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Synchronization and Control in Bistable Oscillator Networks: Towards Epilepsy Regulation

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Abstract—This letter studies networks of diffusively-coupled bistable oscillators. This class of oscillators has been shown before to be suitable for capturing epileptic dynamics, since its stable equilibrium can describe normal neural activity, while its stable limit cycle signifies seizure events. To investigate seizure propagation in brain networks, a sufficient condition to ensure synchronization of the set of oscillators is established. The synchronized network inherits the property of bistability from the single oscillators, i.e., the network can transition between a stable equilibrium and a limit cycle manifold, capturing network-level onsets and offsets of generalized seizures. To prevent the network from converging towards the limit cycle, a controller that targets a subset of the oscillators is proposed, which is proven to globally stabilize the equilibrium. Numerical studies are conducted to validate the theoretical findings.

Index Terms—Networked control systems, biological systems, stability of nonlinear systems.

I. INTRODUCTION

EPILEPSY is one of the most prevalent neurological disorders worldwide, characterized by abrupt seizures accompanied by excessive synchronization between brain regions [1], [2]. Various therapeutic approaches, including medication, surgery, and brain stimulation, have been employed to treat epilepsy [3]. Yet, the efficacy remains limited, largely because the precise, likely individualized, mechanisms underlying epileptic seizures remain unclear. System modeling presents a promising method for advancing current therapies. By constructing models, researchers

aim to gain deeper insights into the principles underlying epilepsy through a lens of dynamical systems, paving the way for developing more adaptive and personalized therapeutic interventions.

This letter makes an effort along this line by investigating epileptic dynamics in networks composed of bistable oscillators. As demonstrated in [4], [5], [6], each bistable oscillator in the network, which allows for the coexistence of a stable equilibrium and a stable limit cycle, well captures the fundamental states of epilepsy. The equilibrium describes normal (*interictal* or *non-ictal*) neural activity without seizures, while the limit cycle signifies abnormal (*ictal*) oscillations associated with seizure events [7]. Transitions between them describe seizure onsets and offsets. This letter studies synchronization in networks of such oscillators, aiming to shed new light on the propagation of seizures in brain networks.

In the context of synchronization in neuronal networks, existing work primarily focuses on dynamics characterized by a single stable limit cycle. In [8], the coupling gains necessary to achieve synchrony are studied by utilizing the property of semi-passivity. In [9], less conservative coupling gains for achieving (cluster) synchronization are established based on contractiveness of the networked neurons. In [10], time-varying gains are employed, following a similar principle. For networked heterogeneous oscillators, the work in [11] employs dynamic coupling rules to achieve synchrony, while a type of funnel coupling is adopted in [12]. In recent work [13], adaptive coupling gains to limit the synchronization error are adopted using calculation of reachable sets. Concerning bistable oscillators, the study in [14] examines how coupling strength and specific initializations influence the synchronized manifold for Hodgkin-Huxley oscillators. The work in [15] presents conditions to achieve phase synchronization for bistable quantum oscillators. Specific to modeling and regulating epileptic dynamics, a class of linear-threshold and linear fractional-order models are employed in [16], [17], [18], [19], while a bilinear one is employed in [20].

In this letter, diffusively coupled bistable oscillators are used to model brain network dynamics (Section II). A sufficient condition for the coupling strength is established under which the oscillators achieve global synchronization (Section III). Unlike most existing studies, the synchronized network exhibits bistability, enabling transitions between the equilibrium and the limit cycle, similar to individual node dynamics. This behavior captures brain-network-level onsets

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and offsets of ictal events. To mitigate the emergence of widespread pathological oscillations, a control strategy akin to pinning control (Section IV) is proposed, which regulates only a subset of oscillators in the network to make the origin (the non-ictal state) of the controlled network become globally stable. Numerical experiments are also performed to validate the findings (Section V).

II. PROBLEM DESCRIPTION

In this letter, the following model to describe the dynamics of individual neuronal populations is considered (see [4]):

$$\dot{x} = -\omega y + x \left(\sigma + 2ab(x^2 + y^2) - b(x^2 + y^2)^2 \right) \quad (1a)$$

$$\dot{y} = \omega x + y \left(\sigma + 2ab(x^2 + y^2) - b(x^2 + y^2)^2 \right) \quad (1b)$$

where $x, y \in \mathbb{R}$ represent the population-averaged firing rate of pyramidal and interneuron ensembles, and $\omega, a, b > 0$, $\sigma \in \mathbb{R}$ are the corresponding parameters. For simplicity of notation, denote $g(x, y) := 2ab(x^2 + y^2) - b(x^2 + y^2)^2$. As shown in [4], the system (1) exhibits a variety of dynamic behavior at different parameter regimes, as documented below.

Lemma 1: Denote the solution to the system (1) for any initial condition $\text{col}(x(0), y(0))$ by $\text{col}(x(t), y(t))$. Then, the following statements hold:

- 1) If $\sigma < -a^2b$, the origin is globally stable.
- 2) If $\sigma = -a^2b$, for any initial state satisfying $x^2(0) + y^2(0) < a$, $\text{col}(x(t), y(t))$ converges to the origin as $t \rightarrow \infty$. For $x^2(0) + y^2(0) \geq a$, the solution converges to a unique limit cycle defined by:

$$\mathcal{C}_1 := \{x, y \in \mathbb{R} \mid x^2 + y^2 = a\}.$$

- 3) If $-a^2b < \sigma < 0$, two limit cycles:

$$\mathcal{C}_2 := \left\{x, y \in \mathbb{R} \mid x^2 + y^2 = a - \sqrt{a^2 + \sigma/b}\right\}$$

$$\mathcal{C}_3 := \left\{x, y \in \mathbb{R} \mid x^2 + y^2 = a + \sqrt{a^2 + \sigma/b}\right\}$$

co-exist. While the solution converges to the origin for any initial state satisfying $x^2(0) + y^2(0) < a - \sqrt{a^2 + \sigma/b}$, the limit cycle $\mathcal{C}_2$ attained for $x^2(0) + y^2(0) = a - \sqrt{a^2 + \sigma/b}$ is unstable, and the solution converges to $\mathcal{C}_3$ for $x^2(0) + y^2(0) > a - \sqrt{a^2 + \sigma/b}$.

- 4) If $\sigma \geq 0$, the solution $\text{col}(x(t), y(t))$ always converges to the limit cycle $\mathcal{C}_3$ except starting from the origin.

To capture the characters of epileptic dynamics, the bistable regime where $-a^2b < \sigma < 0$ is assumed here, as the stable equilibrium at the origin describes background neural activity at a normal state without epileptic seizures, while the stable limit cycle $\mathcal{C}_3$ represents the abnormal oscillating neural activity when seizures take place. External inputs or internal noise can drive transitions between these two states. Following [4], the system (1) in this regime is referred as a *bistable oscillator*, since it behaves like a damped oscillator around the origin and an undamped one near the limit cycle. The oscillation frequency depends on the parameter ω .

In this letter, networks of such bistable oscillators are studied, as the brain is a complex network consisting of a large number of interacting neurons. Specifically, given a

set $\mathcal{N} = \{1, \dots, n\}$ of bistable oscillators (1), which are heterogeneous with regard to the parameters ω, a, b, σ , the following diffusively coupled dynamics is considered:

$$\dot{x}_i = -\omega_i y_i + x_i(\sigma_i + g_i(x_i, y_i)) + k \sum_{j \in \mathcal{N}} \alpha_{ij}(x_j - x_i) \quad (2a)$$

$$\dot{y}_i = \omega_i x_i + y_i(\sigma_i + g_i(x_i, y_i)) + k \sum_{j \in \mathcal{N}} \alpha_{ij}(y_j - y_i) \quad (2b)$$

with $g_i(x_i, y_i) := 2a_i b_i(x_i^2 + y_i^2) - b_i(x_i^2 + y_i^2)^2$, $\alpha_{ij} \in \{0, 1\}$, and $k \in \mathbb{R}_{>0}$ representing the coupling gain. The network can be understood as an undirected graph $\mathcal{G} := (\mathcal{N}, \mathcal{E})$, with the sets of nodes $\mathcal{N}$ and edges $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ (A generalization to directed graphs is provided at the end of Section IV). The adjacency matrix is defined by $A := [\alpha_{ij}]_{n \times n}$, which satisfies $\alpha_{ij} = 1$ whenever an edge $e_{ij} \in \mathcal{E}$ exists, and $\alpha_{ij} = 0$ otherwise. It is assumed that no self-loop exists, i.e., $\alpha_{ii} = 0$ for all $i \in \mathcal{N}$.

Remark 1: In modeling brain networks, various coupling schemes are used to capture interactions between brain areas, including diffusive coupling (e.g., [8], [21], [22]) and additive coupling (e.g., [23], [24]). In this letter, the type of linear diffusive coupling is adopted, as the focus is on modeling the spread of seizure activity across brain networks. Note that incorporating nonlinearity, such as through activation functions inspired by deep neural networks [24], may yield richer network-level dynamics. Exploring such nonlinear extensions remains an important direction for future work.

This letter investigates the network-level behavior of (2) with a focus on synchronization of the oscillator network, since synchronization is closely related to epilepsy. In Section III, conditions will be derived under which synchronization of the oscillators occurs. It turns out, however, that synchronization does not guarantee that the network always stays close to the origin; thus, controllers aiming at globally stabilizing the origin will be investigated in Section IV.

III. SYNCHRONIZATION OF HOMOGENEOUS BISTABLE OSCILLATORS

This section starts with a homogeneous network, in which all bistable oscillators in (2) share the same system parameters $\omega := \omega_i$, $a := a_i$, $b := b_i$, and $\sigma := \sigma_i$ (an initial constraint that is later relaxed). Let $z_i := [x_i, y_i]^T$, such that each node dynamics (without coupling) becomes:

$$\dot{z}_i = f(z_i) := \begin{bmatrix} -\omega y_i + x_i(\sigma + g(x_i, y_i)) \\ \omega x_i + y_i(\sigma + g(x_i, y_i)) \end{bmatrix}. \quad (3)$$

Subsequently, the overall diffusively coupled network system can be written as:

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} f(z_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(z_n) \end{bmatrix} - k(\mathcal{L}(\mathcal{G}) \otimes I_2) \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \quad (4)$$

where $\mathcal{L}(\mathcal{G})$ represents the Laplacian matrix of the graph $\mathcal{G}$ with $\otimes$ denoting the Kronecker product, and $I_2 \in \mathbb{R}^{2 \times 2}$ the identity matrix. Before presenting the main results in this section, the following intermediate results are provided first.

Lemma 2: For any coupling gain $k > 0$ and any initial state $z_i(0) \in \mathbb{R}^2$, $i \in \mathcal{N}$, the solution $\mathbf{z}(t) = \text{col}(z_1(t), \dots, z_n(t))$ of (4) is bounded for all $t \geq 0$.

Proof: Define the function $V(\mathbf{z}) = \frac{1}{2}\mathbf{z}^T\mathbf{z}$. The time derivative of $V(\mathbf{z})$ is given by:

$$\begin{aligned}\dot{V}(\mathbf{z}) &= \sum_{i \in \mathcal{N}} (\sigma + g(x_i, y_i)) (x_i^2 + y_i^2) \\ &\quad - k \sum_{e_{ij} \in \mathcal{E}} ((x_i - x_j)^2 + (y_i - y_j)^2) \\ &\leq \sum_{i \in \mathcal{N}} (\sigma + 2ab(x_i^2 + y_i^2) - b(x_i^2 + y_i^2)^2) (x_i^2 + y_i^2) \\ &\leq \sum_{i \in \mathcal{N}} (-b(x_i^2 + y_i^2 - a)^2 + a^2b + \sigma) (x_i^2 + y_i^2). \quad (5)\end{aligned}$$

As $\sigma \in (-a^2b, 0)$, the derivative $\dot{V}(\mathbf{z}) < 0$ applies whenever $x_i^2 + y_i^2 > a + \sqrt{a^2 + \sigma/b}$ or $x_i^2 + y_i^2 < a - \sqrt{a^2 + \sigma/b}$ hold for all $i \in \mathcal{N}$. Accordingly, $V(\mathbf{z}(t))$ is bounded for all $t \geq 0$, and thus so is $\mathbf{z}(t)$. ■

Note the function $f(z_i)$ is continuously differentiable for all $z_i \in \mathbb{R}^2$. The following lemma establishes a property for the Jacobian matrix of the dynamics in (3):

Lemma 3: The Jacobian matrix $J(z_i) := \frac{\partial f}{\partial z_i}$ satisfies:

$$J^T(z_i) + J(z_i) \leq (2\sigma + 3.6a^2b)I_2, \quad \forall z_i \in \mathbb{R}^2. \quad (6)$$

Proof: For any $z_i \in \mathbb{R}^2$, the Jacobian matrix $J(z_i) = \begin{bmatrix} J_{11}(z_i) & J_{12}(z_i) \\ J_{21}(z_i) & J_{22}(z_i) \end{bmatrix}$ is given by:

$$\begin{aligned}J_{11}(z_i) &= \sigma - (4bx_i^2 - 2ab)(x_i^2 + y_i^2) - 4abx_i^2 - b(x_i^2 + y_i^2)^2 \\ J_{12}(z_i) &= -w - 4bx_iy_i(x_i^2 + y_i^2) + 4abx_iy_i \\ J_{21}(z_i) &= w - 4bx_iy_i(x_i^2 + y_i^2) + 4abx_iy_i \\ J_{22}(z_i) &= \sigma - (4by_i^2 - 2ab)(x_i^2 + y_i^2) - 4aby_i^2 - b(x_i^2 + y_i^2)^2.\end{aligned}$$

The two eigenvalues of $J^T(z_i) + J(z_i)$ can be expressed by:

$$\begin{aligned}\lambda_1 &= -2b(x_i^2 + y_i^2 - a)^2 + 2a^2b + 2\sigma \leq 2a^2b + 2\sigma \\ \lambda_2 &= -10b(x_i^2 + y_i^2 - 0.6a)^2 + 3.6a^2b + 2\sigma \leq 3.6a^2b + 2\sigma.\end{aligned}$$

As $a, b > 0$, the inequality (6) must always hold. ■

Based on Lemma 2 and 3, the following theorem presents a condition for the coupling gain k to ensure asymptotic synchronization of the bistable oscillators in the network.

Theorem 1: Assume that the graph $\mathcal{G}$ is connected and let $\lambda_2(\mathcal{L}(\mathcal{G})) > 0$ denote the second smallest eigenvalue of the Laplacian matrix $\mathcal{L}(\mathcal{G})$. If the coupling gain k satisfies

$$k > (1.8a^2b + \sigma) / \lambda_2(\mathcal{L}(\mathcal{G})), \quad (7)$$

then for any initial state $z_i(0) \in \mathbb{R}^2$, $i \in \mathcal{N}$, the solution of (4) always satisfies:

$$\lim_{t \rightarrow +\infty} \|z_i(t) - z_j(t)\|_2 = 0 \quad (8)$$

for any pair $i, j \in \mathcal{N}$, i.e., the local states are asymptotically synchronized.

Proof: First note that $J(z_i)$ is a continuously differentiable function and that the local solution $z_i(t)$ of the diffusively coupled system (4) remains bounded in a convex set $\mathcal{X} \subseteq \mathbb{R}^2$. The condition (7) is then a direct result of [25, Th. 4],

which shows for the stated conditions that the local states are asymptotically synchronized if:

$$J^T(z_i) + J(z_i) - 2k\lambda_2(\mathcal{L}(\mathcal{G})) < 0 \quad (9)$$

holds for all $z_i \in \mathcal{X}$, $i \in \mathcal{N}$. Note that for any coupling gain k satisfying (7), the inequality (9) always holds according to Lemma 3. Since, in addition, the $z_i(t)$ are bounded for any initialization according to Lemma 2, i.e., the set $\mathcal{X}$ can be arbitrarily large, pairwise asymptotic synchronization as in (8) thus applies for any initial state. ■

While synchronization is guaranteed if (7) is satisfied for the coupling gain, there does interestingly not exist a unique manifold towards which the states converge. As shown later in Fig. 2, the network inherits the bistability of individual oscillators: depending on the initial states, the network can converge to the origin, or to the limit cycle $\mathcal{C}_3$. The origin is the desired state, as it represents that the entire brain is seizure-free. The following theorem constructs a condition for the initial states under which the local states converge to the origin.

Theorem 2: For initial states satisfying $\|z_i(0)\|_2^2 < (a - \sqrt{a^2 + \sigma/b})/n$ for all $i \in \mathcal{N}$ and when using a coupling gain k satisfying (7), all local states $z_i(t)$ synchronize in the origin for $t \rightarrow \infty$.

Proof: If $\|z_i(0)\|_2^2 < (a - \sqrt{a^2 + \sigma/b})/n$ holds for all $i \in \mathcal{N}$, the vector of initial states $\mathbf{z}(0)$ satisfies $\|\mathbf{z}(0)\|_2^2 < a - \sqrt{a^2 + \sigma/b}$. For the function $V(\mathbf{z}) = \frac{1}{2}\mathbf{z}^T\mathbf{z} = \frac{1}{2}\|\mathbf{z}\|_2^2$, the time derivative $\dot{V}(\mathbf{z})$ thus must be negative at the initial time according to (5). As $\|\mathbf{z}\|_2^2 \leq \|\mathbf{z}(0)\|_2^2 < a - \sqrt{a^2 + \sigma/b}$ holds for all $i \in \mathcal{N}$, the derivative must remain negative for all $t \geq 0$ according to (5), until $\|\mathbf{z}\|_2 = 0$ applies, i.e., all local states are stabilized to the origin. ■

This theorem provides an estimate of the region of attraction for the origin of the entire network. However, it is a bit conservative for the following reasons: Firstly, this estimate of the region of attraction is smaller than that of decoupled individual oscillators. Secondly, the size of the admissible initial set decreases with an increasing number n of oscillators in the network. Finally, it relies on the assumption that all local bistable oscillators are all homogeneous (w.r.t. ω , a , b , and σ), since the properties of neuronal populations often vary across different brain areas [26]. In light of these limitations, the next section relaxes the assumptions by considering networks of heterogeneous oscillators and introduces a control strategy to stabilize the origin of such more general networks.

IV. STABILIZATION OF HETEROGENEOUS BISTABLE OSCILLATORS

This section studies the heterogeneous network described by (2). Similar to the homogeneous case in the previous section, the heterogeneous network also exhibits bistability or even multi-stability. This section aims at designing controllers to stabilize the normal state, i.e., the origin. In particular, similar to the concept of pinning control [27], external control inputs are only imposed on a subset of oscillators in the network. Specifically, the controlled system is described by:

$$\dot{z}_i = f_i(z_i) + k \sum_{j \in \mathcal{N}} a_{ij}(z_j - z_i) + u_i \quad (10)$$

with $f_i(z_i) := \begin{bmatrix} -\omega_i y_i + x_i(\sigma_i + g_i(x_i, y_i)) \\ \omega_i x_i + y_i(\sigma_i + g_i(x_i, y_i)) \end{bmatrix}$. Here, the node set $\mathcal{N}$ is decomposed into two subsets $\mathcal{N}_1$ and $\mathcal{N}_2$, where $\mathcal{N}_1$ is

the driver set ($|\mathcal{N}_1| = n_1$) containing the oscillators directly under control, while the set $\mathcal{N}_2$ (with $|\mathcal{N}_2| = n_2$, $n = n_1 + n_2$) consists of the oscillators not directly influenced by the control inputs, thus $u_j = 0$ for all $j \in \mathcal{N}_2$. A state-feedback control law is considered for the oscillators with $i \in \mathcal{N}_1$:

$$u_i = -q_i[x_i \ y_i]^T = -q_i z_i, \text{ with } q_i > 0. \quad (11)$$

Without loss of generality, assume that the oscillators in $\mathcal{N}_1$ are labeled by $1, 2, \dots, n_1$, and use $z_{\mathcal{N}_1} := \text{col}(z_1, \dots, z_{n_1})$ and $z_{\mathcal{N}_2} := \text{col}(z_{n_1+1}, \dots, z_n)$. Furthermore, for f_i given by (3) and the feedback control law given by (11), let:

$$f_{i,c}(z_i) := \begin{bmatrix} -\omega_i y_i + x_i(\sigma_i - q_i + g_i(x_i, y_i)) \\ \omega_i x_i + y_i(\sigma_i - q_i + g_i(x_i, y_i)) \end{bmatrix} \quad (12)$$

denote the controlled dynamics of local oscillators, and let $\mathbf{F}_{c,\mathcal{N}_1} := \text{diag}(f_{1,c}(z_1), \dots, f_{n_1,c}(z_{n_1}))$, $\mathbf{F}_{\mathcal{N}_2} := \text{diag}(f_{n_1+1}(z_{n_1+1}), \dots, f_n(z_n))$. The overall dynamics can be written in the compact form as:

$$\begin{bmatrix} \dot{z}_{\mathcal{N}_1} \\ \dot{z}_{\mathcal{N}_2} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{c,\mathcal{N}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{\mathcal{N}_2} \end{bmatrix} - k(\mathcal{L}(\mathcal{G}) \otimes I_2) \begin{bmatrix} z_{\mathcal{N}_1} \\ z_{\mathcal{N}_2} \end{bmatrix}. \quad (13)$$

Lemma 4: Let $J_{i,c}(z_i) = \frac{\partial f_{i,c}}{\partial z_i}$ for all $i \in \mathcal{N}_1$ and $J_j(z_j) = \frac{\partial f_j}{\partial z_j}$ for all $j \in \mathcal{N}_2$. Then, there exists $\gamma > 0$ such that:

$$J_j^T(z_j) + J_j(z_j) \leq \gamma I_2 \quad (14)$$

holds for all $z_j \in \mathbb{R}^2$ and $j \in \mathcal{N}_2$. If the feedback gain in (11) satisfies $q_i > \sigma_i + 1.8a_i^2 b_i$, there exists $\beta > 0$ such that:

$$J_{i,c}^T(z_i) + J_{i,c}(z_i) \leq (2(\sigma_i - q_i) + 3.6a_i^2 b_i) I_2 \leq -\beta I_2 \quad (15)$$

holds for all $z_i \in \mathbb{R}^2$ and $i \in \mathcal{N}_1$.

Proof: Based on (6), γ in (14) can be set to:

$$\gamma := \max_{j \in \mathcal{N}_2} 3.6a_j^2 b_j + 2\sigma_j, \quad (16)$$

and the inequality:

$$J_{i,c}^T(z_i) + J_{i,c}(z_i) \leq (2\sigma_i + 3.6a_i^2 b_i - 2q_i) I_2 \quad (17)$$

applies with $f_{i,c}(z_i)$ for all $z_i \in \mathbb{R}^2$. Thus if $q_i > \sigma_i + 1.8a_i^2 b_i$ for all $i \in \mathcal{N}_1$, the value of β in (15) can be set to:

$$\beta := \min_{i \in \mathcal{N}_1} |2\sigma_i + 3.6a_i^2 b_i - 2q_i|, \quad (18)$$

what completes the proof. ■

Let the Laplacian matrix $\mathcal{L}(\mathcal{G})$ be decomposed into:

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{12}^T & \mathcal{L}_{22} \end{bmatrix} \quad (19)$$

with $\mathcal{L}_{11} \in \mathbb{R}^{n_1 \times n_1}$, $\mathcal{L}_{12} \in \mathbb{R}^{n_1 \times n_2}$, and $\mathcal{L}_{22} \in \mathbb{R}^{n_2 \times n_2}$. The Jacobian matrix of (13) is then given by:

$$\mathbf{J}(\mathbf{z}) = \begin{bmatrix} \mathbf{J}_{c,\mathcal{N}_1} - k\mathcal{L}_{11} \otimes I_2 & -k\mathcal{L}_{12} \otimes I_2 \\ -k\mathcal{L}_{12}^T \otimes I_2 & \mathbf{J}_{\mathcal{N}_2} - k\mathcal{L}_{22} \otimes I_2 \end{bmatrix} \quad (20)$$

with $\mathbf{J}_{c,\mathcal{N}_1} := \text{diag}(J_{1,c}(z_1), \dots, J_{n_1,c}(z_{n_1}))$ and $\mathbf{J}_{\mathcal{N}_2} := \text{diag}(J_{n_1+1}(z_{n_1+1}), \dots, J_n(z_n))$. Next, some useful properties of the sub-matrices of $\mathcal{L}(\mathcal{G})$ are emphasized.

Lemma 5: If every node in $\mathcal{N}_1$ is connected with at least $c_1 \in \mathbb{N}$ nodes in $\mathcal{N}_2$, then the inequality:

$$\mathcal{L}_{11}^T + \mathcal{L}_{11} \geq 2c_1 I_{n_1} \quad (21)$$

holds for the symmetric matrix $\mathcal{L}_{11}$ in (19). Furthermore, if every node in $\mathcal{N}_2$ is connected with at least $c_2 \in \mathbb{N}$ nodes in $\mathcal{N}_1$, then the following holds:

$$\mathcal{L}_{22}^T + \mathcal{L}_{22} \geq 2c_2 I_{n_2}. \quad (22)$$

Proof: Let $\mathcal{G}_{\mathcal{N}_1} := (\mathcal{V}_{\mathcal{N}_1}, \mathcal{E}_{\mathcal{N}_1})$ denote the subgraph of $\mathcal{G}$ containing only the nodes in $\mathcal{N}_1$ and the edges in between these nodes. Let $\mathcal{L}(\mathcal{G}_{\mathcal{N}_1}) \in \mathbb{R}^{n_1 \times n_1}$ denote the Laplacian matrix of $\mathcal{G}_{\mathcal{N}_1}$, for which it holds that:

$$\mathcal{L}_{11} = \mathcal{L}(\mathcal{G}_{\mathcal{N}_1}) + \text{diag}(\eta_1, \dots, \eta_{n_1}), \quad (23)$$

with $\eta_i \in \mathbb{N}$, $i \in \mathcal{N}_1$, representing the total number of edges connecting each node i in $\mathcal{N}_1$ with nodes in $\mathcal{N}_2$. Since the smallest eigenvalue of $\mathcal{L}(\mathcal{G}_{\mathcal{N}_1})$ is always non-negative and $\eta_i \geq c_1$ applies for all $i \in \mathcal{N}_1$, the smallest eigenvalue of $\mathcal{L}_{11}$ cannot be smaller than c_1 , and similar reasoning applies for $\mathcal{L}_{22}$. ■

Lemma 6: For the matrix $\mathcal{L}_{12}$ in (19), let $c_3 \in \mathbb{N}$ denote the total number of edges connecting the two sets $\mathcal{N}_1$ and $\mathcal{N}_2$. Then, the inequality:

$$\mathcal{L}_{12}^T \mathcal{L}_{12} \leq c_3 I_{n_2} \quad (24)$$

always holds.

Proof: Since $\mathcal{L}_{12}$ belongs to the off-diagonal block of $\mathcal{L}(\mathcal{G})$, the entries of $\mathcal{L}_{12}$ can only be -1 (if a corresponding edge connecting $\mathcal{N}_1$ and $\mathcal{N}_2$ exists) or zero. Let $\mathcal{L}_{12,[i]}$ denote the i -th column of $\mathcal{L}_{12}$. The trace of $\mathcal{L}_{12}^T \mathcal{L}_{12}$ equals to $\sum_{i=1}^{n_2} \mathcal{L}_{12,[i]}^T \mathcal{L}_{12,[i]}$, and is thus also equal to c_3 . As all eigenvalues of $\mathcal{L}_{12}^T \mathcal{L}_{12}$ are real-valued and non-negative, the largest eigenvalue can thus not be larger than c_3 , leading to the inequality (24). ■

Based on the Lemmas 4 to 6, the following theorem establishes a condition for the feedback gains q_i , $i \in \mathcal{N}_1$, implying that the network is globally stabilized to the origin.

Theorem 3: Given a network $\mathcal{G}$ and the coupling gain k , let γ be defined as in (14), as well as the constants c_1 , c_2 , and c_3 as in the Lemmas 5 and 6. If $\gamma - 2kc_2 < 0$ applies for the coupling gain k and the feedback control law (11) satisfies:

$$q_i > \sigma_i + 1.8a_i^2 b_i + (2k^2(c_1 c_2 - c_3) - kc_1 \gamma) / (\gamma - 2kc_2) \quad (25)$$

for all $i \in \mathcal{N}_1$, then the origin of the controlled system (10) is globally exponentially stable.

Proof: The first step of the proof is to show that the sum of the Jacobian matrix $\mathbf{J}(\mathbf{z})$ and its transpose is always negative definite. Note that the sum of $\mathbf{J}_{c,\mathcal{N}_1} - k\mathcal{L}_{11} \otimes I_2$ and its transpose satisfies:

$$\mathbf{J}_{c,\mathcal{N}_1}^T + \mathbf{J}_{c,\mathcal{N}_1} - k(\mathcal{L}_{11}^T + \mathcal{L}_{11}) \otimes I_2 \leq -(\beta + 2kc_1) \otimes I_{2n_1} \quad (26)$$

according to (15) and (21), and thus:

$$\left(\mathbf{J}_{c,\mathcal{N}_1}^T + \mathbf{J}_{c,\mathcal{N}_1} - k(\mathcal{L}_{11}^T + \mathcal{L}_{11}) \otimes I_2 \right)^{-1} \geq \frac{-1}{\beta + 2kc_1} \otimes I_{2n_1} \quad (27)$$

must hold. Likewise, the sum of $\mathbf{J}_{\mathcal{N}_2} - k\mathcal{L}_{22} \otimes I_2$ and its transpose satisfies:

$$\mathbf{J}_{\mathcal{N}_2}^T + \mathbf{J}_{\mathcal{N}_2} - k(\mathcal{L}_{22}^T + \mathcal{L}_{22}) \otimes I_2 \leq (\gamma - 2kc_2) \otimes I_{2n_2} \quad (28)$$

according to (14) and (22). Due to (27) and (24), the inequalities:

$$\begin{aligned} & (2k\mathcal{L}_{12} \otimes I_2)^T \left(\mathbf{J}_{c,\mathcal{N}_1}^T + \mathbf{J}_{c,\mathcal{N}_1} - k(\mathcal{L}_{11}^T + \mathcal{L}_{11}) \otimes I_2 \right)^{-1} \\ & (2k\mathcal{L}_{12} \otimes I_2) \\ & \geq \frac{-4k^2}{\beta + 2kc_1} \mathcal{L}_{12}^T \mathcal{L}_{12} \otimes I_2 \geq \frac{-4k^2 c_3}{\beta + 2kc_1} \otimes I_{2n_2} \end{aligned} \quad (29)$$

hold. According to (28) and (29), the inequalities:

$$\begin{aligned} & \mathbf{J}_{\mathcal{N}_2}^T + \mathbf{J}_{\mathcal{N}_2} - k(\mathcal{L}_{22}^T + \mathcal{L}_{22}) \otimes I_2 \\ & - (2k\mathcal{L}_{12} \otimes I_2)^T \left(\mathbf{J}_{c,\mathcal{N}_1}^T + \mathbf{J}_{c,\mathcal{N}_1} - k(\mathcal{L}_{11}^T + \mathcal{L}_{11}) \otimes I_2 \right)^{-1} \\ & (2k\mathcal{L}_{12} \otimes I_2) \\ & \leq \left(\gamma - 2kc_2 + \frac{4k^2 c_3}{\beta + 2kc_1} \right) \otimes I_{2n_2} < 0 \end{aligned} \quad (30)$$

hold if $\gamma - 2kc_2 < 0$ and:

$$\beta > (4k^2(c_1c_2 - c_3) - 2kc_1\gamma)/(\gamma - 2kc_2). \quad (31)$$

According to (18), the inequality (31) is satisfied if (25) holds for all q_i , $i \in \mathcal{N}_1$. Based on (26), (30) and the Schur complement, the sum of $\mathbf{J}(\mathbf{z})$ and its transpose must satisfy:

$$\mathbf{J}^T(\mathbf{z}) + \mathbf{J}(\mathbf{z}) < -\xi I_{2n} \quad (32)$$

for a $\xi > 0$, and for all $\mathbf{z} \in \mathbb{R}^{2n}$. Furthermore, it is known from Theorem 1 in [28], that the system (13) is *contractive* (i.e., the distance between trajectories starting from two arbitrary initial states exponentially decreases over time) if (32) holds. This fact, together with the fact that the solution of (13) is always bounded (which can be shown in a similar way to Lemma 2), imply that for any pair of different initial states $\mathbf{z}_1(0)$ and $\mathbf{z}_2(0)$ of (13), it holds $\forall t \geq 0$ that:

$$\|\mathbf{z}_1(t) - \mathbf{z}_2(t)\| \leq e^{-\xi t} \|\mathbf{z}_1(0) - \mathbf{z}_2(0)\|. \quad (33)$$

Since $\mathbf{z}(t) = 0$ is a solution of (13) for the initial state $\mathbf{z}(0) = 0$, the solution for any other initial state in $\mathbb{R}^{2n}$ converges exponentially to the origin $\mathbf{z}(t) = 0$ according to (33). ■

Note that the condition (25) can also be used to stabilize the network with $\sigma_i > 0$, i.e., if the equilibrium in the origin is unstable for the local oscillators.

Remark 2: While only undirected graphs have been considered, our obtained results can be easily generalized to directed graphs. If the graph $\mathcal{G}$ is directed, two inequalities similar to (21) and (22) can be identified for $\mathcal{L}_{11}$ and $\mathcal{L}_{22}$, since $\mathcal{L}_{11}^T + \mathcal{L}_{11}$ and $\mathcal{L}_{22}^T + \mathcal{L}_{22}$ remain symmetric for directed graphs. The inequality (24), on the other hand, can be replaced by an inequality involving $(\mathcal{L}_{12} + \mathcal{L}_{21})^T(\mathcal{L}_{12} + \mathcal{L}_{21})$.

Remark 3: If the feedback gains q_i , $i \in \mathcal{N}_1$, are fixed, then for any coupling gain k satisfying:

$$\gamma\beta + 2k(c_1\gamma - c_2\beta) - 4k^2(c_1c_2 - c_3) < 0, \quad (34)$$

the origin of the controlled system (10) is *globally exponentially stable*. This is a direct result from (30), when either by increasing β , or by selecting new systems in $\mathcal{N}_1$ and $\mathcal{N}_2$ to modify the values of γ , c_1 , c_2 and c_3 leads to an expansion of the admissible range of k in (34). A larger admissible range of k enhances the robustness of the stabilization, what is desirable since it is difficult to ensure in practice that all edges are weighted by exactly the same gain.

Remark 4: Note that the feedback controllers designed in this letter generate aperiodic, continuous inputs. In contrast,

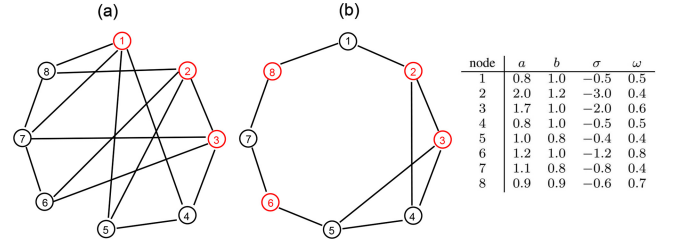


Fig. 1. Eight coupled oscillators with two different topologies. The nodes marked in red are those contained in $\mathcal{N}_1$ while the others belong to $\mathcal{N}_2$.

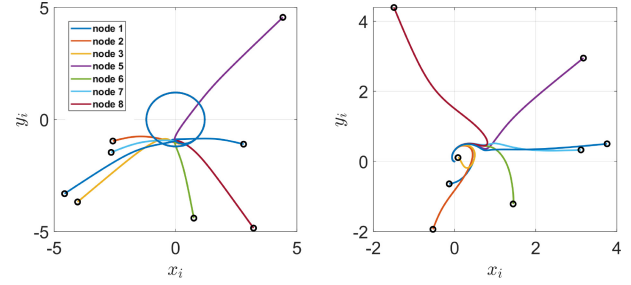


Fig. 2. For $k = 2.73$, the local states converge to the origin for the initialization on the right, while to the limit cycle $\mathcal{C}_3$ for another initialization on the left (i.e., bistability of the synchronized network).

brain stimulation in practice often involves pulse-like signals (e.g., in deep brain stimulation) or continuous, periodic inputs (e.g., in transcranial direct current stimulation). Investigating the medical applicability of stabilizing controls obtained from the proposed schemes is an important aspect of future research.

V. NUMERICAL EXAMPLES

To examine the previous results, consider a set of $n = 8$ homogeneous oscillators of type (1), connected by the graph shown in Fig. 1(a). The parameters of (1) are specified to $a = b = 1$, $\sigma = -0.8$, $\omega = 0.4$. By employing a coupling gain of $k = 2.73$, which satisfies the condition (7), the coupled local dynamics are guaranteed to achieve synchronization for any initial state. However, as illustrated in Fig. 2, the manifold of synchronization depends on the initial states. In the second test, still referring to the graph in Fig. 1(a), the local oscillators are chosen to be heterogeneous with the parameters listed on the right side of Fig. 1. In the absence of control and if $k = 0$, the local states converge to their respective limit cycles (or to the origin), as illustrated in Fig. 3(a).

In case the oscillators are only coupled by a gain $k = 2$, as shown in Fig. 3(c) and (d), they keep oscillating but the limit cycles are different to their local ones, or they converge to the origin, depending on the initial states. Now, assume that nodes 1 through 3 are controlled by (11) with $q_1 = 12$, $q_2 = 16$ and $q_3 = 15$, and let the condition (25) be satisfied with the coupling gain $k = 2$, $\gamma = 2.79$ in Lemma 4, as well as $c_1 = 3$, $c_2 = 2$, and $c_3 = 10$ for the graph under consideration. Then, the local states are stabilized to the origin, as illustrated in Fig. 3(b). According to (34), one can also determine the admissible range $k \in [1.26, 2.87]$ of the coupling gain to achieve global stabilization in this case.

In another test involving the same heterogeneous nodes as before, but now connected by the graph shown in Fig. 1(b), the nodes 2, 3, 6, and 8 are controlled according to (11) with $q_2 = 15$, $q_3 = 13$, $q_6 = 11$ and $q_8 = 10$. With again a coupling gain of $k = 2$ but with the new values of $\gamma = 2.09$, $c_1 = c_2 = 2$, and $c_3 = 8$ resulting from the new graph and the set of

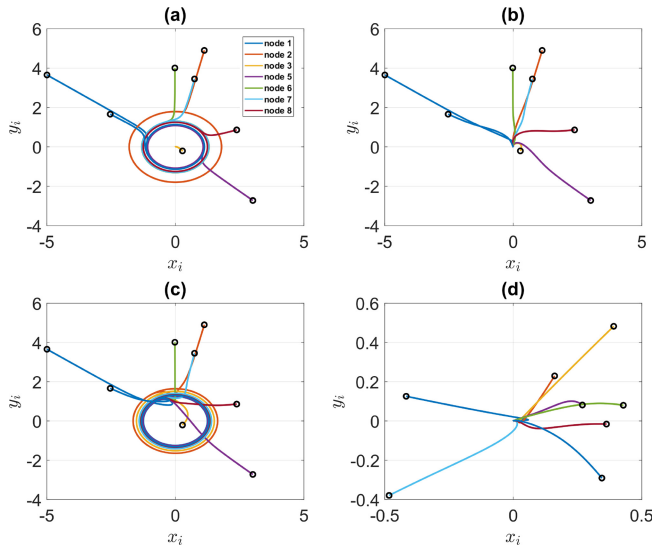


Fig. 3. (a) The oscillators converge to local limit cycles or to the origin when no coupling and control are applied. (b) For a coupling gain $k = 2$ and a control law satisfying (25), all local states are stabilized to the origin. (c) and (d) Different behaviors in the limit are observed for distinct initial states if the oscillators are only coupled by a gain $k = 2$, but no control is present.

nodes to be controlled, the inequality (25) is again satisfied, while the admissible range of k in (34) is expanded to $k \in [0.73, 3.38]$ due to the smaller values of c_3 and γ .

VI. CONCLUSION

In this letter, sufficient conditions to achieve synchronization for networks of bistable neural oscillators are investigated, offering new insights into the emergence and spread of epileptic seizures. In addition, in the case that the local oscillators are heterogeneous, a method following the concept of pinning control has been proposed to stabilize the network at the origin. Conditions were derived that guarantee stabilization by controlling only a portion of the oscillators through external signals. In the context of epilepsy treatment, this finding suggests that overall brain activity can be stabilized by targeting specific brain areas, thereby reducing the side effects associated with treatment. Future work will focus on incorporating more topological properties to enhance the effectiveness of the approach, and on extending the results to nonlinear coupling functions. Furthermore, we aim to develop control strategies inspired by brain stimulation paradigms, capable of delivering stimulation-like inputs (e.g., vibrational signals [29]) to better align with physiologically relevant intervention methods [30].

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