

Attainable Knowledge

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December 14, 2024

Abstract

The article investigates an evidence-based semantics for epistemic logics in which pieces of evidence are interpreted as equivalence relations on the epistemic worlds. It is shown that the properties of knowledge obtained from potentially infinitely many pieces of evidence are described by modal logic S5. At the same time, the properties of knowledge obtained from only a finite number of pieces of evidence are described by modal logic S4. The main technical result is a sound and complete bi-modal logical system that describes properties of these two modalities and their interplay.

1 Introduction

The question of which modal logic is better at capturing the epistemic properties has long been a subject for philosophical discussions. While some think that such properties could be described by modal logic S4 [1, 5], others argued in support of logic S5 that extends S4 by adding Negative Introspection axiom [3]. There have also been suggestions to consider some of the modal systems in between S4 and S5 [6],[7, p.82].

In this article we propose a uniform framework in which S4 and S5 modalities can be interpreted as referring to two different types of knowledge. Our approach is based on the assumption that an agent's perception about the current epistemic state is formed by a (possibly infinite) number of pieces of evidence also commonly referred to as facts, tests, observations, or justifications. The main contribution of this article is the observation that the knowledge formed by a *finite* number of pieces of evidence has modal properties described by modal logic S4, whereas the knowledge formed by a potentially *infinite* number of pieces of evidence has modal properties described by modal logic S5.

We refer to the knowledge formed by a finite number of pieces of evidence as *attainable* knowledge and to the knowledge that formed by a potentially infinite number of pieces of evidence simply as knowledge. The main technical result of this article is a sound and complete logical system with two modalities that capture individual properties of the two types of knowledge as well as the properties that connect these two types of knowledge.

1.1 Grand Hotel Example

Consider the famous Hilbert Grand Hotel that has infinitely many rooms. By a single observation in this setting we mean opening and examining a room in order to establish whether it is empty. If the hotel has vacancies, then at least one room is empty. By opening just a single door from that specific room one can learn that the hotel has vacancies. We write $\Box$ (“hotel has vacancies”) to express this fact. The modality $\Box$ denotes attainable knowledge that can be formed from finitely many observations, in this case just a single observation. On the other hand, if the hotel has no vacancies, then the knowledge about this can not be obtained from examining finitely many rooms: $\neg \Box$ (“hotel has no vacancies”). At the same time, one can learn that hotel is full by examining all rooms in the hotel. We write this as $\Box$ (“hotel has no vacancies”), where modality $\Box$ denotes the knowledge that can be formed from (potentially infinite) set of all available pieces of evidence.

Let us first show that attainable knowledge does not satisfy axiom S5, also known as Negative Introspection axiom:

$$\neg \Box \varphi \rightarrow \Box \neg \Box \varphi. \quad (1)$$

Indeed, consider again the epistemic world w in which Hilbert Grant Hotel is full and let φ be statement “the hotel has vacancies”. Thus, $w \not\models \varphi$. Since by examining hotel rooms one cannot conclude anything false, it follows that $w \not\models \Box \varphi$. Hence, $w \models \neg \Box \varphi$. At the same time, informally, the only reason why $w \models \neg \Box \varphi$ is true is that the hotel is full. Since the latter can not be established through finitely many observations, $w \models \neg \Box \neg \Box \varphi$. More formally, to prove $w \models \neg \Box \neg \Box \varphi$ we need to show that no matter which finite set of evidence is examined, there still will be an epistemic world u indistinguishable from w such that $u \models \Box \varphi$. Indeed, suppose that we have chosen to examine rooms $r_1, \dots, r_n$. Let r be any room different from rooms $r_1, \dots, r_n$. Let u be an epistemic world in which all room except for room r are occupied. Thus, epistemic worlds w and u are indistinguishable through the chosen finite set of examinations. Yet $u \models \Box \varphi$ because in epistemic world u , it is enough to examine a single room r to learn that the hotel has vacancies. This concludes the counterexample for Negative Introspection principle (1).

Let us now consider a weaker form of Negative Introspection axiom used in logic S4.4 [6]:

$$\varphi \rightarrow (\neg \Box \varphi \rightarrow \Box \neg \Box \varphi). \quad (2)$$

The above counterexample for formula (1) does not work as a counterexample for formula (2) because in that setting $w \not\models \varphi$. Nevertheless, principle (2) is not valid in general. Indeed, let us modify the hotel setting by assuming that some rooms in the hotel might have bedbugs. The presence of the bedbugs can be tested when a room is examined. Furthermore, let us assume that once a single room in the hotel becomes infected with bedbugs all guests immediately leave the hotel. Thus, the hotel might have either (a) visitors and bedbugs, or (b) no visitors and no bedbugs, or (c) bedbugs and no visitors. Consider an

epistemic world w in which the hotel has no visitors and no bugs. Let φ be the statement “hotel has no visitors”. Thus, $w \models \varphi$. Note that there are two ways to verify that hotel has no visitors: either by examining all rooms to observe that they are all vacant or by examining a single room that contains bedbugs. Since in the epistemic world w the hotel is not infected with bugs, the only way to verify that the hotel is empty in this world is to examine all rooms. Thus, $w \models \neg \Box \varphi$. To finish the counterexample for formula (2), it suffices to show that $w \not\models \Box \neg \Box \varphi$. In other words, we need to prove that after opening a finite set of rooms $r_1, \dots, r_n$ we will not be able to distinguish epistemic state w from an epistemic state u such that $u \models \Box \varphi$. Indeed, let r be any room different from rooms $r_1, \dots, r_n$ and let u be the epistemic world in which room r is infected with bedbugs. Note that $u \models \Box \varphi$ because in epistemic state u it is enough to examine the bug-infected room r to conclude that the hotel is empty.

1.2 Formal Semantics of Evidence

To make our Grand Hotel example more formal, we need to introduce the formal semantics of evidence-based knowledge. Since the purpose of the pieces of evidence is to distinguish epistemic worlds, the pieces of evidence can be viewed as equivalence relations on the worlds. When an agent takes into account several pieces of evidence, equivalence relations corresponding to these pieces intersect to form the equivalence relation of the agent. In the second Grant Hotel example above, each room is in one of the three states: occupied, vacant without bedbugs, and vacant with bedbugs. An epistemic world can be described by specifying the state of each room. The observation that consists of examining room r can distinguish two epistemic worlds in which room r is in different states. It can not distinguish two epistemic worlds in which room r is in the same state.

In other words, we assemble agent’s knowledge from several pieces of evidence in exactly the same way as how distributed knowledge [3] of a group is assembled from the individual observations of the members of the group. Thus, the logical system developed in this article could also be used to describe two different forms of group knowledge by an infinite group of agents: modality $\Box$ represents the standard distributed knowledge by the whole group and modality $\Box$ represents the distributed knowledge by a finite subgroup of the whole group.

The formal evidence semantics described above is similar to the one for the budget-constrained knowledge proposed by Naumov and Tao [8]. It is different from the neighbourhood semantics of evidence investigated by van Benthem and Pacuit [11], and the probabilistic approach of Halpern and Pucella [4].

1.3 Logical System

In this article we propose a sound and complete logical system that describes universal properties of knowledge modality $\Box$ and attainable knowledge modality $\Box$. The axioms involving only modality $\Box$ are exactly those forming modal logic S5:

1. Truth: $\Box\varphi \rightarrow \varphi$,
2. Positive Introspection: $\Box\varphi \rightarrow \Box\Box\varphi$,
3. Negative Introspection: $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$,
4. Distributivity: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

Later we do not list Positive Introspection principle among axioms of our system because, just like in the case of logic S5, Positive Introspection is derivable from the other axioms. We prove this in Lemma 1. The axioms involving only modality $\Box$ are exactly those forming modal logic S4:

1. Attainable Truth: $\Box\varphi \rightarrow \varphi$,
2. Attainable Positive Introspection: $\Box\varphi \rightarrow \Box\Box\varphi$,
3. Attainable Distributivity: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

Finally, there appears to be two independent principles that capture the interplay of the two modalities:

1. Monotonicity: $\Box\varphi \rightarrow \Box\varphi$,
2. Mixed Negative Introspection: $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$.

The first of these principles states that any knowledge that can be formed from a finite subset of observations can also be formed on the bases of the whole set of observations. The second principle is significantly more interesting. We have seen in our first Grant Hotel example that Negative Introspection is not true for attainable knowledge. Namely an agent in a fully occupied hotel can not learn that statement $\Box$ (“the hotel has vacancies”) is false by examining only finitely many rooms. It can learn this, however, by examining all room in the hotel and this is exactly what Mixed Negative Introspection principle claims. Surprisingly, Mixed Negative Introspection principle is provable from the other axioms of our logical system. We show this in Lemma 2. As a result, Mixed Negative Introspection is not included as an axiom of our system. Additionally, it is easy to see that Attainable Truth principle follows from Truth axiom and Monotonicity axiom. For this reason, we do not list Attainable Truth as one of our axioms either. Finally, although one can state two forms of Necessitation inference rule: one for $\Box$ modality and another for $\Box$, the former follows from the latter and Monotonicity axiom. Thus, our system, in addition to Modus Ponens, only includes Attainable Necessitation inference rule.

1.4 Outline

This article is organized as following. In Section 2 we define the syntax and the semantics of our logical system. In Section 3 we list the axioms and the inference rules of this system. In Section 4 we present two examples of formal proofs in our system, including a proof of Mixed Negative Introspection principle mentioned

above. In Section 5 we prove the soundness of our logical system with respect to our evidence-based semantics. In Section 6 we establish the completeness of our system. Section 7 concludes.

2 Syntax and Semantics

In this section we describe the formal syntax and the formal evidence-based semantics of our logical system. Throughout the rest of the article we assume a fixed infinite set of propositional variables.

Definition 1 *The set Φ of all formulae φ is defined by the Backus-Naur form $\varphi := p \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid \Box\varphi$, where p represents propositional variables.*

Definition 2 *A Kripke model with evidence is $\langle W, E, \{\sim_e\}_{e \in E}, \pi \rangle$, where*

1. W is a set of “epistemic worlds”,
2. E is an arbitrary “evidence” set,
3. $\sim_e$ is an “indistinguishability” equivalence relation on W for each $e \in E$,
4. π is a function that maps propositional variables into subsets of W .

For instance, in our first Grand Hotel example, an epistemic world is a function $\mathbb{N} \rightarrow \{\text{vacant}, \text{occupied}\}$ that assigns a state to each room in the hotel. The set of evidence is $\mathbb{N}$, where evidence with number $r \in \mathbb{N}$ corresponds to examining room with number r in this hotel. Epistemic worlds w_1 and w_2 are $\sim_r$ -equivalent if room number r has the same state in both of the worlds. In other words, $w_1 \sim_r w_2$ if $w_1(r) = w_2(r)$. Finally a function π may, for example, map propositional variable p into the set of all epistemic worlds representing a nonempty hotel: $\pi(p) = \{w \in W \mid \exists r \in \mathbb{N} \text{ such that } w(r) = \text{occupied}\}$.

In this article, we write $w_1 \sim_F w_2$ if $w_1 \sim_e w_2$ for each $e \in F$.

Definition 3 *For any formula $\varphi \in \Phi$ and any epistemic world $w \in W$ of a Kripke model $\langle W, E, \{\sim_e\}, \pi \rangle$, let the satisfiability relation $w \Vdash \varphi$ be defined as follows,*

1. $w \Vdash p$ if $w \in \pi(p)$,
2. $w \Vdash \neg\varphi$ if $w \not\Vdash \varphi$,
3. $w \Vdash \varphi \rightarrow \psi$ if $w \not\Vdash \varphi$ or $w \Vdash \psi$,
4. $w \Vdash \Box\varphi$ if there is a finite $F \subseteq E$ such that $w \sim_F u$ implies $u \Vdash \varphi$ for each $u \in W$,
5. $w \Vdash \Box\varphi$ if $u \Vdash \varphi$ for each $u \in W$ such that $w \sim_E u$.

3 Axioms

In addition to propositional tautologies in language Φ , our logical system consists of the following axioms:

1. Truth: $\Box\varphi \rightarrow \varphi$,
2. Negative Introspection: $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$,
3. Distributivity: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$,
4. Monotonicity: $\Box\varphi \rightarrow \Box\varphi$,
5. Attainable Positive Introspection: $\Box\varphi \rightarrow \Box\Box\varphi$,
6. Attainable Distributivity: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

We say that formula φ is a theorem in our logical system and write $\vdash \varphi$ if formula φ is derivable from the axioms of our systems using Modus Ponens and Attainable Necessitation inference rules:

$$\frac{\varphi, \quad \varphi \rightarrow \psi}{\psi} \quad \frac{\varphi}{\Box\varphi}.$$

We write $X \vdash \varphi$ if formula φ is derivable from the theorems of our logical systems and an additional set of axioms X using only Modus Ponens inference rule.

4 Examples

The soundness of our logical system is shown in Section 5. In this section we give two examples of formal proofs in our system. The first example, which will be used later in the proof of the completeness, is the standard observation that Positive Introspection principle is derivable from the axioms of modal logic S5. Since our system does not contain Necessitation rule for modality $\Box$, the proof is using Attainable Necessitation inference rule and Monotonicity axiom instead.

Lemma 1 $\vdash \Box\varphi \rightarrow \Box\Box\varphi$.

Proof. Note that formula $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$ is an instance of Negative Introspection axiom. Thus, $\vdash \neg\Box\neg\Box\varphi \rightarrow \Box\varphi$ by the law of contrapositive in the propositional logic. Hence, $\vdash \Box(\neg\Box\neg\Box\varphi \rightarrow \Box\varphi)$ by Attainable Necessitation inference rule. Thus, $\vdash \Box(\neg\Box\neg\Box\varphi \rightarrow \Box\varphi)$ by Monotonicity axiom and Modus Ponens inference rule. Thus, by Distributivity axiom and Modus Ponens inference rule,

$$\vdash \Box\neg\Box\neg\Box\varphi \rightarrow \Box\Box\varphi. \quad (3)$$

At the same time, $\Box\neg\Box\varphi \rightarrow \neg\Box\varphi$ is an instance of Truth axiom. Thus, $\vdash \Box\varphi \rightarrow \neg\Box\neg\Box\varphi$ by contraposition. Hence, taking into account the following

instance of Negative Introspection axiom $\neg\Box\neg\Box\varphi \rightarrow \Box\neg\Box\neg\Box\varphi$, one can conclude that $\vdash \Box\varphi \rightarrow \Box\neg\Box\neg\Box\varphi$. The latter, together with statement (3), implies the statement of the lemma by the laws of propositional reasoning. $\square$

Our next and perhaps more interesting example is the proof of Mixed Negative Introspection principle mentioned in Section 1.3 of the introduction.

Lemma 2 $\vdash \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$.

Proof. By Truth axiom, $\vdash \Box\Box\varphi \rightarrow \Box\varphi$. Thus, by the law of contrapositive in the propositional logic, $\vdash \neg\Box\varphi \rightarrow \neg\Box\Box\varphi$. At the same time, by Negative Introspection axiom, $\vdash \neg\Box\Box\varphi \rightarrow \Box\neg\Box\Box\varphi$. Hence, by the laws of propositional reasoning,

$$\vdash \neg\Box\varphi \rightarrow \Box\neg\Box\Box\varphi. \quad (4)$$

Note that $\vdash \Box\varphi \rightarrow \Box\Box\varphi$ by Attainable Positive Introspection axiom and $\vdash \Box\Box\varphi \rightarrow \Box\varphi$ by Monotonicity axiom. Thus, by the laws of propositional reasoning, $\vdash \Box\varphi \rightarrow \Box\Box\varphi$. Hence, by the law of contrapositive in the propositional logic, $\vdash \neg\Box\Box\varphi \rightarrow \neg\Box\varphi$. Then, $\vdash \Box(\neg\Box\Box\varphi \rightarrow \neg\Box\varphi)$ by Attainable Necessitation inference rule. At the same time,

$$\vdash \Box(\neg\Box\Box\varphi \rightarrow \neg\Box\varphi) \rightarrow \Box(\neg\Box\Box\varphi \rightarrow \neg\Box\varphi)$$

by Monotonicity axiom. Thus, $\vdash \Box(\neg\Box\Box\varphi \rightarrow \neg\Box\varphi)$ by Modus Ponens inference rule. Hence, $\vdash \Box\neg\Box\Box\varphi \rightarrow \Box\neg\Box\varphi$ by Distributivity axiom and Modus Ponens inference rule. Therefore, $\vdash \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$, by the laws of propositional reasoning taking statement (4) into account. $\square$

5 Soundness

In this section we prove the soundness of our logical system with respect to the evidence-based semantics introduced in Section 2.

Theorem 1 *If $\vdash \varphi$, then $w \Vdash \varphi$ for each epistemic world $w \in W$ of each Kripke model $\langle W, E, \{\sim_e\}_{e \in E}, \pi \rangle$.*

The soundness of propositional tautologies and Modus Ponens inference rule is straightforward. Below we prove the soundness of each of the remaining axioms and Attainable Necessitation inference rule as separate lemmas.

Lemma 3 *If $w \Vdash \Box\varphi$, then $w \Vdash \Box\Box\varphi$.*

Proof. By Definition 3, the assumption $w \Vdash \Box\varphi$ implies that there is a finite $F \subseteq E$ such that $u \Vdash \varphi$ for each $u \in W$ where $w \sim_F u$.

Again by Definition 3, it suffices to show that $v \Vdash \Box\varphi$ for all $v \in W$ such that $w \sim_F v$. To establish this, it is enough to prove that $u \Vdash \varphi$ for all $u \in W$ such that $v \sim_F u$. Note that $w \sim_F v \sim_F u$. Thus, $w \sim_F u$ because $\sim_f$ is an equivalence relation for each element $f \in F$. Therefore, $u \Vdash \varphi$ by the choice of set F . $\square$

Lemma 4 *If $w \Vdash \Box(\varphi \rightarrow \psi)$ and $w \Vdash \Box\varphi$, then $w \Vdash \Box\psi$.*

Proof. By Definition 3, the assumption $w \Vdash \Box(\varphi \rightarrow \psi)$ implies that there is a finite $F_1 \subseteq E$ such that $u \Vdash \varphi \rightarrow \psi$ for each $u \in W$ such that $w \sim_{F_1} u$. Similarly, the assumption $w \Vdash \Box\varphi$ implies that there is a finite $F_2 \subseteq E$ such that $u \Vdash \varphi$ for each $u \in W$ such that $w \sim_{F_2} u$.

Let $F = F_1 \cup F_2$. It suffices to show that $u \Vdash \psi$ for each $u \in W$ such that $w \sim_F u$. Indeed, statement $w \sim_F u$ implies that $w \sim_{F_1} u$ and $w \sim_{F_2} u$. Hence, $u \Vdash \varphi \rightarrow \psi$ and $u \Vdash \varphi$ due to the choice of sets F_1 and F_2 . Therefore, $u \Vdash \psi$ by Definition 3. $\square$

Lemma 5 *If $w \Vdash \Box\varphi$, then $w \Vdash \varphi$.*

Proof. By Definition 3, assumption $w \Vdash \Box\varphi$ implies that $u \Vdash \varphi$ for all $u \in W$ such that $w \sim_E u$. Note that $w \sim_e w$ for all $e \in E$ because $\sim_e$ is an equivalence relation. Hence, $w \sim_E w$. Therefore, $w \Vdash \varphi$. $\square$

Lemma 6 *If $w \Vdash \neg\Box\varphi$, then $w \Vdash \Box\neg\Box\varphi$.*

Proof. By Definition 3, assumption $w \Vdash \neg\Box\varphi$ implies that there is $u \in W$ such that $w \sim_E u$ and $u \nVdash \varphi$.

Consider any $v \in W$ such that $w \sim_E v$. By Definition 3, to prove $w \Vdash \Box\neg\Box\varphi$, it suffices to show that $v \Vdash \neg\Box\varphi$. Note that $w \sim_E u$ and $w \sim_E v$. Thus, $v \sim_E u$ due to $\sim_e$ being an equivalence relation for each $e \in E$. Recall that $u \nVdash \varphi$. Hence, $v \nVdash \Box\varphi$ by Definition 3. Therefore, $v \Vdash \neg\Box\varphi$ again by Definition 3. $\square$

Lemma 7 *If $w \Vdash \Box(\varphi \rightarrow \psi)$ and $w \Vdash \Box\varphi$, then $w \Vdash \Box\psi$.*

Proof. Consider any $u \in W$ such that $w \sim_E u$. By Definition 3, it suffices to show that $u \Vdash \psi$. Indeed, by Definition 3 assumptions $w \Vdash \Box(\varphi \rightarrow \psi)$ and $w \Vdash \Box\varphi$ imply that $u \Vdash \varphi \rightarrow \psi$ and $u \Vdash \varphi$. Therefore, $u \Vdash \psi$, again by Definition 3. $\square$

Lemma 8 *If $w \Vdash \Box\varphi$, then $w \Vdash \Box\varphi$.*

Proof. Consider any $u \in W$ such that $w \sim_E u$. By Definition 3, it suffices to prove that $u \Vdash \varphi$. By the same definition, the assumption $w \Vdash \Box\varphi$ implies that there is finite $F \subseteq E$ such that $v \Vdash \varphi$ for each $v \in W$ such that $w \sim_F v$. Note that statement $w \sim_E u$ implies that $w \sim_F u$. Therefore, $u \Vdash \varphi$. $\square$

Lemma 9 *If $w \Vdash \varphi$ for each epistemic world $w \in W$ of each Kripke model $\langle W, E, \{\sim_e\}_{e \in E}, \pi \rangle$, then $w \Vdash \Box\varphi$ for each epistemic world $w \in W$ of each Kripke model $\langle W, E, \{\sim_e\}_{e \in E}, \pi \rangle$.*

Proof. Consider any epistemic world $w \in W$ of an arbitrary Kripke model $\langle W, E, \{\sim_e\}_{e \in E}, \pi \rangle$. By Definition 3, it suffices to show that there is $F \subseteq E$ such that $u \Vdash \varphi$ for each $u \in W$ where $w \sim_F u$. Indeed, let $F = \emptyset$. Note that $u \Vdash \varphi$ for each $u \in W$ due to the assumption of the lemma. $\square$
This concludes the proof of the soundness of our logical system.

6 Completeness

In this section we prove the completeness of our logical system. Throughout the section we use two operations on sequences. If w is a sequence $(x_1, x_2, \dots, x_n)$ and u is a sequence $(y_1, y_2, \dots, y_m)$, then by concatenation $w :: u$ of these two sequences we mean sequence $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$. By head $hd(w)$ of a nonempty sequence $w = (x_1, x_2, \dots, x_n)$ we mean element x_n . For example, $(a, b) :: (c) = (a, b, c)$ and $hd(a, b, c) = c$.

Theorem 2 *For any formula $\varphi_0 \in \Phi$, if $w \Vdash \varphi_0$ for each epistemic world $w \in W$ of each Kripke model $\langle W, E, \{\sim_e\}_{e \in E}, \pi \rangle$, then $\vdash \varphi_0$.*

The rest of this section is dedicated to the proof of Theorem 2. Suppose that $\not\vdash \varphi_0$. We first define a “canonical” Kripke model $\langle W_\infty, W_\infty, \{\sim_e\}_{e \in W_\infty}, \pi \rangle$ and later show that $w \not\vdash \varphi_0$ for some $w \in W_\infty$. Note that the set of epistemic worlds in the canonical model is identical to the evidence set. One can think that all pieces of evidence related in some sense to a given epistemic world are combined together into a single evidence associated with this world. The evidence associated with world w is simply referred to as evidence w .

We define set W_∞ of epistemic worlds using the “unravelling” technique [10]. Informally, this set consists of sequences of the form $(X_0, \epsilon_1, X_1, \dots, \epsilon_n, X_n)$, where $X_0, \dots, X_n$ are maximal consistent sets of formulae and each of $\epsilon_1, \dots, \epsilon_n$ is either a finite subset of W_∞ or symbol $*$. In what follows, the case when ϵ is a finite subset of W_∞ will form a $\Box$ -accessibility relation and the case $\epsilon = *$ will form both $\Box$ -accessibility and $\Box$ -accessibility relations. Such sequences can be visualised, see Figure 1, as paths in an infinite collection of infinite trees whose vertices are maximal consistent sets of formulae and whose edges are labeled with the ϵ ’s described above.

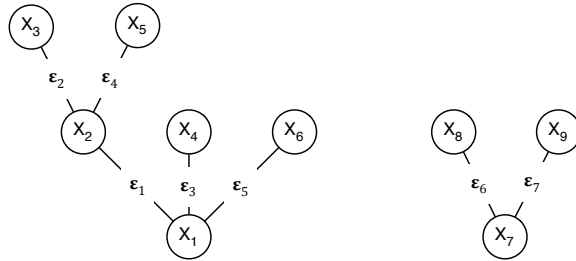


Figure 1: Trees of sequences.

Formally, set W_∞ is specified as the union of a recursively defined infinite sequence of sets $W_0, W_1, W_2, \dots$. These sets represent different stages of building the infinite set of infinite tree partially depicted in Figure 1. Note, however, that stages do not correspond to the levels of the trees. Generally speaking, vertices at the same level are not created at the same stage.

Definition 4 *A sequence of sets $W_0, W_1, \dots$ is defined recursively as follows*

1. W_0 is the set of all single-element sequences (X) , where X is a maximal consistent subset of Φ ,
2. W_{n+1} contains all sequences of the form $w::(\epsilon, Y)$ such that
 - (a) $w \in \bigcup_{i \leq n} W_i$,
 - (b) ϵ is either symbol $*$ or a finite subset of $\bigcup_{i \leq n} W_i$,
 - (c) Y is a maximal consistent subset of Φ ,
 - (d) if $\epsilon = *$, then $\{\varphi \mid \Box \varphi \in \text{hd}(w)\} \subseteq Y$,
 - (e) if $\epsilon \in \bigcup_{i \leq n} W_i$, then $\{\varphi \mid \Box \varphi \in \text{hd}(w)\} \subseteq Y$.

Definition 5 $W_\infty = \bigcup_i W_i$.

Lemma 10 *Set W_∞ is infinite.*

Proof. Recall that $\not\models \varphi_0$ by the choice of statement φ_0 . Thus, set $\{\neg \varphi_0\}$ is consistent. Since our language contains infinitely many propositional variables, set $\{\neg \varphi\}$ could be extended to a maximal consistent set in an infinitely many ways. Hence, set W_0 is infinite by Definition 4. Therefore, set W_∞ is infinite by Definition 5. $\square$

Informally, two sequences are $\sim_e$ -equivalent if they start with the same prefix and once they deviate all subsequent ϵ 's either are equal to $*$ or contain element e , see Figure 2 (a). The formal definition is below.

Definition 6 *For any epistemic world $w = (X_0, \epsilon_1, X_1, \dots, \epsilon_n, X_n)$, any epistemic world $u = (X'_0, \epsilon'_1, X'_1, \dots, \epsilon'_m, X'_m)$, and any $e \in W_\infty$, let $w \sim_e u$ if there is k such that*

1. $0 \leq k \leq \min\{n, m\}$,
2. $X_i = X'_i$ for all i such that $0 \leq i \leq k$,
3. $\epsilon_i = \epsilon'_i$ for all i such that $0 < i \leq k$,
4. for all i , if $k < i \leq n$, then either $e \in \epsilon_i$ or $\epsilon_i = *$,
5. for all i , if $k < i \leq m$, then either $e \in \epsilon'_i$ or $\epsilon'_i = *$.

Definition 7 $\pi(p) = \{w \in W_\infty \mid p \in \text{hd}(w)\}$.

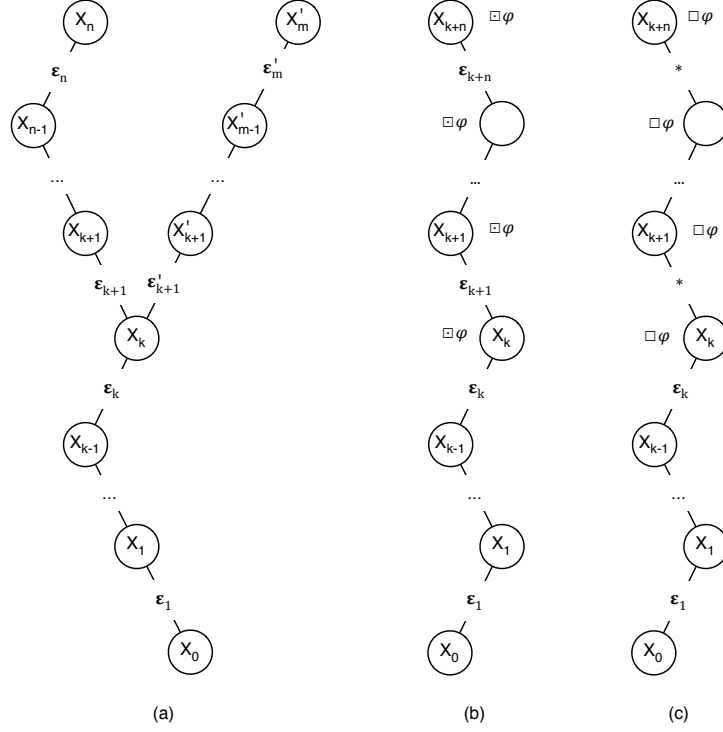


Figure 2: Illustrations: (a) for Definition 6, where ϵ_i and ϵ'_i either are equal to $*$ or contain e for each $i \geq k$, (b) for Lemma 11, and (c) for Lemma 12 and Lemma 13.

The canonical Kripke model $\langle W_\infty, W_\infty, \{\sim_e\}_{e \in W_\infty}, \pi \rangle$ is now fully defined. Next we establish several properties of this model that are necessary for our proof of the completeness.

First we show that if set $hd(w)$ contains a formula $\Box\varphi$, then so does set $hd(u)$ for each descendant u of vertex w .

Lemma 11 *For any $k \geq 0$, any $n \geq 0$, and any*

$$(X_0, \epsilon_1, X_1, \dots, \epsilon_k, X_k, \epsilon_{k+1}, X_{k+1}, \dots, \epsilon_{k+n}, X_{k+n}) \in W_\infty,$$

if $\Box\varphi \in X_k$, then $\Box\varphi \in X_{k+n}$.

Proof. We prove the lemma by induction on n , see Figure 2 (b). If $n = 0$, then assumption $\Box\varphi \in X_k$ implies that $\Box\varphi \in X_{k+n}$. If $n > 0$, then $\Box\varphi \in X_{k+n-1}$ by the induction hypothesis. Hence, $X_{k+n-1} \vdash \Box\Box\varphi$ by Attainable Positive Introspection axiom.

Case I: $\epsilon_{k+n} = *$. Note that $X_{k+n-1} \vdash \Box\Box\varphi$ implies $X_{k+n-1} \vdash \Box\Box\varphi$ by Monotonicity axiom. Hence, $\Box\Box\varphi \in X_{k+n-1}$ due to the maximality of set X_{k+n-1} . Then, $\Box\varphi \in X_{k+n}$ by Definition 4 and due to the assumption $\epsilon_{k+n} = *$.

Case II: $\epsilon_{k+n} \neq *$. Statement $X_{k+n-1} \vdash \Box \Box \varphi$ implies that $\Box \Box \varphi \in X_{k+n-1}$ due to the maximality of set X_{k+n-1} . Thus, $\Box \varphi \in X_{k+n}$ by Definition 4. $\square$

We next show that if $hd(w)$ contains a formula $\Box \varphi$, then so does $hd(u)$ for each descendant u of vertex w reachable through edges all of which are labeled by symbol $*$.

Lemma 12 *For any $k \geq 0$ and $n \geq 0$, and any*

$$(X_0, \epsilon_1, X_1, \dots, \epsilon_k, X_k, \epsilon_{k+1}, X_{k+1}, \dots, \epsilon_{k+n}, X_{k+n}) \in W_\infty,$$

*if $\Box \varphi \in X_k$ and $\epsilon_i = *$ for all i such that $k < i \leq k+n$, then $\Box \varphi \in X_{k+n}$.*

Proof. We prove the lemma by induction on n , see Figure 2 (c). If $n = 0$, then assumption $\Box \varphi \in X_k$ implies that $\Box \varphi \in X_{k+n}$. If $n > 0$, then $\Box \varphi \in X_{k+n-1}$ by the induction hypothesis. Hence, $X_{k+n-1} \vdash \Box \Box \varphi$ by Lemma 1. Thus, $\Box \varphi \in X_{k+n}$ by Definition 4 and due to the assumption $\epsilon_{k+n} = *$. $\square$

The next lemma is a converse of Lemma 12. It shows that if $hd(u)$ contains a formula $\Box \varphi$ where u is a descendant of vertex w reachable through edges all of which are labeled by symbol $*$, then so does $hd(w)$.

Lemma 13 *For any $k \geq 0$, any $n \geq 0$, and any*

$$(X_0, \epsilon_1, X_1, \dots, \epsilon_k, X_k, \epsilon_{k+1}, X_{k+1}, \dots, \epsilon_{k+n}, X_{k+n}) \in W_\infty,$$

*if $\Box \varphi \in X_{k+n}$ and $\epsilon_i = *$ for all i such that $k < i \leq k+n$, then $\Box \varphi \in X_k$.*

Proof. We prove the statement of the lemma by induction on n , see again Figure 2 (c). If $n = 0$, then assumption $\Box \varphi \in X_{k+n}$ implies that $\Box \varphi \in X_k$. We assume now that $n > 0$.

Case I: $\Box \varphi \notin X_{k+n-1}$. Thus, $\neg \Box \varphi \in X_{k+n-1}$ due to the maximality of the set X_{k+n-1} . Hence, $X_{k+n-1} \vdash \Box \neg \Box \varphi$ by Negative Introspection axiom. Then, $\Box \neg \Box \varphi \in X_{k+n-1}$ due to the maximality of the set X_{k+n-1} . Hence, $\neg \Box \varphi \in X_{k+n}$ by Definition 4 and because $\epsilon_{k+n} = *$. Thus, $\Box \varphi \notin X_{k+n}$ due to the consistency of the set X_{k+n} , which is a contradiction with the assumption of the lemma.

Case II: $\Box \varphi \in X_{k+n-1}$. Thus, $\Box \varphi \in X_k$ by the induction hypothesis. $\square$

The next two lemmas are relatively standard lemmas for a completeness proof of a modal logic. Their proofs show how a sequence representing an epistemic world can be extended to produce different types of child nodes on the trees in Figure 1.

Lemma 14 *For any $w \in W_\infty$, any $\neg \Box \varphi \in hd(w)$, and any finite $F \subseteq W_\infty$, there is $u \in W_\infty$ such that $w \sim_F u$ and $\neg \varphi \in hd(u)$.*

Proof. We first show that the following set is consistent:

$$Y_0 = \{\neg \varphi\} \cup \{\psi \mid \Box \psi \in hd(w)\}.$$

Assume the opposite. Thus, there must exist $\Box\psi_1, \dots, \Box\psi_n \in hd(w)$ such that $\psi_1, \dots, \psi_n \vdash \varphi$. Hence, by the deduction theorem for propositional logic,

$$\vdash \psi_1 \rightarrow (\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots).$$

Then, by Attainable Necessitation inference rule,

$$\vdash \Box(\psi_1 \rightarrow (\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots)).$$

By Attainable Distributivity axiom and Modus Ponens inference rule,

$$\vdash \Box\psi_1 \rightarrow \Box(\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots).$$

By Modus Ponens inference rule,

$$\Box\psi_1 \vdash \Box(\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots).$$

By repeating the last two steps $n - 1$ times,

$$\Box\psi_1, \dots, \Box\psi_n \vdash \Box\varphi.$$

Hence, $hd(w) \vdash \Box\varphi$ by the choice of formulae $\psi_1, \dots, \psi_n$. Thus, $\neg \Box\varphi \notin hd(w)$ due to the consistency of the set $hd(w)$, which contradicts the assumption of the lemma. Therefore, set Y_0 is consistent.

Let Y be any maximal consistent extension of set Y_0 and let u be the sequence $w :: (F, Y)$. We next show that $u \in W_\infty$. Indeed, since $w \in W_\infty$, by Definition 5, there must exist $n_1 \geq 0$ such that $w \in W_{n_1}$. At the same time, since set F is a *finite* subset of W_∞ , by Definition 5, there must exist $n_2 \geq 0$ such that $F \subseteq \bigcup_{i \leq n_2} W_i$. Let $n = \max\{n_1, n_2\}$. Thus, $w \in \bigcup_{i \leq n} W_i$ and $F \subseteq \bigcup_{i \leq n} W_i$. Hence, $w :: (F, Y) \in W_{n+1}$ by Definition 4. Therefore, $u = w :: (F, Y) \in W_\infty$ by Definition 5.

Finally, $w \sim_F u$ by Definition 6. To finish the proof of the lemma, note that $\neg\varphi \in Y_0 \subseteq Y = hd(u)$. $\square$

Lemma 15 *For any $w \in W_\infty$ and any $\neg\Box\varphi \in hd(w)$, there is $u \in W_\infty$ such that $w \sim_{W_\infty} u$ and $\neg\varphi \in hd(u)$.*

Proof. We first show that the following set is consistent:

$$Y_0 = \{\neg\varphi\} \cup \{\psi \mid \Box\psi \in hd(w)\}.$$

Assume the opposite. Thus, there must exist formulae $\Box\psi_1, \dots, \Box\psi_n \in hd(w)$ and $\psi_1, \dots, \psi_n \vdash \varphi$. Hence, by the deduction theorem for the propositional logic,

$$\vdash \psi_1 \rightarrow (\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots).$$

Then, by Attainable Necessitation inference rule,

$$\vdash \Box(\psi_1 \rightarrow (\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots)).$$

By Monotonicity axiom and Modus Ponens Inference rule,

$$\vdash \Box(\psi_1 \rightarrow (\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots)).$$

By Distributivity axiom and Modus Ponens inference rule,

$$\vdash \Box\psi_1 \rightarrow \Box(\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots).$$

By Modus Ponens inference rule,

$$\Box\psi_1 \vdash \Box(\psi_2 \rightarrow \dots (\psi_n \rightarrow \varphi) \dots).$$

By repeating the last two steps $n - 1$ times,

$$\Box\psi_1, \dots, \Box\psi_n \vdash \Box\varphi.$$

Hence, $hd(w) \vdash \Box\varphi$ by the choice of formulae $\psi_1, \dots, \psi_n$. Thus, $\neg\Box\varphi \notin hd(w)$ due to the consistency of the set $hd(w)$, which contradicts the assumption of the lemma. Therefore, set Y_0 is consistent.

Let Y be any maximal consistent extension of set Y_0 and let u be sequence $w :: (*, Y)$. We next show that $u \in W_\infty$. Indeed, since $w \in W_\infty$, by Definition 5, there must exist $n \geq 0$ such that $w \in W_n$. Hence, $w :: (*, Y) \in W_{n+1}$ by Definition 4. Therefore, $u = w :: (*, Y) \in W_\infty$ by Definition 5.

Finally, let us observe that $w \sim_e u$ for each $e \in W_\infty$ by Definition 6. Thus, $w \sim_{w_\infty} u$. To finish the proof of the lemma, note that $\neg\varphi \in Y_0 \subseteq Y = hd(u)$. $\square$

By Definition 4, elements of set W_∞ are sequences of the form

$$(X_0, \epsilon_1, X_1, \dots, \epsilon_k, X_k),$$

where ϵ_i is either symbol $*$ or a subset of W_∞ . One might wonder if elements of W_∞ are wellfounded. In other words, is it possible for an element $w \in W_\infty$ to be a member of one of its own ϵ_i ? Lemma 17 shows that such elements do not exist. This is a very important observation for our proof of completeness. Lemma 16 is essentially a different form of Lemma 17 which is easier to prove by induction.

Lemma 16 *For any $k \geq 0$, any $n \geq 0$, any $t \geq 0$, and any*

$$w = (X_0, \epsilon_1, X_1, \dots, \epsilon_k, X_k, \epsilon_{k+1}, X_{k+1}, \dots, \epsilon_{k+n}, X_{k+n}) \in W_t,$$

*if $\epsilon_k \neq *$, then $\epsilon_k \subseteq \bigcup_{i=0}^{t-1} W_i$.*

Proof. We prove this statement by induction on n . First, assume that $n = 0$. Thus, $w = (X_0, \epsilon_1, X_1, \dots, \epsilon_k, X_k)$. Hence, By Definition 4, assumptions $w \in W_t$ and $\epsilon_k \neq *$ imply that $\epsilon_k \subseteq \bigcup_{i=0}^{t-1} W_i$.

Suppose now that $n > 0$. By Definition 4, the assumption $w \in W_t$ implies that

$$(X_0, \epsilon_1, X_1, \dots, \epsilon_k, X_k, \epsilon_{k+1}, X_{k+1}, \dots, \epsilon_{k+n-1}, X_{k+n-1}) \in W_{t-1}.$$

Thus, by the induction hypothesis, $\epsilon_k \subseteq \bigcup_{i=0}^{t-2} W_i$. Therefore, $\epsilon_k \subseteq \bigcup_{i=0}^{t-1} W_i$. $\square$

Lemma 17 $w \notin \epsilon_k$ for each $w = (X_0, \epsilon_1, X_1, \dots, \epsilon_n, X_n) \in W_\infty$ and each $k \leq n$.

Proof. By Definition 5, assumption $w \in W_\infty$ implies that there is $t \geq 0$ such that $w \in W_t$. Let m be the smallest m such that $w \in W_m$. Thus, $w \notin \bigcup_{i=0}^{m-1} W_i$. At the same time, $\epsilon_k \subseteq \bigcup_{i=0}^{m-1} W_i$ by Lemma 16. Therefore, $w \notin \epsilon_k$. $\square$

The next lemma puts together the pieces of the proof that we have developed. It connects the satisfiability of a formula in an epistemic world of the canonical Kripke model with the maximal consistent sets out of which the world is constructed.

Lemma 18 $w \Vdash \varphi$ iff $\varphi \in hd(w)$.

Proof. We prove the lemma by induction on the structural complexity of formula φ . If formula φ is an atomic proposition, then the required follows from Definition 7 and Definition 3. If formula φ is a negation or an implication, then the required follows from Definition 3 and the maximality and the consistency of the set $hd(w)$ in the standard way.

Suppose now that formula φ has the form $\Box\psi$.

($\Rightarrow$) If $\Box\psi \notin hd(w)$, then $\neg \Box\psi \in hd(w)$ due to the maximality of the set $hd(w)$. To prove $w \not\Vdash \Box\psi$, by Definition 3, we need to show that for any finite set $F \subseteq W_\infty$ there is $u \in W_\infty$ such that $w \sim_F u$ and $u \not\Vdash \psi$. Indeed, by Lemma 14, there is $u \in W_\infty$ such that $w \sim_F u$ and $\neg\psi \in hd(u)$. Thus, $\psi \notin hd(u)$ due to the consistency of the set $hd(u)$. Therefore, $u \not\Vdash \psi$ by the induction hypothesis.

($\Leftarrow$) Suppose that $\Box\psi \in hd(w)$. Thus, $hd(w) \vdash \Box \Box\psi$ by Attainable Positive Introspection axiom. Hence, $hd(w) \vdash \Box \Box\psi$ by Monotonicity axiom. Then,

$$\Box \Box\psi \in hd(w) \tag{5}$$

due to the maximality of the set $hd(w)$.

By Definition 3, it suffices to show that $u \Vdash \psi$ for all $u \in W_\infty$ such that $w \sim_w u$. By Definition 6, assumption $w \sim_w u$ implies that if $w = (X_0, \epsilon_1, X_1, \dots, \epsilon_n, X_n)$ and $u = (X'_0, \epsilon'_1, X'_1, \dots, \epsilon'_m, X'_m)$, then there is k such that,

1. $0 \leq k \leq \min\{n, m\}$,
2. $X_i = X'_i$ for all i such that $0 \leq i \leq k$,
3. $\epsilon_i = \epsilon'_i$ for all i such that $0 < i \leq k$,
4. for all i , if $k < i \leq n$, then either $w \in \epsilon_i$ or $\epsilon_i = *$,
5. for all i , if $k < i \leq m$, then either $w \in \epsilon'_i$ or $\epsilon'_i = *$.

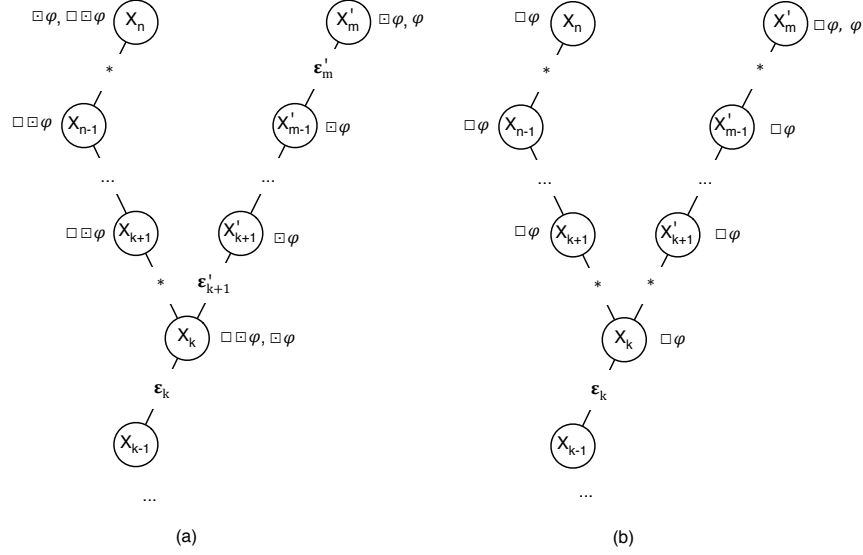


Figure 3: Illustrations for the proof of Lemma 18.

By Lemma 17, $w \notin \epsilon_i$ for each $i \leq n$. Thus, from the condition 4 above, $\epsilon_i = *$ for each i such $k < i \leq n$, see Figure 3 (a). Hence, by Lemma 13, statement (5) implies that $\Box \Box \psi \in X_k$. Thus, $X_k \vdash \Box \psi$ by Truth axiom. Then, $\Box \psi \in X_k$ due to the maximality of set X_k . Therefore, $\psi \in X'_m = hd(u)$ by Lemma 11 and condition 5 above. Therefore, $u \Vdash \psi$ by the induction hypothesis.

Finally, let formula φ have the form $\Box \psi$.

($\Rightarrow$) If $\Box \psi \notin hd(w)$, then $\neg \Box \psi \in hd(w)$ due to the maximality of the set $hd(w)$. To prove $w \not\Vdash \Box \psi$, by Definition 3, we need to show that $u \not\Vdash \psi$ for some $u \in W_\infty$ such that $w \sim_{W_\infty} u$. Indeed, by Lemma 15, there is $u \in W_\infty$ such that $w \sim_{W_\infty} u$ and $\neg \psi \in hd(u)$. Thus, $\psi \notin hd(u)$ due to the consistency of the set $hd(u)$. Therefore, $u \not\Vdash \psi$ by the induction hypothesis.

($\Leftarrow$) Suppose that $\Box \psi \in hd(w)$. By Definition 3, it suffices to show that $u \Vdash \psi$ for all $u \in W_\infty$ such that $w \sim_{W_\infty} u$. Indeed, let $w = (X_0, \epsilon_1, X_1, \dots, \epsilon_n, X_n)$ and $u = (X'_0, \epsilon'_1, X'_1, \dots, \epsilon'_m, X'_m)$. By Definition 6, assumption $w \sim_{W_\infty} u$ implies that for each $e \in W_\infty$ there is integer k_e such that

1. $0 \leq k_e \leq \min\{n, m\}$,
2. $X_i = X'_i$ for all i such that $0 \leq i \leq k_e$,
3. $\epsilon_i = \epsilon'_i$ for all i such that $0 < i \leq k_e$,
4. for all i , if $k_e < i \leq n$, then either $w \in \epsilon_i$ or $\epsilon_i = *$,
5. for all i , if $k_e < i \leq m$, then either $w \in \epsilon'_i$ or $\epsilon'_i = *$.

By Lemma 10, set W_∞ is infinite and, thus, it is nonempty. Hence, set $\{k_e \mid e \in W_\infty\}$ is nonempty. Set $\{k_e \mid e \in W_\infty\}$ is finite because $0 \leq k_e \leq \min\{n, m\}$ for each $e \in W_\infty$. Let k be the maximal element of the set $\{k_e \mid e \in W_\infty\}$. Hence,

1. $0 \leq k \leq \min\{n, m\}$,
2. $X_i = X'_i$ for all i such that $0 \leq i \leq k$,
3. $\epsilon_i = \epsilon'_i$ for all i such that $0 < i \leq k$,
4. for all i , if $k < i \leq n$, then either $W_\infty \subseteq \epsilon_i$ or $\epsilon_i = *$,
5. for all i , if $k < i \leq m$, then either $W_\infty \subseteq \epsilon'_i$ or $\epsilon'_i = *$.

By Lemma 10, set W_∞ is infinite yet sets ϵ'_i and ϵ_i are finite for each i by Definition 4. Hence, $W_\infty \not\subseteq \epsilon_i$ and $W_\infty \not\subseteq \epsilon'_i$ for each i . Thus, conditions 4 and 5 above imply that $\epsilon_i = *$ for all i such that $k < i \leq n$ and $\epsilon_i = *$ for all i such that $k < i \leq m$, see Figure 3 (b). Thus, by Lemma 13, assumption $\Box\psi \in hd(w) = X_n$ implies that $\Box\psi \in X_k = X'_k$. Therefore, $\psi \in X'_m = hd(u)$, by Lemma 12. $\square$

We are now ready to finish the proof of Theorem 2. Recall that $\not\models \varphi_0$. Consider any maximal consistent set X_0 such that $\varphi_0 \notin X_0$. Let w_0 be the single-element sequence (X_0) . Thus, $w_0 \not\models \varphi_0$ by Lemma 18. This concludes the proof of the theorem.

7 Conclusion

In this article we have shown that knowledge obtained from infinitely many pieces of evidence has properties captured by modal logic S5 and knowledge obtained from finitely many pieces of evidence has properties described by modal logic S4. The main technical result is a sound and complete propositional bimodal logic that captures properties of both of these types of knowledge and their interplay.

A natural next step is to consider first-order logic with the same two modalities. Note that the knowledge modality $\Box$ satisfies Barcan Formula [2] $\forall x \Box\varphi \rightarrow \Box\forall x \varphi$ because this formula is derivable from S5 axioms stated in the first-order modal language [9]. At the same time, attainable knowledge modality $\Box$ does not satisfy Barcan Formula $\forall x \Box \varphi \rightarrow \Box\forall x \varphi$. Indeed, if variable x ranges over an infinite domain and each value of x in this domain has a distinct single evidence e_x that justifies $\varphi(x)$, then $\forall x \Box \varphi$ is true, but $\Box\forall x \varphi$ is not. A complete axiomatization of the interplay of these two modalities in the first-order language remains an open problem.

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