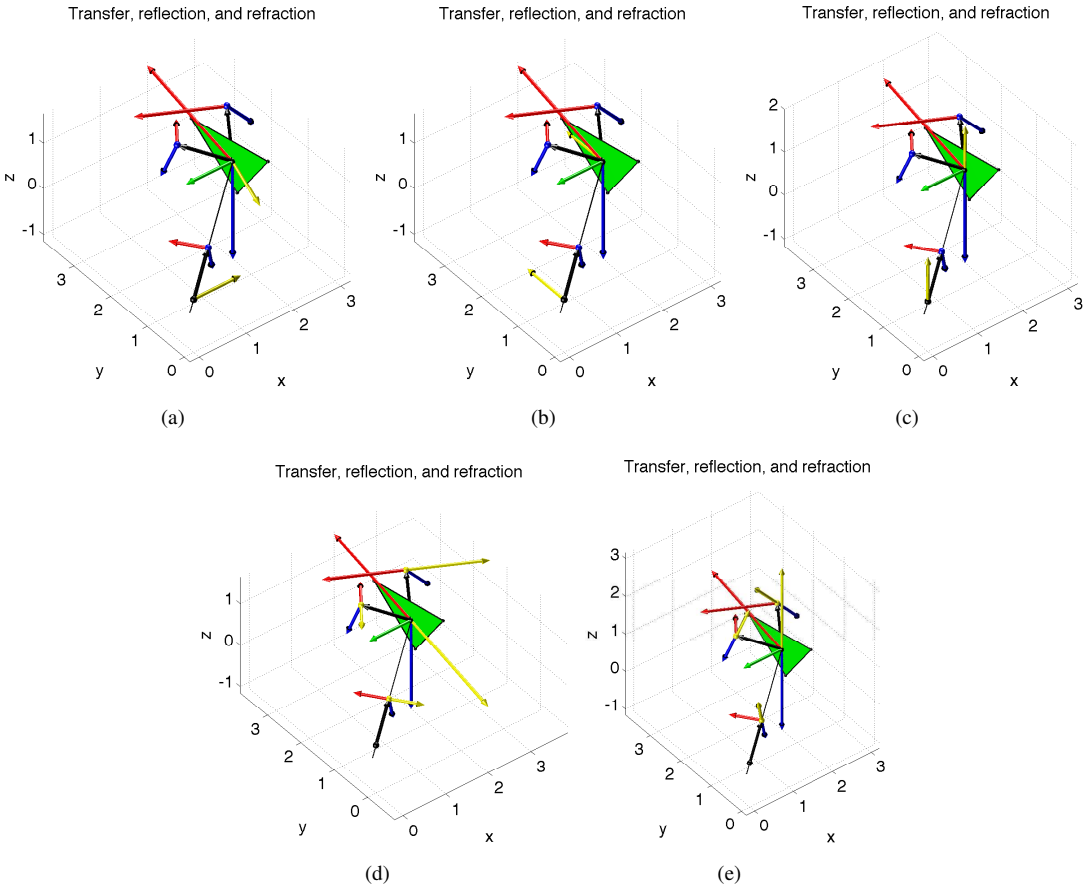
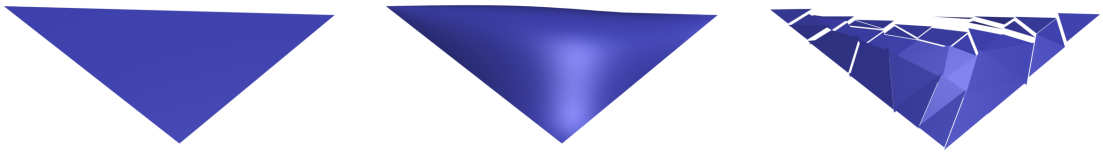


Figure 3: Time differentials for Flat surfaces. Yellow arrows denote imposed and resulting time derivatives. Subfigures (a)-(c) shows imposed velocities in three orthogonal directions on the origin, $\boldsymbol{P}$, (d)-(e) shows imposed rotational velocities in viewing direction $\boldsymbol{V}$, and (f)-(h) shows imposed velocities in three orthogonal directions on one of the vertices.



(a) (b) (c)

Figure 4: Phong shading assumes fish scale geometry. A triangle, 4(a), shaded with Phong's model, 4(b), expresses a complexity not supported by the real geometry. One way of conceptualizing this model is to think of the triangle as consisting of fish scales, 4(c); in this mindset every point on the surface of the triangle is associated with an independent local plane or fish scale whose normal is interpolated from the corners of the triangle.

3.2 Phong Shaded Surface

Phong shading uses a triangle as a base geometry, but imposes varying normals across it. Since the flatness of the triangle contradicts the changing normals, we prefer to think of this as a fish scale model as illustrated in Figure 4.

Phong shading assumes a plane represented by the 3 vertices of a triangle, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, and $\boldsymbol{Q}_2$, and corresponding vertex normals $\boldsymbol{N}_0$, $\boldsymbol{N}_1$, and $\boldsymbol{N}_2$. To calculate the intersection of the view ray with the triangle we use the Flat surface model (23), calculate the flat normal, $\boldsymbol{N}_{\text{flat}}$, by (31), and we find the point of intersection by solving (25). For reflection and refraction we construct a linearly interpolated normal from the three vertex normals. The flat normal and interpolated vertex normal most often will not coincide, and, as a consequence, $\frac{\partial \boldsymbol{Q}^{\text{phong}}}{\partial \boldsymbol{V}}$ will not span the triangle. Therefore, we calculate $\mathrm{d}\boldsymbol{Q}^{\text{phong}}$=$\mathrm{d}\boldsymbol{Q}^{\text{flat}}$ by (35) and simply write $\boldsymbol{Q}$ and $\mathrm{d}\boldsymbol{Q}$ in the remaining text.

To interpolate the vertex normals at the point of intersection, $\boldsymbol{Q}$, we calculate the Barycentric coordinates,

$$\boldsymbol{Q} = \lambda_0 \boldsymbol{Q}_0 + \lambda_1 \boldsymbol{Q}_1 + \lambda_2 \boldsymbol{Q}_2, \tag{36}$$

where $\lambda_i \geq 0$ are homogeneous Barycentric coordinates such that $\lambda_0 + \lambda_1 + \lambda_2 = 1$. The Barycentric coordinates are then used to interpolate the vertex normals as,

$$\boldsymbol{n} = \lambda_0 \boldsymbol{N}_0 + \lambda_1 \boldsymbol{N}_1 + \lambda_2 \boldsymbol{N}_2, \tag{37a}$$

$$\boldsymbol{N} = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|_2}. \tag{37b}$$

Note that the Barycentric coordinates are local to the triangle, and their differentials $\mathrm{d}\lambda_i$ may be used to estimate the change in $\boldsymbol{Q}$ in terms of the triangle. However, this requires algorithmic care near the border of the triangle, where $\lambda_i + \Delta\lambda_i$ may fall outside the triangle.

Assuming a ray passing through $\boldsymbol{P}$ with direction $\boldsymbol{V}$, which intersects a triangle within vertices $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, and $\boldsymbol{Q}_2$, then $0 \leq \lambda_i \leq 1$, and we may find the Barycentric coordinates as follows:

$$\boldsymbol{E}_1 = \boldsymbol{Q}_1 - \boldsymbol{Q}_0, \tag{38a}$$

$$\boldsymbol{E}_2 = \boldsymbol{Q}_2 - \boldsymbol{Q}_0, \tag{38b}$$

$$\boldsymbol{T} = \boldsymbol{P} - \boldsymbol{Q}_0, \tag{38c}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{E}_1 & \boldsymbol{E}_2 & -\boldsymbol{V} \end{bmatrix}, \tag{38d}$$

in which case we may write the point of intersection as the system of linear equations, $\boldsymbol{T} = \boldsymbol{A}[\lambda_1, \lambda_2, s]^T$. We solve for the unknown in this system of equations using Cramer's rule: consider the matrices $\boldsymbol{A}_i$, $i = 1, 2$ where the i'the column of matrix $\boldsymbol{A}$ has been replaced by $\boldsymbol{T}$ such that $\boldsymbol{A}_1 = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{E}_2 & -\boldsymbol{V} \end{bmatrix}$, and $\boldsymbol{A}_2 = \begin{bmatrix} \boldsymbol{E}_1 & \boldsymbol{T} & -\boldsymbol{V} \end{bmatrix}$, in that case the solutions are,

$$\lambda_0 = 1 - \lambda_1 - \lambda_2, \tag{39a}$$

$$\lambda_i = \frac{|\boldsymbol{A}_i|}{|\boldsymbol{A}|}, \; i = 1, 2. \tag{39b}$$

The differential, $\mathrm{d}\boldsymbol{N}$, is now found to be,

$$\mathrm{d}\boldsymbol{N} = \frac{\boldsymbol{n}^T\boldsymbol{n}\boldsymbol{I}_3 - \boldsymbol{n}\boldsymbol{n}^T}{\left(\boldsymbol{n}^T\boldsymbol{n}\right)^{3/2}}\mathrm{d}\boldsymbol{n}, \tag{40a}$$

$$\mathrm{d}\boldsymbol{n} = \boldsymbol{N}_0\mathrm{d}\lambda_0 + \lambda_0\mathrm{d}\boldsymbol{N}_0 + \boldsymbol{N}_1\mathrm{d}\lambda_1 + \lambda_1\mathrm{d}\boldsymbol{N}_1 + \boldsymbol{N}_2\mathrm{d}\lambda_2 + \lambda_2\mathrm{d}\boldsymbol{N}_2, \tag{40b}$$

and the differentials of the Barycentric coordinates are found as follows:

$$\mathrm{d}\lambda_0 = -\mathrm{d}\lambda_1 - \mathrm{d}\lambda_2, \tag{41a}$$

$$\mathrm{d}\lambda_i = \frac{|\boldsymbol{A}|\,\mathrm{d}\,|\boldsymbol{A}_i| - |\boldsymbol{A}_i|\,\mathrm{d}\,|\boldsymbol{A}|}{|\boldsymbol{A}|^2},\ i = 1\ldots 2. \tag{41b}$$

Using the triple product form of the determinant, $\left|\begin{bmatrix}\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c}\end{bmatrix}\right| = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$, that any interchange in columns of a matrix flips the sign of its determinant, and linearity of the cross and dot products, we may write,

$$\mathrm{d}\,|\boldsymbol{A}| = \mathrm{d}\left|\begin{bmatrix}\boldsymbol{E}_1 & \boldsymbol{E}_2 & -\boldsymbol{V}\end{bmatrix}\right| \tag{42a}$$

$$= (\boldsymbol{E}_2\times\boldsymbol{E}_1)^T\,\mathrm{d}\boldsymbol{V} + (\boldsymbol{V}\times\boldsymbol{E}_2)^T\,\mathrm{d}\boldsymbol{E}_1 - (\boldsymbol{V}\times\boldsymbol{E}_1)^T\,\mathrm{d}\boldsymbol{E}_2, \tag{42b}$$

$$= \boldsymbol{\psi}^T\mathrm{d}\boldsymbol{V} + \boldsymbol{\zeta}^T\mathrm{d}\boldsymbol{E}_1 + \boldsymbol{\gamma}^T\mathrm{d}\boldsymbol{E}_2, \tag{42c}$$

$$\mathrm{d}\,|\boldsymbol{A}_1| = \mathrm{d}\left|\begin{bmatrix}\boldsymbol{T} & \boldsymbol{E}_2 & -\boldsymbol{V}\end{bmatrix}\right|, \tag{42d}$$

$$= (\boldsymbol{E}_2\times\boldsymbol{T})^T\,\mathrm{d}\boldsymbol{V} + (\boldsymbol{V}\times\boldsymbol{E}_2)^T\,\mathrm{d}\boldsymbol{T} - (\boldsymbol{V}\times\boldsymbol{T})^T\,\mathrm{d}\boldsymbol{E}_2, \tag{42e}$$

$$= \boldsymbol{\kappa}^T\mathrm{d}\boldsymbol{V} + \boldsymbol{\zeta}^T\mathrm{d}\boldsymbol{T} + \boldsymbol{\tau}^T\mathrm{d}\boldsymbol{E}_2, \tag{42f}$$

$$\mathrm{d}\,|\boldsymbol{A}_2| = \mathrm{d}\left|\begin{bmatrix}\boldsymbol{E}_1 & \boldsymbol{T} & -\boldsymbol{V}\end{bmatrix}\right|, \tag{42g}$$

$$= (\boldsymbol{T}\times\boldsymbol{E}_1)^T\,\mathrm{d}\boldsymbol{V} + (\boldsymbol{V}\times\boldsymbol{T})^T\,\mathrm{d}\boldsymbol{E}_1 - (\boldsymbol{V}\times\boldsymbol{E}_1)^T\,\mathrm{d}\boldsymbol{T}, \tag{42h}$$

$$= \boldsymbol{\omega}^T\mathrm{d}\boldsymbol{V} - \boldsymbol{\tau}^T\mathrm{d}\boldsymbol{E}_1 + \boldsymbol{\gamma}^T\mathrm{d}\boldsymbol{T}, \tag{42i}$$

where $\boldsymbol{\psi} = \boldsymbol{E}_2\times\boldsymbol{E}_1$, $\boldsymbol{\tau} = -\boldsymbol{V}\times\boldsymbol{T}$, $\boldsymbol{\gamma} = -\boldsymbol{V}\times\boldsymbol{E}_1$, $\boldsymbol{\zeta} = \boldsymbol{V}\times\boldsymbol{E}_2$, $\boldsymbol{\kappa} = \boldsymbol{E}_2\times\boldsymbol{T}$, and $\boldsymbol{\omega} = \boldsymbol{T}\times\boldsymbol{E}_1$. Thus,

$$\mathrm{d}\lambda_1 = \frac{|\boldsymbol{A}|\left(\boldsymbol{\kappa}^T\mathrm{d}\boldsymbol{V} + \boldsymbol{\zeta}^T\mathrm{d}\boldsymbol{T} + \boldsymbol{\tau}^T\mathrm{d}\boldsymbol{E}_2\right) - |\boldsymbol{A}_1|\left(\boldsymbol{\psi}^T\mathrm{d}\boldsymbol{V} + \boldsymbol{\zeta}^T\mathrm{d}\boldsymbol{E}_1 + \boldsymbol{\gamma}^T\mathrm{d}\boldsymbol{E}_2\right)}{|\boldsymbol{A}|^2} \tag{43a}$$

$$= \frac{\left(|\boldsymbol{A}|\,\boldsymbol{\kappa}^T - |\boldsymbol{A}_1|\,\boldsymbol{\psi}^T\right)\mathrm{d}\boldsymbol{V} + |\boldsymbol{A}|\,\boldsymbol{\zeta}^T\mathrm{d}\boldsymbol{T} - |\boldsymbol{A}_1|\,\boldsymbol{\zeta}^T\mathrm{d}\boldsymbol{E}_1 + \left(|\boldsymbol{A}|\,\boldsymbol{\tau}^T - |\boldsymbol{A}_1|\,\boldsymbol{\gamma}^T\right)\mathrm{d}\boldsymbol{E}_2}{|\boldsymbol{A}|^2}, \tag{43b}$$

$$\mathrm{d}\lambda_2 = \frac{|\boldsymbol{A}|\left(\boldsymbol{\omega}^T\mathrm{d}\boldsymbol{V} - \boldsymbol{\tau}^T\mathrm{d}\boldsymbol{E}_1 + \boldsymbol{\gamma}^T\mathrm{d}\boldsymbol{T}\right) - |\boldsymbol{A}_2|\left(\boldsymbol{\psi}^T\mathrm{d}\boldsymbol{V} + \boldsymbol{\zeta}^T\mathrm{d}\boldsymbol{E}_1 + \boldsymbol{\gamma}^T\mathrm{d}\boldsymbol{E}_2\right)}{|\boldsymbol{A}|^2} \tag{43c}$$

$$= \frac{\left(|\boldsymbol{A}|\,\boldsymbol{\omega}^T - |\boldsymbol{A}_2|\,\boldsymbol{\psi}^T\right)\mathrm{d}\boldsymbol{V} + |\boldsymbol{A}|\,\boldsymbol{\gamma}^T\mathrm{d}\boldsymbol{T} - \left(|\boldsymbol{A}|\,\boldsymbol{\tau}^T + |\boldsymbol{A}_2|\,\boldsymbol{\zeta}^T\right)\mathrm{d}\boldsymbol{E}_1 - |\boldsymbol{A}_2|\,\boldsymbol{\gamma}^T\mathrm{d}\boldsymbol{E}_2}{|\boldsymbol{A}|^2}. \tag{43d}$$

Gathering terms we get,

$$
\begin{aligned}
\mathrm{d}\boldsymbol{N} = & \\
& \boldsymbol{J}\Big(\lambda_0 \mathrm{d}\boldsymbol{N}_0 + \lambda_1 \mathrm{d}\boldsymbol{N}_1 + \lambda_2 \mathrm{d}\boldsymbol{N}_2 \\
& + \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)}{|\boldsymbol{A}|^2}\left(\left(|\boldsymbol{A}|\,\boldsymbol{\kappa}^T - |\boldsymbol{A}_1|\,\boldsymbol{\psi}^T\right)\mathrm{d}\boldsymbol{V} + |\boldsymbol{A}|\,\boldsymbol{\zeta}^T \mathrm{d}\boldsymbol{T} - |\boldsymbol{A}_1|\,\boldsymbol{\zeta}^T \mathrm{d}\boldsymbol{E}_1 + \left(|\boldsymbol{A}|\,\boldsymbol{\tau}^T - |\boldsymbol{A}_1|\,\boldsymbol{\gamma}^T\right)\mathrm{d}\boldsymbol{E}_2\right) \\
& + \frac{(\boldsymbol{N}_2 - \boldsymbol{N}_0)}{|\boldsymbol{A}|^2}\left(\left(|\boldsymbol{A}|\,\boldsymbol{\omega}^T - |\boldsymbol{A}_2|\,\boldsymbol{\psi}^T\right)\mathrm{d}\boldsymbol{V} + |\boldsymbol{A}|\,\boldsymbol{\gamma}^T \mathrm{d}\boldsymbol{T} - \left(|\boldsymbol{A}|\,\boldsymbol{\tau}^T + |\boldsymbol{A}_2|\,\boldsymbol{\zeta}^T\right) d\boldsymbol{E}_1 - |\boldsymbol{A}_2|\,\boldsymbol{\gamma}^T \mathrm{d}\boldsymbol{E}_2\right)\Big)
\end{aligned}
\tag{44a}
$$

$$
\begin{aligned}
= \boldsymbol{J}\Big(& \lambda_0 \mathrm{d}\boldsymbol{N}_0 + \lambda_1 \mathrm{d}\boldsymbol{N}_1 + \lambda_2 \mathrm{d}\boldsymbol{N}_2 \\
& + \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)\left(|\boldsymbol{A}|\,\boldsymbol{\kappa}^T - |\boldsymbol{A}_1|\,\boldsymbol{\psi}^T\right) + (\boldsymbol{N}_2 - \boldsymbol{N}_0)\left(|\boldsymbol{A}|\,\boldsymbol{\omega}^T - |\boldsymbol{A}_2|\,\boldsymbol{\psi}^T\right)}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{V} \\
& + \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)\,|\boldsymbol{A}|\,\boldsymbol{\zeta}^T + (\boldsymbol{N}_2 - \boldsymbol{N}_0)\,|\boldsymbol{A}|\,\boldsymbol{\gamma}^T}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{T} \\
& + \frac{-(\boldsymbol{N}_1 - \boldsymbol{N}_0)\,|\boldsymbol{A}_1|\,\boldsymbol{\zeta}^T - (\boldsymbol{N}_2 - \boldsymbol{N}_0)\left(|\boldsymbol{A}|\,\boldsymbol{\tau}^T + |\boldsymbol{A}_2|\,\boldsymbol{\zeta}^T\right)}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{E}_1 \\
& + \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)\left(|\boldsymbol{A}|\,\boldsymbol{\tau}^T - |\boldsymbol{A}_1|\,\boldsymbol{\gamma}^T\right) - (\boldsymbol{N}_2 - \boldsymbol{N}_0)\,|\boldsymbol{A}_2|\,\boldsymbol{\gamma}^T}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{E}_2\Big),
\end{aligned}
\tag{44b}
$$

where $\boldsymbol{J} = \frac{\boldsymbol{n}^T\boldsymbol{n}\boldsymbol{I}_3 - \boldsymbol{n}\boldsymbol{n}^T}{(\boldsymbol{n}^T\boldsymbol{n})^{3/2}}$. Since $\mathrm{d}\boldsymbol{E}_1 = \mathrm{d}\boldsymbol{Q}_1 - \mathrm{d}\boldsymbol{Q}_0$, $\mathrm{d}\boldsymbol{E}_2 = \mathrm{d}\boldsymbol{Q}_2 - \mathrm{d}\boldsymbol{Q}_0$, $\mathrm{d}\boldsymbol{T} = \mathrm{d}\boldsymbol{P} - \mathrm{d}\boldsymbol{Q}_0$, we find that

$$
\begin{aligned}
\mathrm{d}\boldsymbol{N} = \boldsymbol{J}\Big(& \lambda_0 \mathrm{d}\boldsymbol{N}_0 + \lambda_1 \mathrm{d}\boldsymbol{N}_1 + \lambda_2 \mathrm{d}\boldsymbol{N}_2 \\
& + \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)\left(|\boldsymbol{A}|\,\boldsymbol{\kappa}^T - |\boldsymbol{A}_1|\,\boldsymbol{\psi}^T\right) + (\boldsymbol{N}_2 - \boldsymbol{N}_0)\left(|\boldsymbol{A}|\,\boldsymbol{\omega}^T - |\boldsymbol{A}_2|\,\boldsymbol{\psi}^T\right)}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{V} \\
& + \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)\,|\boldsymbol{A}|\,\boldsymbol{\zeta}^T + (\boldsymbol{N}_2 - \boldsymbol{N}_0)\,|\boldsymbol{A}|\,\boldsymbol{\gamma}^T}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{P} \\
& + \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)\left(|\boldsymbol{A}_1|\left(\boldsymbol{\zeta}^T + \boldsymbol{\gamma}^T\right) - |\boldsymbol{A}|\left(\boldsymbol{\zeta}^T + \boldsymbol{\tau}^T\right)\right)}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{Q}_0 \\
& + \frac{(\boldsymbol{N}_2 - \boldsymbol{N}_0)\left(|\boldsymbol{A}_2|\left(\boldsymbol{\gamma}^T + \boldsymbol{\zeta}^T\right) - |\boldsymbol{A}|\left(\boldsymbol{\gamma}^T - \boldsymbol{\tau}^T\right)\right)}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{Q}_0 \\
& + \frac{-(\boldsymbol{N}_1 - \boldsymbol{N}_0)\,|\boldsymbol{A}_1|\,\boldsymbol{\zeta}^T - (\boldsymbol{N}_2 - \boldsymbol{N}_0)\left(|\boldsymbol{A}|\,\boldsymbol{\tau}^T + |\boldsymbol{A}_2|\,\boldsymbol{\zeta}^T\right)}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{Q}_1 \\
& + \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)\left(|\boldsymbol{A}|\,\boldsymbol{\tau}^T - |\boldsymbol{A}_1|\,\boldsymbol{\gamma}^T\right) - (\boldsymbol{N}_2 - \boldsymbol{N}_0)\,|\boldsymbol{A}_2|\,\boldsymbol{\gamma}^T}{|\boldsymbol{A}|^2}\mathrm{d}\boldsymbol{Q}_2\Big)
\end{aligned}
\tag{45a}
$$

$$
\begin{aligned}
= \boldsymbol{J}\Big(& \left(\Delta_1\left(\boldsymbol{\kappa}^T - \lambda_1\boldsymbol{\psi}^T\right) + \Delta_2\left(\boldsymbol{\omega}^T - \lambda_2\boldsymbol{\psi}^T\right)\right)\mathrm{d}\boldsymbol{V} \\
& + \left(\Delta_1\boldsymbol{\zeta}^T + \Delta_2\boldsymbol{\gamma}^T\right)\mathrm{d}\boldsymbol{P} \\
& + \lambda_0 \mathrm{d}\boldsymbol{N}_0 + \lambda_1 \mathrm{d}\boldsymbol{N}_1 + \lambda_2 \mathrm{d}\boldsymbol{N}_2 \\
& + \left(\Delta_1\left(\lambda_1\left(\boldsymbol{\zeta}^T + \boldsymbol{\gamma}^T\right) - \boldsymbol{\zeta}^T - \boldsymbol{\tau}^T\right) + \Delta_2\left(\lambda_2\left(\boldsymbol{\gamma}^T + \boldsymbol{\zeta}^T\right) - \boldsymbol{\gamma}^T + \boldsymbol{\tau}^T\right)\right)\mathrm{d}\boldsymbol{Q}_0 \\
& + \left(-\Delta_1\lambda_1\boldsymbol{\zeta}^T - \Delta_2\left(\boldsymbol{\tau}^T + \lambda_2\boldsymbol{\zeta}^T\right)\right)\mathrm{d}\boldsymbol{Q}_1 \\
& + \left(\Delta_1\left(\boldsymbol{\tau}^T - \lambda_1\boldsymbol{\gamma}^T\right) - \Delta_2\lambda_2\boldsymbol{\gamma}^T\right)\mathrm{d}\boldsymbol{Q}_2\Big),
\end{aligned}
\tag{45b}
$$

where $\Delta_i = (\boldsymbol{N}_i - \boldsymbol{N}_0) / |\boldsymbol{A}|$. Hence, we identify the Jacobians, $\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{V}} = \boldsymbol{J}\left(\Delta_1\left(\boldsymbol{\kappa}^T - \lambda_1\boldsymbol{\psi}^T\right) + \Delta_2\left(\boldsymbol{\omega}^T - \lambda_2\boldsymbol{\psi}^T\right)\right)$, $\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{P}} = \boldsymbol{J}\left(\Delta_1\boldsymbol{\zeta}^T + \Delta_2\boldsymbol{\gamma}^T\right)$, $\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{N}_i} = \lambda_i\boldsymbol{J}$, $\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{Q}_0} = \boldsymbol{J}\left(\Delta_1\left(\lambda_1\left(\boldsymbol{\zeta}^T + \boldsymbol{\gamma}^T\right) - \boldsymbol{\zeta}^T - \boldsymbol{\tau}^T\right) + \Delta_2\left(\lambda_2\left(\boldsymbol{\gamma}^T + \boldsymbol{\zeta}^T\right) - \boldsymbol{\gamma}^T + \boldsymbol{\tau}^T\right)\right)$, $\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{Q}_1} = \boldsymbol{J}\left(-\Delta_1\lambda_1\boldsymbol{\zeta}^T - \Delta_2\left(\boldsymbol{\tau}^T + \lambda_2\boldsymbol{\zeta}^T\right)\right)$, and $\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{Q}_2} = \boldsymbol{J}\left(\Delta_1\left(\boldsymbol{\tau}^T - \lambda_1\boldsymbol{\gamma}^T\right) - \Delta_2\lambda_2\boldsymbol{\gamma}^T\right)$.

Expanding $\mathrm{d}\boldsymbol{N}$ in (17) we find that

$$\mathrm{d}\boldsymbol{W}^{\text{phong}}_{\text{reflect}} = \left(\frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{reflect}}}{\partial \boldsymbol{V}} + \frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{reflect}}}{\partial \boldsymbol{N}}\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{V}}\right)\mathrm{d}\boldsymbol{V} + \frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{reflect}}}{\partial \boldsymbol{N}}\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{P}}\,\mathrm{d}\boldsymbol{P} + \frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{reflect}}}{\partial \boldsymbol{N}}\sum_{i=0}^{2}\left(\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{N}_i}\,\mathrm{d}\boldsymbol{N}_i + \frac{\partial \boldsymbol{N}}{\partial \boldsymbol{Q}_i}\,\mathrm{d}\boldsymbol{Q}_i\right) \tag{46a}$$

where superscript on the Jacobians implies the same form as (17) but with use of the Phong normal, and where the all remaining terms have been have been evaluated in (17) and (45b). Likewise, expanding $\mathrm{d}\boldsymbol{N}$ in (22) we find that

$$\begin{aligned}\mathrm{d}\boldsymbol{W}^{\text{phong}}_{\text{refract}} = &\left(\frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{refract}}}{\partial \boldsymbol{V}} + \frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{refract}}}{\partial \boldsymbol{N}}\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{V}}\right)\mathrm{d}\boldsymbol{V} + \frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{refract}}}{\partial \boldsymbol{N}}\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{P}}\,\mathrm{d}\boldsymbol{P} \\ &+ \frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{refract}}}{\partial \boldsymbol{N}}\sum_{i=0}^{2}\left(\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{N}_i}\,\mathrm{d}\boldsymbol{N}_i + \frac{\partial \boldsymbol{N}}{\partial \boldsymbol{Q}_i}\,\mathrm{d}\boldsymbol{Q}_i\right) + \frac{\partial \boldsymbol{W}^{\text{phong}}_{\text{refract}}}{\partial \eta}\mathrm{d}\eta\end{aligned} \tag{47a}$$

where superscript on the Jacobians implies the same form as (22) but with use of the Phong normal, and where the all remaining terms have been have been evaluated in (22) and (45b).

4 Conclusion

In this work, we have evaluated the complete first order spatiotemporal structure of light's interaction with dielectric materials as reflection and refraction. In contrast to earlier work, [Igehy, 1999, Schjøth et al., 2007], we only assume that the geometry is given as a piecewise smooth surface. The derivation allows for easy extension to other parameters than viewing directions, and parallel rays are briefly treated as a special case as well as velocities on both the view point, direction and surfaces. Finally, we give two examples in complete detail of common shading models, flat and Phong.

Conceptually, we model ray bundles instead of rays and obvious applications are ray tracing and photon splatting, but the methodology is naturally and easily extended to all phenomena well approximated by first order Taylor series. Our generalization offer more accurate and faithful reconstruction of ray bundles in space and time.

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