arXiv:2507.16107v1 [stat.ME] 21 Jul 2025

Recursive Equations For Imputation Of Missing Not At Random Data With Sparse Pattern Support

Trung Phung TPHUNG1@JHU.EDU
Department of Computer Science
Johns Hopkins University
Baltimore, MD 21218, USA

Kyle Reese KREESE15@JHU.EDU
Department of Computer Science
Johns Hopkins University
Baltimore, MD 21218, USA

Ilya Shpitser ILYAS@CS.JHU.EDU
Department of Computer Science
Johns Hopkins University
Baltimore, MD 21218, USA

Rohit Bhattacharya RB17@WILLIAMS.EDU
Department of Computer Science
Williams College
Williamstown, MA 01267, USA

Abstract

A common approach for handling missing values in data analysis pipelines is multiple imputation via software packages such as MICE (Van Buuren and Groothuis-Oudshoorn, 2011) and Amelia (Honaker et al., 2011). These packages typically assume the data are missing at random (MAR), and impose parametric or smoothing assumptions upon the imputing distributions in a way that allows imputation to proceed even if not all missingness patterns have support in the data. Such assumptions are unrealistic in practice, and induce model misspecification bias on any analysis performed after such imputation.

In this paper, we provide a principled alternative. Specifically, we develop a new characterization for the full data law in graphical models of missing data. This characterization is constructive, is easily adapted for the calculation of imputation distributions for both MAR and MNAR (missing not at random) mechanisms, and is able to handle lack of support for certain patterns of missingness. We use this characterization to develop a new imputation algorithm—Multivariate Imputation via Supported Pattern Recursion (MISPR)—which uses Gibbs sampling, by analogy with the Multivariate Imputation with Chained Equations (MICE) algorithm, but which is consistent under both MAR and MNAR settings, and is able to handle missing data patterns with no support without imposing additional assumptions beyond those already imposed by the missing data model itself.

In simulations, we show MISPR obtains comparable results to MICE when data are MAR, and superior, less biased results when data are MNAR. Our characterization and imputation algorithm based on it are a step towards making principled missing data methods more practical in applied settings, where the data are likely both MNAR and sufficiently high dimensional to yield missing data patterns with no support at available sample sizes.

Keywords: imputation, graphical models, pattern mixture factorization, Gibbs sampling

©2025 Trung Phung, Kyle Reese, Ilya Shpitser, Rohit Bhattacharya.

License: CC-BY 4.0, see https://creativecommons.org/licenses/by/4.0/.

1 Introduction

Missing data is a pervasive problem in applied data analyses across a variety of disciplines in the empirical and social sciences (Pigott, 2001; McKnight et al., 2007; Eekhout et al., 2012; Berchtold, 2019). Currently, imputation using software packages such as MICE (Van Buuren and Groothuis-Oudshoorn, 2011) and Amelia (Honaker et al., 2011) is one of the most popular approaches for analyzing datasets that exhibit missingness (Sinharay et al., 2001; Azur et al., 2011; Sun et al., 2023). However, these methods assume the data are missing at random (MAR), and impose parametric or smoothing assumptions upon the imputing distributions in a way that allows imputation to proceed even if not all missingness patterns have support in the data. Such assumptions are unrealistic in practice, and induce model misspecification bias on any analysis performed after such imputation.

A promising alternative approach to missing data modeling in complex multivariate settings is based on graphical models. This approach provides a number of advantages for analysis, including a concise representation of the observed data likelihood, an intuitive causal interpretation of missingness mechanisms, as well as developed methods for identification theory (Robins and Gill, 1997; Daniel et al., 2012; Bhattacharya et al., 2019; Mohan and Pearl, 2021; Malinsky et al., 2021; Nabi et al., 2020, 2022; Miao et al., 2024). Finally, graphical models can easily incorporate methods that address model uncertainty, either based on learning a graph using model selection procedures (Gain and Shpitser, 2018; Tu et al., 2019), or developing identification criteria that are sound in a potentially large union model of missingness processes (Mathur et al., 2025).

Despite these advantages and a growing recognition of limitations of existing imputation based approaches (Sterne et al., 2009; Perkins et al., 2018; Mathur and Shpitser, 2025), the use of graphical modeling methods for missing data in practical analyses has been limited. This is because of additional methodological complexity of such methods: functionals identified under MNAR tend to have a complex form, and lead to semiparametric or weighting estimators that are challenging to implement. Finally, the complexity of MNAR models of missingness makes it challenging to address missingness patterns with no support, a common complication in high dimensional applications with missing data. All these complications mean that existing imputation algorithms are far more likely to be incorporated into an analysis pipeline, due to their ease of implementation and use, and despite their known methodological disadvantages.

In this paper, we address these limitations of graphical models of missing data, and in doing so, provide a principled alternative to existing imputation methods. Specifically, we develop a new characterization for the full data law in graphical models of missing data. This characterization is constructive, is easily adapted for the calculation of imputation distributions for both MAR and MNAR (missing not at random) mechanisms, and is able to handle lack of support for certain patterns of missingness. We use this characterization to develop a new imputation algorithm—Pattern mixture Imputaton with Gibbs Sampling (MISPR)—which uses Gibbs sampling, by analogy with the Multivariate Imputation with Chained Equations (MICE) algorithm, but which is consistent under both MAR and MNAR settings, and is able to handle missing data patterns with no support without imposing additional assumptions beyond those already imposed by the missing data model itself. More specifically, we show how identification and imputation is possible by leveraging non-

parametric restrictions encoded by a missing data graph when only the complete case pattern and small subsets of other missingness patterns are present. For completeness, we also provide an example of how to conduct these tasks without any complete case patterns.

In simulations, we show MISPR obtains comparable results to MICE when the data are MAR, and superior, less biased results when the data are MNAR. Our characterization and imputation algorithm based on it are a step towards making principled missing data methods more practical in applied settings, where the data are likely both MNAR and sufficiently high dimensional to yield missing data patterns with no support at available sample sizes.

The remainder of this paper is organized as follows. In Section 2, we introduce notation, formalize our problem statement, and review the classes of missing data graphs considered in this paper. Section 3 presents PM-ID, our new constructive characterization of the full data law of these models and contrasts it with the identification approaches used in prior work (in particular, Nabi et al. (2020) and Malinsky et al. (2021)) on graphical models that are not constructive. Section 4 extends our constructive characterization to settings with positivity violations, i.e., settings where certain patterns of missingness have no support. Section 5 describes our imputation algorithm based on this constructive characterization. Section 6 provides numerical comparisons to MICE. Section 7 provides detailed qualitative comparisons of our work to other work on MNAR missing data models, and finally, Section 8 concludes the paper.

2 Problem Setup and Description Of Models

In this section, we formalize our problem statement, introduce relevant concepts related to graphical models of missing data, and provide basic intuition on how they are used.

2.1 Missing Data Models

Consider a missing data problem with J fully observed random variables $O = \{O_1, \ldots, O_J\}$, whose values are always observed in every data sample, and K potentially missing random variables $X^{(1)} = \{X_1^{(1)}, \ldots, X_K^{(1)}\}$, whose values are missing in at least one of the sample. To describe the data generation process with missingness, we use two additional sets of observed random variables: the missingness indicators $R = \{R_1, \ldots, R_K\}$ and the observed proxies $X = \{X_1, \ldots, X_K\}$. The $X^{(1)}$ are considered hidden, while the observed data sample consists of values of (O, X, R), determined via the missing data analogue of *consistency* in causal inference: $X_k = X_k^{(1)}$ if $R_k = 1$ and $X_k =?$ if $R_k = 0$. Each $O_j \in O$ takes values in $\mathcal{O}_j$, each $X_k^{(1)} \in X^{(1)}$ takes values in $\mathcal{X}_k^{(1)}$, and these state spaces are unrestricted. By definition, the state space of each missingness indicator $R_k \in R$ is $\{0, 1\}$, and the state space $\mathcal{X}_k$ of each observed proxy $X_k \in X$ is the state space of the corresponding potentially missing random variable augmented with the special value "?" to indicate missingness, $\mathcal{X}_k = \mathcal{X}_k^{(1)} \cup \{?\}$.

A missing data model is a set of distributions defined over $X^{(1)} \cup O \cup R \cup X$ and satisfying missing data consistency. That is, the joint distribution $p(X^{(1)}, O, R, X)$ may be written as

$$p(X^{(1)}, O, R, X) = p(X^{(1)}, O, R) \times \prod_{k=1}^{K} p(X_k \mid R_k, X_k^{(1)}), \tag{1}$$

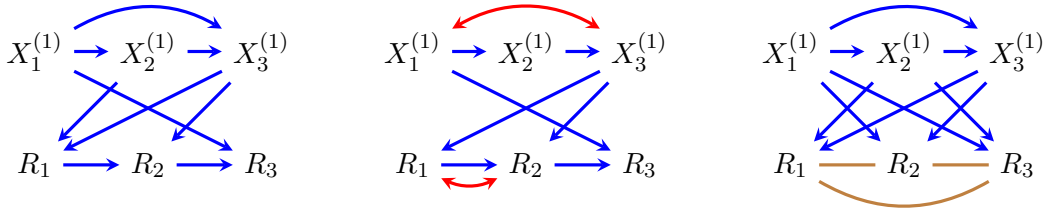


Figure 1: Different kinds of missing data graphs: (a) A missing data DAG; (b) A missing data ADMG; (c) A missing data CG. Though each graph has different causal implications, all of them imply identification of the full data law as they encode the following set of independences for all $i \in \{1, 2, 3\}$: $X_i^{(1)} \perp\!\!\!\perp R_i \mid X_{-i}^{(1)}, R_{-i}$.

where each factor $p(X_k \mid R_k, X_k^{(1)})$ is deterministically defined by consistency. Following previous conventions (Bhattacharya et al., 2019; Nabi et al., 2022), the non-deterministic portion $p(X^{(1)}, O, R)$ is often referred to as the *full law*, while the fully observed portion $p(O, X, R)$ is referred to as the *observed law*. The full law can be further factorized as $p(X^{(1)}, O, R) = p(X^{(1)}, O) \times p(R \mid X^{(1)}, O)$, where $p(X^{(1)}, O)$ is known as the *target law* and $p(R \mid X^{(1)}, O)$ is known as the *missingness mechanism* or *propensity score*. Since X are defined via fixed functions from R and $X^{(1)}$, knowing the full law is equivalent to knowing the entire joint distribution $p(X^{(1)}, O, R, X)$.

For brevity, we sometimes use $[K] = \{1, \ldots, K\}$ as a shorthand for the set of indices of all missing variables. Further, if $Z \subseteq [K]$ is an index subset, we use the notation $R_Z, X_Z^{(1)}, X_Z$ to denote the corresponding subset of variables, e.g., $R_Z = \{R_k \mid k \in Z\}$. Finally, for simplicity, we take statements such as $p(R = 1 \mid X^{(1)}, O) > 0$ where no specific values are specified for some of the random variables to mean that the statement holds for all possible values of those random variables (e.g., $X^{(1)}, O$ in this case).

Goal The goal of our work is to design identification and imputation algorithms for Missing Not At Random (MNAR) models—missing data models in which the missingness status of variables may depend on values that are themselves missing—that may also exhibit positivity violations. We pursue imputation as a goal, because it is often helpful as a pre-processing step in missing data problems as it allows downstream analyses to proceed as if there had been no missingness. Most often, parameters of interest are functions of the target law $p(X^{(1)}, O)$, but occasionally they can be functions of the entire full law $p(X^{(1)}, O, R)$, for example in causal discovery applications (Gain and Shpitser, 2018; Tu et al., 2019; Liu and Constantinou, 2022). Note also that the target law is identified—i.e., can be expressed as a function of the observed law $p(O, X, R)$—if the full law is identified. Thus, in this work, we consider full law identification, and propose an imputation method for a wide class of MNAR models with full laws that factorize according to a graphical model.

2.2 Graphical Models of Missing Data

A graph $\mathcal{G}(V, E)$ consists of a set of vertices V and edges E. Similar to causal inference, much progress has been made in non-parametric missing data identification theory by representing

missing random variables and their missingness indicators as vertices on a graph and using edges to encode both substantive mechanistic explanations of missingness and conditional independence relations between the missing variables and their missingness indicators. In particular, these conditional independences arise due to the absence of edges in the graph.

Thus far, three classes of graphs have been primarily used in the missing data literature: missing data DAGs (m-DAGs), missing data ADMGs (m-ADMGs), and missing data chain graphs (m-CGs), collectively referred to as *m-graphs*. We describe these three classes here, and note that the identification and imputation methods described in following sections can be applied to datasets that come from distributions that factorize (or equivalently, satisfy the Markov property) according to any of these three graph classes.

Missing Data DAGs

The most commonly used graphical models are those associated with directed acyclic graphs (DAGs). A DAG $\mathcal{G}(V, E)$ is a graph where the set of edges E are restricted to be directed ($\rightarrow$) and no directed cycles (a series of edges that point in the same direction and have the same vertex as the start and end points, i.e., $V_i \rightarrow \cdots \rightarrow V_i$) are allowed.

In a missing data DAG, which we shorten to m-DAG following the convention in Mohan and Pearl (2021), the vertices V can be partitioned into the missing variables $X^{(1)}$, fully observed variables O, missingness indicators R, and observed proxies X. That is, for an m-DAG $\mathcal{G}(V, E)$, we have $V = X^{(1)} \cup O \cup R \cup X$. In addition to acyclicity, m-DAGs have two more restrictions, (i) Missingness indicators cannot have outgoing edges to variables in $X^{(1)}$ or O, and (ii) The only edges incident to each observed proxy X_i are $X_i^{(1)} \rightarrow X_i \leftarrow R_i$, which is used to enforce the consistency assumption of missing data models.

Figure 1(a) shows an example of a valid m-DAG with three missing variables and no fully observed variables. For brevity, we omit the portions of the graph that relate to the observed proxies, since they correspond to fully deterministic portions of the models and are not relevant for describing our identification and imputation algorithms.

The primary advantage of graphs is that they provide a concise picture of independence assumptions. Let $\text{pa}_{\mathcal{G}}(V_i)$ denote the parents of a vertex V_i in $\mathcal{G}$, i.e., $\text{pa}_{\mathcal{G}}(V_i) = \{V_j \mid V_j \rightarrow V_i \text{ in } \mathcal{G}\}$. The statistical model of an m-DAG $\mathcal{G}(V, E)$ is the set of distributions $p(V)$ that factorize as $p(V) = \prod_{i=1}^{|V|} p(V_i \mid \text{pa}_{\mathcal{G}}(V_i))$, with each conditional factor of an observed proxy $p(X_k \mid R_k, X_k^{(1)})$ appearing in this factorization being deterministic as before.

The factorization implies a set of independences that can be read off an m-DAG using d-separation (Pearl, 1988). More directly relevant for our work, however, is a smaller set of independences that is similar to the set used to describe the local Markov property of DAG models (Pearl, 1988; Lauritzen, 1996). The local Markov property provides an equivalent (often simpler) description of these models. The independences are described using Markov blankets — the Markov blanket of a variable V_i is the smallest set of variables in V that makes V_i conditionally independent of all other variables in the graph. In DAGs, the Markov blanket of a variable V_i is known to be the parents of V_i, the children of V_i, and the co-parents of the children of V_i (Pearl, 1988). That is, given a DAG $\mathcal{G}$, the Markov blanket can be described in terms of the sets defined in Table 1.

Set name	**Notation**	**Definition**
Parents, Children	$\text{pa}_{\mathcal{G}}(V_i), \text{ch}_{\mathcal{G}}(V_i)$	$\{V_j \mid V_j \to V_i \text{ in } \mathcal{G}\}, \{V_j \mid V_i \to V_j \text{ in } \mathcal{G}\}$
Co-parents	$\text{co-pa}_{\mathcal{G}}(V_i)$	$\{V_j \mid V_j \to V_k \leftarrow V_i \text{ in } \mathcal{G}\}$
Markov blanket	$\text{mb}_{\mathcal{G}}(V_i)$	$\text{pa}_{\mathcal{G}}(V_i) \cup \text{ch}_{\mathcal{G}}(V_i) \cup \text{co-pa}_{\mathcal{G}}(V_i)$

Table 1: Definition of a Markov blanket in DAGs.

This gives us the following set of local independences for every variable $V_i \in V$

$$V_i \perp\!\!\!\perp V \setminus \text{mb}_{\mathcal{G}}(V_i) \mid \text{mb}_{\mathcal{G}}(V_i). \tag{2}$$

As an example, if we apply the above property to the missing variables of the m-DAG model in Figure 1(a), we get the following independences:

$$X_1^{(1)} \perp\!\!\!\perp R_1 \mid X_2^{(1)}, X_3^{(1)}, R_2, R_3$$
$$X_2^{(1)} \perp\!\!\!\perp R_2, R_3 \mid X_1^{(1)}, X_3^{(1)}, R_1$$
$$X_3^{(1)} \perp\!\!\!\perp R_3 \mid X_1^{(1)}, X_2^{(1)}, R_1, R_2$$

In later sections we see how these independences help with identification and imputation.

We end this subsection with a note on causal interpretations of m-DAGs. Interpreted causally, a directed edge $V_i \to V_j$ in $\mathcal{G}$ indicates that V_i is potentially a direct cause of V_j relative to other variables on the graph (Spirtes et al., 2000; Pearl, 2009). This makes the MNAR nature of certain models fairly clear. For example, in Figure 1(a), the edge $X_1^{(1)} \to R_3$ clearly indicates that the missing values of one variable can cause the missingness of another. m-DAGs have appeared in numerous works, such as Mohan et al. (2013); Saadati and Tian (2019); Bhattacharya et al. (2019); Nabi et al. (2020), and Chen et al. (2023).

Missing Data ADMGs

Missing data acyclic directed mixed graphs (m-ADMGs) are a useful extension of m-DAGs when some variables are not just missing but completely unobserved. Consider an m-DAG with vertices that can be partitioned into a set $V = X^{(1)} \cup O \cup R \cup X$ consisting of variables that are either fully observed or missing, and a set H consisting of variables that are always unobserved or hidden. Given such an m-DAG $\mathcal{G}(V \cup H, E)$, an m-ADMG $\mathcal{G}(V, E')$ over the partially observed vertices is a mixed graph consisting of directed ($\to$) and bidirected ($\leftrightarrow$) edges obtained by applying the latent projection operator described in Verma and Pearl (1990). Latent projection of an m-DAG onto an m-ADMG maintains the acylicity property as well as the two additional restrictions of m-DAGs: No directed edges of the form $R_i \to X_j^{(1)}$ or $R_i \to O_j$ (though bidirected edges between these variables are permitted in m-ADMGs), and each observed proxy X_k has only two parents $X_k^{(1)}$ and R_k. From a causal perspective, a directed edge $V_i \to V_j$ in an m-ADMG maintains the same interpretation as before; a bidirected edge $V_i \leftrightarrow V_j$ can be interpreted (wlog) as the existence of one or more

Set name	**Notation**	**Definition**
District	$\text{dis}_{\mathcal{G}}(V_i)$	$\{V_j \mid V_j \text{ has a bidirected path to } V_i \text{ in } \mathcal{G}\} \cup \{V_i\}$
Boundary	$\text{bd}_{\mathcal{G}}(V_i)$	$\text{pa}_{\mathcal{G}}(V_i) \cup \text{dis}_{\mathcal{G}}(V_i) \cup \left(\bigcup_{D_i \in \text{dis}_{\mathcal{G}}(V_i)} \text{pa}_{\mathcal{G}}(D_i) \right)$
Markov blanket	$\text{mb}_{\mathcal{G}}(V_i)$	$\text{bd}_{\mathcal{G}}(V_i) \cup \text{ch}_{\mathcal{G}}(V_i) \cup \left(\bigcup_{C_i \in \text{ch}_{\mathcal{G}}(V_i)} \text{bd}_{\mathcal{G}}(C_i) \right) \setminus \{V_i\}$

Table 2: Definition of a Markov blanket in ADMGs.

unobserved confounders $V_i \leftarrow H_k \rightarrow V_j$ between the two observed or partially observed variables (Evans, 2018). An example of an m-ADMG is shown in Figure 1(b).

One of the main benefits of m-ADMGs is that they enable modeling of the (partially) observed margin $p(V)$ without explicitly considering parameterizations of latent variable models. The margin $p(V)$ satisfies a recursive factorization with respect to $\mathcal{G}$, known as the nested Markov factorization (Richardson et al., 2023), that is significantly more complex than the DAG factorization. For the purpose of full law identification and imputation, however, it is again sufficient to focus on a small set of independences similar to an ordinary local Markov property for ADMGs. The extension of Markov blankets from DAGs to ADMGs is shown in Table 2.[1] Given an m-ADMG $\mathcal{G}(V, E)$, this modified definition of the Markov blanket yields a set of independences with the same form as those in (2) for each partially observed variable $V_i \in V$.

Missing Data CGs

A chain graph (CG) $\mathcal{G}(V, E)$ is a mixed graph consisting of directed ($\rightarrow$) and undirected ($-$) edges such that there are no partially directed cycles (a series of edges from V_i back to itself such that at least one directed edge is present and every edge is either undirected or a directed edge pointing away from V_i.) Chain graphs have been useful in modeling interference and dependent data problems (Ogburn et al., 2020; Peña, 2020; Bhattacharya et al., 2020; Tchetgen Tchetgen et al., 2021). From a causal perspective, directed edges in a CG maintain the same interpretation as before, while an undirected edge $V_i - V_j$ can be interpreted as a potential causal relationship involving feedback between the variables V_i and V_j (Lauritzen, 1996).

Missing data CGs (m-CGs) have become an important tool for analyzing the limits of non-parametric identification in MNAR graphical models. In particular, Shpitser (2016) and Malinsky et al. (2021) describe an m-CG model[2], known as the *no self-censoring model*, where each missing variable $X_i^{(1)}$ is permitted to cause the missingness of any other variable except itself, and all missingness indicators are pairwise connected via undirected edges. That is, the only contraints arising in the missingness mechanism of the model are due to the absence of self-censoring edges $X_i^{(1)} \rightarrow R_i$. Soon after, Nabi et al. (2020) showed that all m-DAGs and m-ADMGs with K missing variables and identified full data laws are in

1. The definition of boundary in Table 2 is the same as the definition of an *ordered* Markov blanket that was used to define an ordered local Markov property in Richardson (2003).
2. Sadinle and Reiter (2017) independently described the same model without an m-CG representation.

Set name	**Notation**	**Definition**
Neighbors	$\text{nb}_{\mathcal{G}}(V_i)$	$\{V_j \mid V_j - V_i \text{ in } \mathcal{G}\}$
Co-parents	$\text{co-pa}_{\mathcal{G}}(V_i)$	$\{V_j \mid V_j \to \circ \leftarrow V_i \text{ in } \mathcal{G}$ where $\circ$ is one or more vertices connected by an undirected path$\}$
Markov blanket	$\text{mb}_{\mathcal{G}}(V_i)$	$\text{pa}_{\mathcal{G}}(V_i) \cup \text{nb}_{\mathcal{G}}(V_i) \cup \text{ch}_{\mathcal{G}}(V_i) \cup \text{co-pa}_{\mathcal{G}}(V_i)$

Table 3: Definition of a Markov blanket in CGs.

fact statistical sub-models of a no self-censoring m-CG defined on as many variables, i.e., they imply the same or more independence constraints as the corresponding m-CG. A three variable no self-censoring model is shown in Figure 1(c). Though they have very different causal implications, the result in Nabi et al. (2020) implies that the m-DAG and m-ADMG in Figures 1(a, b) are statistical sub-models of the m-CG in Figure 1(c). In the following sections, we rely on this observation quite frequently to unify the imputation theory in our paper, so that it applies equally well to any missing data model that factorizes according to an m-DAG, m-ADMG, or m-CG with an identified full data law.

We end by noting relevant statistical properties of CGs. The statistical model of a CG $\mathcal{G}(V, E)$ is the set of distributions $p(V)$ that satisfy a two-level factorization involving undirected connected components of $\mathcal{G}$ (Lauritzen, 1996).[3] Once again, in our paper, we rely only on a local Markov property of m-CGs defined using an appropriately modified definition of the Markov blanket shown in Table 3. Given an m-CG $\mathcal{G}(V, E)$, applying this definition provides a list of constraints for each $V_i \in V$ of the form shown in (2).

3 New Constructive Identification Method For Strictly Positive Full Laws

In this section, we present a new identification method for the full law of missing data models based on m-graphs. To motivate the need for this new method, we briefly summarize the identification strategy and proof used in prior work by Nabi et al. (2020) and Malinsky et al. (2021). While this previous method is sound and complete for identification of the full law in m-DAGs and m-ADMGs[4], we show that it is not constructive, and thus difficult to use in practice. In contrast, we show that our method is constructive, and it remains sound and complete, allowing for the design of an imputation algorithm in Section 5. Further, this constructive argument is easily extended to scenarios with positivity violations, as we show in Section 4.

The key difference between the two methods lies in the factorizations used. Nabi et al. (2020) and Malinsky et al. (2021) use an odds ratio factorization (Chen, 2007) of the missingness mechanism, while we use a pattern mixture factorization (Little, 1993) of the entire full law. Henceforth, for brevity, we will refer to the former as the OR-ID (odds ratio iden-

3. A more general version of the factorization is provided in Shpitser (2023).
4. It has not been explicitly shown to be complete for m-CGs, but rather it was shown by Nabi et al. (2020) that all m-DAGs and m-ADMGs with identified full laws are sub-models of a no self-censoring m-CG model studied in Shpitser (2016), Sadinle and Reiter (2017), and Malinsky et al. (2021).

tification) method and the latter as the PM-ID (pattern mixture identification) method. As will be demonstrated, PM-ID allows both identification and imputation to be formulated in terms of extrapolation densities that can be derived explicitly in terms of the observed law.

Both OR-ID and the new PM-ID algorithm we present in this section require strict positivity of the full law. We relax this assumption in Section 4, where we discuss certain common positivity violations in missing data problems.

3.1 Graphical Missing Data Models with Identified Full Laws

As mentioned earlier, we focus on missing data models with identified full laws, and in this section, will focus on ones that have strictly positive full laws. More formally, consider a missing data model $p(X^{(1)}, O, R, X)$ that factorizes according to an m-graph $\mathcal{G}(V, E)$, where $V = X^{(1)} \cup O \cup R \cup X$. We make the following positivity assumptions that have been standard in prior work on missing data graphs. For some small constant $\epsilon > 0$, we have

(P1) Positivity of the missingness mechanism: $p(R \mid X^{(1)}, O) \geq \epsilon$.

(P2) Positivity of the target law: $p(X^{(1)}, O) \geq \epsilon$.

Note by our convention, these conditions hold for all values of $R, X^{(1)}, O$, and (P1, P2) together imply strict positivity of the full law $p(X^{(1)}, O, R)$.

From prior work in Bhattacharya et al. (2019), Nabi et al. (2020), and Malinsky et al. (2021), we know that under positivity assumptions (P1, P2), an m-graph $\mathcal{G}$ implies identification of the full law $p(X^{(1)}, O, R)$ if the following structural assumption[5] is satisfied,

(S1) No missingness indicator lies in the Markov blanket of the variable it indicates.

That is, for all $i \in [K]$, we have $R_i \notin \text{mb}_{\mathcal{G}}(X_i^{(1)})$,

where the definition of $\text{mb}_{\mathcal{G}}(R_i)$ is inferred from the appropriate Tables 1, 2, or 3 depending on whether $\mathcal{G}$ is an m-DAG, m-ADMG, or m-CG respectively.

For m-DAG and m-ADMG models satisfying strict positivity, Nabi et al. (2020) further show that the graphical criterion (S1) is sound and complete for identification of the full law: If $\mathcal{G}$ satisfies (S1), then the full law is provably identified; if $\mathcal{G}$ does not satisfy (S1), then the full law is provably *not* identified.

It is easy to confirm that all three m-graphs in Figure 1 satisfy (S1), and thus imply identification of the full law. Particularly, the following set of conditional independences can be derived as a direct consequence of (S1) and the local Markov properties of m-graphs described in the previous section: Since each missingness indicator does not appear in the Markov blanket of the corresponding variable it indicates,

$$\text{for all } i \in [K], \text{ we have } X_i^{(1)} \perp\!\!\!\perp R_i \mid X_{-i}^{(1)}, R_{-i}, O. \tag{3}$$

These independences play a key role in non-parametric identification of the full law in all three models. We demonstrate this first through a non-constructive identification method (OR-ID) from prior work, and then with a novel constructive identification method (PM-ID).

5. (S1) here is a succinct and equivalent description of previous graphical criteria posed in Nabi et al. (2020) phrased in terms of absence of self-censoring edges and *colluders* (colliders of the form $X_i^{(1)} \to R_j \leftarrow R_i$) for m-DAGs, and absence of self-censoring edges and *colluding paths* (paths between $X_i^{(1)}$ and R_i where every intermediate vertex is a collider $\to \circ \leftarrow$, $\leftrightarrow \circ \leftarrow$, or $\to \circ \leftrightarrow$) for m-ADMGs.

3.2 Non-constructive method: OR-ID (Nabi et al., 2020; Malinsky et al., 2021)

OR-ID uses an odds ratio factorization (Chen, 2007) of $p(R \mid X^{(1)}, O)$ to prove identifiability of the full law whenever (P1, P2, S1) hold. First, by Bayes rule the full law factorizes as,

$$p(X^{(1)}, O, R) = \frac{p(X^{(1)}, O, R = 1)}{p(R = 1 \mid O, X^{(1)})} \cdot p(R \mid X^{(1)}, O).$$

The numerator in the above fraction is a function of the observed law by consistency, $p(X^{(1)}, O, R = 1) = p(X, O, R = 1)$. Thus, identification of the full law can be reduced to identification of the missingness mechanism $p(R \mid X^{(1)}, O)$ for all possible values of $R, X^{(1)}, O$. For any arbitrary choice of reference values for random variables A and B, the odds ratio factorization of a joint conditional distribution $p(A, B \mid C)$ is (Chen, 2007),

$$p(A, B \mid C) = \frac{1}{Z(C)} \cdot p(A \mid B = b_0, C) \cdot p(B \mid A = a_0, C) \cdot \mathrm{OR}(A, B \mid C),$$

where the odds ratio function $\mathrm{OR}(A, B \mid C)$ is defined as,

$$\mathrm{OR}(A, B \mid C) = \frac{p(A \mid B, C)}{p(A = a_0 \mid B, C)} \cdot \frac{p(A = a_0 \mid B = b_0, C)}{p(A \mid B = b_0, C)},$$

and $Z(C) = \sum_{A,B} p(A \mid B = b_0, C) \cdot p(B \mid A = a_0, C) \cdot \mathrm{OR}(A, B \mid C)$ is a normalizing term.

When the odds ratio factorization is applied recursively to the missingness mechanism with reference values of 1 for each R_k, it gives us the following factorization of $p(R \mid X^{(1)}, O)$,

$$p(R \mid X^{(1)}, O) = \frac{1}{Z(X^{(1)}, O)} \cdot \prod_{k=1}^{K} p(R_k \mid R_{-k} = 1, X^{(1)}, O) \cdot \prod_{k=2}^{K} \mathrm{OR}(R_k, R_{\prec k} \mid R_{\succ k} = 1, X^{(1)}, O),$$

where $Z(X^{(1)}, O)$ is a normalizing term as before and the sets $R_{-k}, R_{\prec k}, R_{\succ k}$ are defined as $R \setminus \{R_k\}, \{R_1, \ldots, R_{k-1}\}$, and $\{R_{k+1}, \ldots, R_K\}$ respectively.

Nabi et al. (2020) and Malinsky et al. (2021) provide an argument for why each of the terms in this factorization are functions of observed data. Rather than describing the identification argument in full generality, we show a specific example of it applied to Figures 1(a, b, c), and demonstrate why it is not constructive. This makes OR-ID difficult to use in practical settings, as directly fitting the observed data likelihood of missing data models can be computationally intractable even for small problems (due to the normalizing function and the need to marginalize over the missing variables in each row of data). Malinsky et al. (2021) provide semiparametric estimators of the missingness mechanism, but these estimators are also not scalable and require fairly involved estimation subroutines that are not easily tailored for different missing data graphs. Moreover, the non-parametric influence function may not be efficient in the working m-graph model in cases where it is a submodel of the corresponding no self-censoring model studied by them.

3.2.1 An example of identification using OR-ID

Example 1 *We apply OR-ID to Figures 1(a, b, c) as an example of how it works and why it does not provide an explicit functional for estimation. The arguments below apply to all*

three m-graphs. The odds ratio factorization of the missingness mechanism $p(R \mid X^{(1)})$ *is,*

$$p(R \mid X^{(1)}) = \frac{1}{Z(X^{(1)})} \cdot p(R_1 \mid R_{23} = 1, X^{(1)}) p(R_2 \mid R_{13} = 1, X^{(1)}) p(R_3 \mid R_{12} = 1, X^{(1)}) \\ \times \mathrm{OR}(R_2, R_1 \mid R_3 = 1, X^{(1)}) \, \mathrm{OR}(R_{12}, R_3 \mid X^{(1)}).$$

Further decomposition of the term $\mathrm{OR}(R_{12}, R_3 \mid X^{(1)})$ *gives*

$$p(R \mid X^{(1)}) = \frac{1}{Z(X^{(1)})} \cdot p(R_1 \mid R_{23} = 1, X^{(1)}) p(R_2 \mid R_{13} = 1, X^{(1)}) p(R_3 \mid R_{12} = 1, X^{(1)}) \\ \times \mathrm{OR}(R_2, R_1 \mid R_3 = 1, X^{(1)}) \, \mathrm{OR}(R_2, R_3 \mid R_1 = 1, X^{(1)}) \, \mathrm{OR}(R_1, R_3 \mid R_2 = 1, X^{(1)}) \\ \times \frac{\mathrm{OR}(R_1, R_2 \mid R_3, X^{(1)})}{\mathrm{OR}(R_1, R_2 \mid R_3 = 1, X^{(1)})}.$$

First, we identify the univariate conditional factors. We have,

$$p(R_1 \mid R_{23} = 1, X^{(1)}) = p(R_1 \mid R_{23} = 1, X^{(1)}_{23}) = p(R_1 \mid R_{23} = 1, X_{23}),$$

where the first equality follows from the independence $R_1^{(1)} \perp\!\!\!\perp X_1 \mid X_{23}^{(1)}, R_{23}$ *appearing in* (3)*, and the second equality follows from consistency. A similar argument yields the identification of* $p(R_2 \mid R_{13} = 1, X^{(1)})$ *and* $p(R_3 \mid R_{12} = 1, X^{(1)})$*.*

Second, to identify the odds ratios, we use their symmetry property and write them in two ways. For example, consider $\mathrm{OR}(R_1, R_2 \mid R_3 = 1, X^{(1)})$ *written in two ways,*

$$\begin{aligned} \mathrm{OR}(R_1, R_2 \mid R_3 = 1, X^{(1)}) &= \frac{p(R_1 \mid R_2, R_3 = 1, X^{(1)})}{p(R_1 = 1 \mid R_2, R_3 = 1, X^{(1)})} \cdot \frac{p(R_1 = 1 \mid R_2 = 1, R_3 = 1, X^{(1)})}{p(R_1 \mid R_2 = 1, R_3 = 1, X^{(1)})} \\ &= \frac{p(R_2 \mid R_1, R_3 = 1, X^{(1)})}{p(R_2 = 1 \mid R_1, R_3 = 1, X^{(1)})} \cdot \frac{p(R_2 = 1 \mid R_1 = 1, R_3 = 1, X^{(1)})}{p(R_2 \mid R_1 = 1, R_3 = 1, X^{(1)})}. \end{aligned}$$

From the first equality we can infer that this odds ratio is not a function of $X_1^{(1)}$*, as* $R_1 \perp\!\!\!\perp X_1^{(1)} \mid X_{23}^{(1)}, R_{23}$ *due to* (3)*. From the second equality, we can infer that the odds ratio is not a function of* $X_2^{(1)}$*, as* $R_2 \perp\!\!\!\perp X_2^{(1)} \mid X_{23}^{(1)}, R_{23}$*. Together with the previous claim, this implies that the odds ratio depends only on the joint distribution* $p(R_1, R_2, X_3^{(1)}, R_3 = 1)$*, which is identified by consistency. Similar arguments lead to the identification of the other two odds ratios* $\mathrm{OR}(R_2, R_3 \mid R_1 = 1, X^{(1)})$ *and* $\mathrm{OR}(R_1, R_3 \mid R_2 = 1, X^{(1)})$*.*

Third, the 3-way interaction term can be written in three ways, due to symmetry,

$$\frac{\mathrm{OR}(R_1, R_2 \mid R_3, X^{(1)})}{\mathrm{OR}(R_1, R_2 \mid R_3 = 1, X^{(1)})} = \frac{\mathrm{OR}(R_2, R_3 \mid R_1, X^{(1)})}{\mathrm{OR}(R_2, R_3 \mid R_1 = 1, X^{(1)})} = \frac{\mathrm{OR}(R_1, R_3 \mid R_2, X^{(1)})}{\mathrm{OR}(R_1, R_3 \mid R_2 = 1, X^{(1)})}.$$

By the independences in (3)*, the term on the left does not depend on* $X_{12}^{(1)}$*, the middle term does not depend on* $X_{23}^{(1)}$*, and the right one does not depend on* $X_{13}^{(1)}$*. Thus, the interaction term does not depend on any of the missing variables* $X_{123}^{(1)}$*, and is identified.*

Finally, identification of all these terms implies identification of the normalization function $Z(X^{(1)})$*, and thus the propensity score* $p(R \mid X^{(1)})$ *and the full law* $p(X^{(1)}, R)$*.*

3.2.2 OR-ID is non-constructive

Although OR-ID establishes identification for all terms in the factorization, it does not provide explicit expressions for the odds ratios and the higher order interaction terms in terms of the observed law. For example, despite the fact that OR-ID establishes that $\mathrm{OR}(R_1, R_2 \mid R_3 = 1, X^{(1)})$ is a function of the observed joint distribution $p(R_1, R_2, R_3 = 1, X_3^{(1)}) = p(R_1, R_2, R_3 = 1, X_3)$, we have the following inequality:

$$\mathrm{OR}(R_1, R_2 \mid R_3 = 1, X^{(1)}) \neq \frac{p(R_1 \mid R_2, R_3 = 1, X_3^{(1)})}{p(R_1 = 1 \mid R_2, R_3 = 1, X_3^{(1)})} \cdot \frac{p(R_1 = 1 \mid R_2 = 1, R_3 = 1, X_3^{(1)})}{p(R_1 \mid R_2 = 1, R_3 = 1, X_3^{(1)})},$$

In other words, OR-ID establishes that this odds ratio can be expressed as some function of observed data $f(p(R_1, R_2, R_3 = 1, X_3))$, however, it does not tell us exactly what this function f is. Further, the above inequality demonstrates this function f could be quite complicated in general, as the odds ratios often do not "collapse" into simpler atomic expressions involving only observed data; see Didelez et al. (2010) and Didelez and Stensrud (2022) for some special cases where odds ratios are collapsible in the presence of selection bias. The same non-collapsibility issue is true for the other odds ratio and interaction terms appearing in the OR factorization. This motivates our new constructive identification method.

3.3 PM-ID: A constructive identification method

PM-ID is an identification method that uses a pattern mixture factorization (Little, 1993) of the full law. In what follows, we call an ordered tuple $r = (r_1, \ldots, r_K)$ a *missingness pattern*, and use $\mathcal{R} = \{0, 1\}^K$ to denote the set of all possible missingness patterns. For each missingness pattern $r \in \mathcal{R}$, the set $\mathbb{M}(r) := \{i \in [K] \mid r_i = 0\}$ contains the indices of all the missing variables that are unobserved in the pattern r, while the set $\mathbb{O}(r) = [K] \setminus \mathbb{M}(r)$ contains the indices of all the missing variables that are observed in the pattern.

For any missingness pattern $r \in \mathcal{R}$, the pattern mixture factorization (PM factorization) of the full law at r is given by,

$$\begin{aligned} p(X^{(1)}, O, R = r) &= p(X^{(1)}_{\mathbb{M}(r)} \mid X^{(1)}_{\mathbb{O}(r)}, O, R = r) \cdot p(X^{(1)}_{\mathbb{O}(r)} \mid O, R = r) \cdot p(O, R = r) \qquad \text{(chain rule)} \\ &= \underbrace{p(X^{(1)}_{\mathbb{M}(r)} \mid X_{\mathbb{O}(r)}, O, R = r)}_{\text{extrapolation density}} \cdot \underbrace{p(X_{\mathbb{O}(r)} \mid O, R = r)}_{\text{interpolation density}} \cdot p(O, R = r). \qquad \text{(consistency)} \end{aligned} \tag{4}$$

The PM factorization is well-defined if $p(X_{\mathbb{O}(r)}, O, R = r)$ is positive, which is guaranteed by positivity assumptions (P1, P2).

The term $p(X_{\mathbb{O}(r)} \mid O, R = r)$ in (4) is known as the *interpolation density* for pattern r, while the term $p(X^{(1)}_{\mathbb{M}(r)} \mid X^{(1)}_{\mathbb{O}(r)}, O, R = r)$ is known as the *extrapolation density*. Since the interpolation density always consists of variables that are observed in the given pattern, it is always identified by consistency. Identifiability of the extrapolation density, however, must be established based on model assumptions, and the full joint distribution $p(X^{(1)}, O, R = r)$ for a certain pattern r is identified if and only if the corresponding extrapolation density is identified. Thus, another way to establish identifiability of the full law $p(X^{(1)}, O, R)$ that is distinct from the strategy used by OR-ID, is to argue that the extrapolation density $p(X^{(1)}_{\mathbb{M}(r)} \mid X^{(1)}_{\mathbb{O}(r)}, O, R = r)$ is identified for all possible missingness patterns $r \in \mathcal{R}$.

This is the premise for the PM-ID algorithm proposed here. We also establish in Section 5 that knowing the extrapolation density for a certain missingness pattern r is sufficient for imputation of data rows matching the pattern r. Hence, the extrapolation density plays a key role in our identification and imputation strategies. The following example makes the intuition behind using the PM factorization for identification more concrete, and leads to a general algorithm in the next subsection.

3.3.1 An example of constructive identification with PM factorization

Example 2 *Consider the m-DAG in Figures 1(a, b, c). We compute explicit identifying functionals of $p(X^{(1)}, R = r)$ for every pattern $r \in \{0,1\}^3$ according to a partial order—starting with the complete case pattern $R = 111$, followed by any patterns where only one variable is unobserved (e.g., $R = 011$), then patterns where only two variables are unobserved (e.g, $R = 001$), and finally the pattern where no variable is observed $R = 000$.*

1. *The complete case pattern $R = 111$: By consistency, $p(X^{(1)}, R = 111) = p(X, R = 111)$.*

2. *Patterns with only one unobserved variable: Say we first process the pattern $R = 011$ (the ordering between these patterns is not important). The PM factorization gives us,*

$$p(X^{(1)}, R = 011) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 011) \cdot p(X_{23} \mid R = 011) \cdot p(R = 011).$$

The extrapolation density is identified as,

$$p(X_1^{(1)} \mid X_{23}^{(1)}, R = 011) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 111) = p(X_1 \mid X_{23}, R = 111),$$

where the first equality follows from $X_1^{(1)} \perp\!\!\!\perp R_1 \mid X_{23}^{(1)}, R_{23}$ and the second follows from consistency. The identification arguments for $R = 101$ and $R = 110$ are similar, using the independences $X_2^{(1)} \perp\!\!\!\perp R_2 \mid X_{13}^{(1)}, R_{13}$ and $X_3^{(1)} \perp\!\!\!\perp R_3 \mid X_{12}^{(1)}, R_{12}$ respectively. For example, for the pattern $R = 101$ we have,

$$\begin{aligned} p(X^{(1)}, R = 101) &= p(X_2^{(1)} \mid X_{13}^{(1)}, R = 101) \cdot p(X_{13} \mid R = 101) \cdot p(R = 101), \\ &= p(X_2 \mid X_{13}, R = 111) \cdot p(X_{13} \mid R = 101) \cdot p(R = 101). \end{aligned}$$

That is, identification of the joint distribution of patterns with one unobserved variable follows by re-expressing them as functions of the complete case pattern. In the next step of PM-ID, we express the patterns with two unobserved variables as functions of the (now identified patterns) with only one unobserved variable.

3. *Patterns with two unobserved variables: Consider the pattern $R = 001$ and its corresponding PM factorization,*

$$p(X^{(1)}, R = 001) = p(X_{12}^{(1)} \mid X_3^{(1)}, R = 001) \cdot p(X_3 \mid R = 001) \cdot p(R = 001).$$

A key to identifying such patterns is that the extrapolation density is identified if $p(X_1^{(1)} \mid X_{23}^{(1)}, R = 001)$ and $p(X_2^{(1)} \mid X_{13}^{(1)}, R = 001)$ are identified, as these conditional distributions are sufficient for Gibbs sampling from the joint distribution (Geman and Geman,

1984). Henceforth, we refer to such conditionals as Gibbs factors. We identify the Gibbs factors for this extrapolation density using different conditional independences as follows,

$$p(X_1^{(1)} \mid X_{23}^{(1)}, R = 001) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 101), \quad (X_1^{(1)} \perp\!\!\!\perp R_1 \mid X_{23}^{(1)}, R_{23})$$
$$p(X_2^{(1)} \mid X_{13}^{(1)}, R = 001) = p(X_2^{(1)} \mid X_{13}^{(1)}, R = 011). \quad (X_2^{(1)} \perp\!\!\!\perp R_2 \mid X_{13}^{(1)}, R_{13})$$

The conditional distributions on the right hand side of these equations are identified, as they are functions of the joint distributions $p(X^{(1)}, R = 101)$ and $p(X^{(1)}, R = 011)$ that were identified in the previous step.

We can repeat this argument for pattern $R = 010$: The extrapolation density $p(X_{13}^{(1)} \mid X_2^{(1)}, R = 010)$ also consists of two Gibbs factors identified as follows,

$$p(X_1^{(1)} \mid X_{23}^{(1)}, R = 010) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 110), \quad (X_1^{(1)} \perp\!\!\!\perp R_1 \mid X_{23}^{(1)}, R_{23})$$
$$p(X_3^{(1)} \mid X_{12}^{(1)}, R = 010) = p(X_3^{(1)} \mid X_{12}^{(1)}, R = 011), \quad (X_3^{(1)} \perp\!\!\!\perp R_3 \mid X_{12}^{(1)}, R_{12})$$

where the conditionals on the right hand side are again functions of joint distributions involving only one unobserved variable that were identified in the previous step.

Finally for pattern $R = 100$, the Gibbs factors for the extrapolation density $p(X_{23}^{(1)} \mid X_1^{(1)}, R = 100)$ can also be identified as,

$$p(X_2^{(1)} \mid X_{13}^{(1)}, R = 100) = p(X_2^{(1)} \mid X_{13}^{(1)}, R = 110), \quad (X_2^{(1)} \perp\!\!\!\perp R_2 \mid X_{13}^{(1)}, R_{13})$$
$$p(X_3^{(1)} \mid X_{12}^{(1)}, R = 100) = p(X_3^{(1)} \mid X_{12}^{(1)}, R = 101). \quad (X_3^{(1)} \perp\!\!\!\perp R_3 \mid X_{12}^{(1)}, R_{12})$$

4. *For the final pattern $R = 000$, the extrapolation density is $p(X^{(1)} \mid R = 000)$. This is identified because all three Gibbs factors can be expressed as functions of of $p(X^{(1)}, R = 100)$, $p(X^{(1)}, R = 010)$ and $p(X^{(1)}, R = 001)$ that were identified in the previous step.*

$$p(X_1^{(1)} \mid X_{23}^{(1)}, R = 000) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 100), \quad (X_1^{(1)} \perp\!\!\!\perp R_1 \mid X_{23}^{(1)}, R_{23})$$
$$p(X_2^{(1)} \mid X_{13}^{(1)}, R = 000) = p(X_2^{(1)} \mid X_{13}^{(1)}, R = 010), \quad (X_2^{(1)} \perp\!\!\!\perp R_2 \mid X_{13}^{(1)}, R_{13})$$
$$p(X_3^{(1)} \mid X_{12}^{(1)}, R = 000) = p(X_3^{(1)} \mid X_{12}^{(1)}, R = 001). \quad (X_3^{(1)} \perp\!\!\!\perp R_3 \mid X_{12}^{(1)}, R_{12})$$

Since $p(X^{(1)}, R = r)$ is identified for every $r \in \mathcal{R}$, the full law is identified.

3.3.2 The PM-ID Algorithm

The example in the previous subsection provides the following insights that we use to phrase a general identification algorithm based on the pattern mixture factorization:

1. The distribution $p(X^{(1)}, O, R = r)$ for a specific pattern $r \in \mathcal{R}$ is identified if and only if the corresponding extrapolation density $p(X_{\mathbb{M}(r)}^{(1)} \mid X_{\mathbb{O}(r)}^{(1)}, O, R = r)$ is identified.

2. To identify an extrapolation density $p(X_{\mathbb{M}(r)}^{(1)} \mid X_{\mathbb{O}(r)}^{(1)}, O, R = r)$, it is sufficient to identify the Gibbs factors $p(X_i^{(1)} \mid X_{-i}^{(1)}, O, R = r)$ for each index $i \in \mathbb{M}(r)$.

3. Identification of each Gibbs factor for a pattern r can be established by using one of the independences in (3) to show that it is equal to the Gibbs factors of another pattern $\widetilde{r}$ whose joint $p(X^{(1)}, O, R = \widetilde{r})$ is already known to be identified. That is, one way to set up a recursive identification procedure is to process patterns according to a partial order—starting with the complete case pattern—where $\widetilde{r} \prec r$ if identification of one of the Gibbs factors of the extrapolation density of r relies on identifying $p(X^{(1)}, O, R = \widetilde{r})$, and the patterns are incomparable otherwise.

These core ideas of the PM-ID algorithm—in particular the partial order on patterns and borrowing information from earlier patterns to identify individual Gibbs factors of the current pattern—can be encoded succinctly using a *pattern DAG* defined as follows.[6]

Definition 1 (Pattern DAG) *Given an m-graph $\mathcal{G}(V, E)$, a pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$ is a directed acyclic multigraph (i.e., a directed acyclic graph with the possibility of multiple edges between the same vertices) satisfying the following three properties: (i) The vertex set $\widetilde{V}$ is a subset of all possible missingness patterns, i.e., $\widetilde{V} \subseteq \mathcal{R}$. (ii) The edge set $\widetilde{E}$ consists of labeled edges of the form $r \to^i \widetilde{r}$, where $i \in [K]$ is an index and $r, \widetilde{r} \in \mathcal{R}$ are two distinct missingness patterns. (iii) If there exists an edge $r \to^i \widetilde{r}$ in the pattern DAG $\mathcal{H}$ then it must be the case that for any distribution p that factorizes according to the m-graph $\mathcal{G}$,*

$$p(X_i^{(1)} \mid X_{-i}^{(1)}, O, R = r) = p(X_i^{(1)} \mid X_{-i}^{(1)}, O, R = \widetilde{r}). \tag{5}$$

Property (iii) tries to link independences in the m-graph $\mathcal{G}$ to identification of Gibbs factors in the extrapolation density of $\widetilde{r}$. While the equality in (5) is symmetric, the asymmetry of the edge $r \to^i \widetilde{r}$ is important: The directed edges in $\mathcal{H}$ are used to form a partial order of identification ensuring the pattern r is identified before processing $\widetilde{r}$. This ordering then justifies borrowing information from $p(X^{(1)}, O, R = r)$ in order to identify the i^{th} Gibbs factor in the extrapolation density of $p(X^{(1)}, O, R = \widetilde{r})$ based on (5).

Given an m-graph $\mathcal{G}$, there are several ways to create a pattern DAG satisfying Definition 1. However, an arbitrary pattern DAG may not necessarily aid in identification using PM factorization. We consider a pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$ to be *PM-ID* compatible if,

(I1) The complete case pattern is present in $\mathcal{H}$ as a root node (has no incoming edges).

(I2) For every $r \in \widetilde{V}$ and every $i \in \mathbb{M}(r)$, there is at least one directed path from the complete case pattern to r such that the last edge on the path is of the form $\circ \to^i r$.

Figures 2(b) and (c) show examples of pattern DAGs that satisfy the properties listed in Definition 1 with respect to the m-graph in Figure 2(a). The pattern DAG in Figure 2(b) is also PM-ID compatible, as it satisfies (I1, I2). In contrast, the pattern DAG in Figure 2(c) does not satisfy (I2), as there is no directed path from the complete case pattern to the pattern 00 that ends with an edge labeled with the missing index 2.

Intuitively, pattern DAGs that are PM-ID compatible set up a recursive identification argument by linking Gibbs factors of each pattern to ones found in previously identified

6. Strictly speaking, per Definition 1, the term pattern DAG here refers to multigraphs and not just simple DAGs. The interpretation of the pattern DAGs we use here is also different than the pattern graphs introduced in Chen (2022). We elaborate on these differences in Section 7.

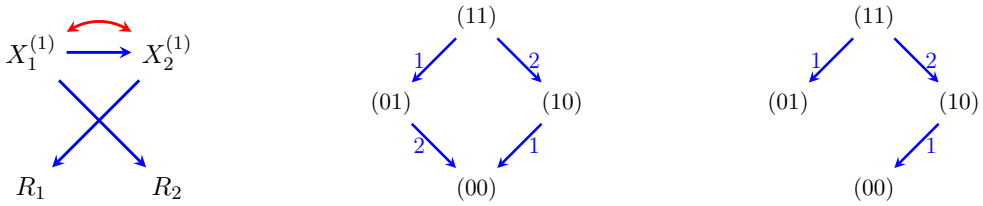


Figure 2: Example demonstrating PM-ID compatibility of pattern DAGs: (a) An m-graph $\mathcal{G}$; (b) A pattern DAG defined wrt $\mathcal{G}$ that is PM-ID compatible; (c) A pattern DAG defined wrt $\mathcal{G}$ this is not PM-ID compatible due to the absence of a directed path from the complete case pattern 11 to the pattern 00 ending with an edge labeled with the index 2.

patterns, with the complete case pattern serving as the base case that is identified simply by consistency. The following theorem formalizes this intuition.[7]

Theorem 2 (PM-ID Algorithm Using A Pattern DAG) *Given a missing data model $p(X^{(1)}, O, R, X)$ that satisfies (P1, P2) and factorizes according to an m-graph $\mathcal{G}(V, E)$, let $\mathcal{H}(\widetilde{V}, \widetilde{E})$ be any PM-ID compatible pattern DAG with respect to $\mathcal{G}$. Then the joint distribution $p(X^{(1)}, O, R = r)$ for all patterns $r \in \widetilde{V}$ are identified by processing them according to any valid topological ordering $\prec_{\mathcal{H}}$ of $\mathcal{H}$ as follows.*

- *The first pattern in $\prec_{\mathcal{H}}$ is the complete case pattern, which is identified by consistency as $p(X^{(1)}, O, R = 1) = p(X, O, R = 1)$.*
- *For subsequent patterns $r \in \prec_{\mathcal{H}}$, the extrapolation density $p(X^{(1)}_{\mathbb{M}(r)} \mid X^{(1)}_{\mathbb{O}(r)}, O, R = r)$, and hence the joint $p(X^{(1)}, O, R = r)$, is identified as for each $i \in \mathbb{M}(r)$ we have,*

$$p(X_i^{(1)} \mid X^{(1)}_{\mathbb{M}(r)\setminus\{i\}}, X^{(1)}_{\mathbb{O}(r)}, O, R = r) = p(X_i^{(1)} \mid X^{(1)}_{\mathbb{M}(r)\setminus\{i\}}, X^{(1)}_{\mathbb{O}(r)}, O, R = \widetilde{r}), \quad (6)$$

where $\widetilde{r}$ is any parent of r such that $\widetilde{r} \to^i r$ exists in $\mathcal{H}$.

Proof We prove this result using induction.

Base case The complete case pattern is indeed the first pattern processed in any valid topological ordering $\prec_{\mathcal{H}}$ due to the requirement (I1) and (I2) for $\mathcal{H}$ to be PM-ID compatible. Further, $p(X^{(1)}, O, R = 1)$ is indeed identified by consistency.

Induction hypothesis and induction step For any subsequent pattern in $r \in \prec_{\mathcal{H}}$, suppose all previous patterns in the ordering have been identified. By requirement (I2) of PM-ID compatibility we know for every Gibbs factor in the extrapolation density $p(X^{(1)}_{\mathbb{M}(r)} \mid X^{(1)}_{\mathbb{O}(r)}, O, R = r)$, there exists at least one parent pattern $\widetilde{r}$ of r such that (6) holds. The

7. This result will also prove useful for the extension of PM-ID with positivity violations, where $\widetilde{V} \neq \mathcal{R}$. We note that we are not the first to provide a constructive argument for full law identification based on PM factorization—Sadinle and Reiter (2017) provide one for full law identification of the no self-censoring model. However, Theorem 2 is more general in the sense that it can be used for constructing the full law using only a subset of patterns with positive support, as we show in Section 4.

Algorithm 1 PM-ID Construction

Require: m-graph $\mathcal{G}$, set of all possible missingness patterns $\mathcal{R}$

1: Initialize a pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$ with vertex set $\widetilde{V} = \mathcal{R}$ and edge set $\widetilde{E} = \emptyset$
2: **for** $r \in \mathcal{R}$ **do**
3: ▷ *For every unobserved variable, check if the Gibbs factors are functions of patterns with one less zero:* $p(X_i^{(1)} \mid X_{-i}^{(1)}, O, R = r) = p(X_i^{(1)} \mid X_{-i}^{(1)}, O, R_i = 1, R_{-i} = r_{-i})$
4: **for** $i \in \mathbb{M}(r)$ **do**
5: **if** $R_i \notin \text{mb}_{\mathcal{G}}(X_i^{(1)})$ **then**
6: Add an edge $\widetilde{r} \to^i r$ to $\mathcal{H}$, where $\mathbb{M}(r) \setminus \mathbb{M}(\widetilde{r}) = \{i\}$
7: **else**
8: **return** "fail"
9: **return** $\mathcal{H}(\widetilde{V}, \widetilde{E})$

positivity assumptions (P1, P2) guarantee that these Gibbs factors are well defined. Further, because every $\widetilde{r}$ is a parent of r, we also know $\widetilde{r} \prec_{\mathcal{H}} r$. That is, by the induction hypothesis and (I2), all Gibbs factors of the extrapolation density of pattern r can be expressed as functions of previously identified patterns. Since all Gibbs factors of the extrapolation density of pattern r are identified, we can conclude that $p(X^{(1)}, O, R = r)$ is identified. The correctness of the base case and this induction step concludes identifiability of all patterns $r \in \widetilde{V}$. ■

An immediate corollary of Theorem 2 is that given an m-graph $\mathcal{G}(V, E)$, if one can construct a PM-ID compatible pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$ that contains all missingness patterns, i.e., $\widetilde{V} = \mathcal{R}$, then the associated full law $p(X^{(1)}, O, R)$ is identified. There are many ways to construct such a pattern DAG for a given m-graph $\mathcal{G}$. From Section 3.2 we have seen that independences of the form (3) are sufficient for identifying strictly positive full laws. Thus, a simple approach to pattern DAG construction would focus on using just these constraints.

Algorithm 1 presents one such method. The PM-ID construction algorithm initializes an empty pattern DAG with vertices $\widetilde{V} = \mathcal{R}$ and then for every pattern $r \in \mathcal{R}$ and every index $i \in \mathbb{M}(r)$ attempts to add an edge $\widetilde{r} \to^i r$ by checking if $\mathcal{G}$ implies $X_i^{(1)} \perp\!\!\!\perp R_i \mid X_{-i}^{(1)}, R_{-i}, O$, where $\widetilde{r}$ is the pattern in $\mathcal{R} \setminus \{r\}$ where i is 1 instead of 0, i.e., a pattern with one less zero such that $\mathbb{M}(r) \setminus \mathbb{M}(\widetilde{r}) = \{i\}$. When all these edge additions are successful, the pattern DAG $\mathcal{H}$ returned by the PM-ID construction algorithm encodes a recursive identification argument via Theorem 2 for each pattern r of the full law in terms of the set of all patterns $\widetilde{r}$ that contain precisely one less zero than r.

It is easy to check that the PM-ID construction algorithm is successful if and only if the m-graph $\mathcal{G}$ satisfies the structural constraint (S1) due to the check in line 5. Further, when the algorithm returns a pattern DAG $\mathcal{H}$, it is indeed PM-ID compatible: For (I1), the complete case pattern is guaranteed to be a root node of $\mathcal{H}$ as a requirement for adding an edge $\widetilde{r} \to^i r$ in Algorithm 1 is that $\mathbb{M}(r) \setminus \mathbb{M}(\widetilde{r}) = \{i\}$. This is never true for the complete case pattern since $\mathbb{M}(r) = \emptyset$ when r is the complete case pattern. For (I2), let $\prec_{\mathcal{H}}$ be any valid topological ordering of the patterns in $\mathcal{H}$. By (I1), we know the complete case pattern is the first element of $\prec_{\mathcal{H}}$ and is the base case pattern that trivially satisfies the criterion in (I2). For any subsequent pattern $r \in \prec_{\mathcal{H}}$, suppose all previous patterns satisfy the criterion

in (I2). We know from line 6 of Algorithm 1 that an edge $\widetilde{r} \to^i r$ is added for every $i \in \mathbb{M}(r)$, where each $\widetilde{r}$ is a pattern with one less zero. Further, it must be the case that $\widetilde{r} \prec_{\mathcal{H}} r$ due to these edges. By the induction hypothesis, we conclude that (I2) also holds for r, as each edge $\widetilde{r} \to^i r$ concludes a directed path from the complete case pattern to r for each index $i \in \mathbb{M}(r)$ through a pattern $\widetilde{r}$ that already satisfies (I2). These observations lead to the following results about the PM-ID algorithm.

Corollary 3 (Soundness of PM-ID) *Suppose a missing data model $p(X^{(1)}, O, R, X)$ satisfies the positivity conditions (P1, P2) and factorizes according to an m-graph $\mathcal{G}(V, E)$ that satisfies (S1). Then the full law $p(X^{(1)}, O, R)$ is identified by applying the PM-ID algorithm described in Theorem 2 to the pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$ returned by Algorithm 1.*

Proof Follows from the fact that $\mathcal{H}$ is PM-ID compatible and $\widetilde{V} = \mathcal{R}$. ■

Corollary 4 (Completeness of PM-ID for m-DAGs and m-ADMGs) *Given an m-DAG or m-ADMG $\mathcal{G}(V, E)$, if the pattern DAG construction procedure in Algorithm 1 returns "fail", then there exist missing data models $p(X^{(1)}, O, R, X)$ satisfying $(P1, P2)$ and factorizing according to $\mathcal{G}$ whose full laws are provably not identified.*

Proof Follows from the fact that Algorithm 1 returns "fail" if and only if $\mathcal{G}$ does not satisfy (S1), and (S1) is known to be a necessary and sufficient condition for full law identification in m-DAGs and m-ADMGs satisfying (P1, P2) (Bhattacharya et al., 2019; Nabi et al., 2020). ■

Corollaries 3 and 4 together imply that PM-ID is sound and complete for full law identification in missing data models of m-DAGs and m-ADMGs satisfying positivity, similar to OR-ID. However, in contrast to OR-ID, the PM-ID algorithm also provides explicit identifying functionals. That is, PM-ID has the same identification capabilites as OR-ID, but is also constructive. To demonstrate this, we revisit the previous example, but this time we use the pattern DAG construction in Algorithm 1 and apply PM-ID to it.

3.3.3 An example of identification using PM-ID With Pattern DAGs

Example 3 *We revisit the m-graphs in Figures 1(a, b, c). Applying Algorithm 1 to any of these m-graphs gives us the pattern DAG in Figure 3. Explicit identifying functionals encoded by this pattern DAG are obtained according to a topological order $\prec_{\mathcal{H}}$ per the PM-ID algorithm in Theorem 2 and are shown in display* (7).

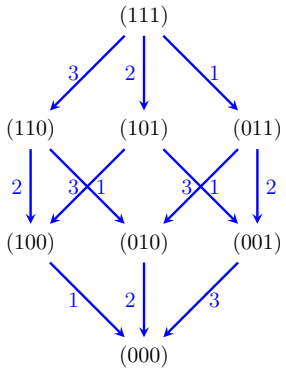


Figure 3: The pattern DAG $\mathcal{H}$ obtained by applying the PM-ID pattern DAG construction method in Algorithm 1 to any of the m-graphs shown in Figure 1.

$$
\begin{aligned}
&p(X^{(1)}, r = 111) = p(X, r = 111)\\
&p(X^{(1)}, r = 110) = p(X_3^{(1)} \mid X_{12}^{(1)}, r = 110)p(X_{12}, r = 110)\\
&\qquad \textit{where: } (111) \to^3 (110) \;\; \textit{means} \;\; p(X_3^{(1)} \mid X_{12}^{(1)}, r = 110) = p(X_3 \mid X_{12}, r = 111)\\
&p(X^{(1)}, r = 101) = p(X_2^{(1)} \mid X_{13}^{(1)}, r = 101)p(X_{13}, r = 101)\\
&\qquad \textit{where: } (111) \to^2 (101) \;\; \textit{means} \;\; p(X_2^{(1)} \mid X_{13}^{(1)}, r = 101) = p(X_2 \mid X_{13}, r = 111)\\
&p(X^{(1)}, r = 011) = p(X_1^{(1)} \mid X_{23}^{(1)}, r = 011)p(X_{23}, r = 011)\\
&\qquad \textit{where: } (111) \to^1 (011) \;\; \textit{means} \;\; p(X_1^{(1)} \mid X_{23}^{(1)}, r = 011) = p(X_1 \mid X_{23}, r = 111)\\
&p(X^{(1)}, r = 100) = p(X_{23}^{(1)} \mid X_1^{(1)}, r = 100)p(X_1, r = 100)\\
&\qquad \textit{where: } \begin{cases} (101) \to^2 (100) \;\; \textit{means} \;\; p(X_2^{(1)} \mid X_{13}^{(1)}, r = 100) = p(X_2^{(1)} \mid X_{13}^{(1)}, r = 101) \\ (110) \to^3 (100) \;\; \textit{means} \;\; p(X_3^{(1)} \mid X_{12}^{(1)}, r = 100) = p(X_3^{(1)} \mid X_{12}^{(1)}, r = 110) \end{cases}\\
&p(X^{(1)}, r = 010) = p(X_{13}^{(1)} \mid X_2^{(1)}, r = 010)p(X_2, r = 010)\\
&\qquad \textit{where: } \begin{cases} (110) \to^1 (010) \;\; \textit{means} \;\; p(X_1^{(1)} \mid X_{23}^{(1)}, r = 010) = p(X_1^{(1)} \mid X_{23}^{(1)}, r = 110) \\ (011) \to^3 (010) \;\; \textit{means} \;\; p(X_3^{(1)} \mid X_{12}^{(1)}, r = 010) = p(X_3^{(1)} \mid X_{12}^{(1)}, r = 011) \end{cases}\\
&p(X^{(1)}, r = 001) = p(X_{12}^{(1)} \mid X_3^{(1)}, r = 001)p(X_3, r = 001)\\
&\qquad \textit{where: } \begin{cases} (110) \to^1 (001) \;\; \textit{means} \;\; p(X_1^{(1)} \mid X_{23}^{(1)}, r = 001) = p(X_1^{(1)} \mid X_{23}^{(1)}, r = 110) \\ (011) \to^2 (001) \;\; \textit{means} \;\; p(X_2^{(1)} \mid X_{13}^{(1)}, r = 001) = p(X_2^{(1)} \mid X_{13}^{(1)}, r = 011) \end{cases}\\
&p(X^{(1)}, r = 000) :\\
&\qquad \textit{where: } \begin{cases} (100) \to^1 (000) \;\; \textit{means} \;\; p(X_1^{(1)} \mid X_{23}^{(1)}, r = 000) = p(X_1^{(1)} \mid X_{23}^{(1)}, r = 100) \\ (010) \to^2 (000) \;\; \textit{means} \;\; p(X_2^{(1)} \mid X_{13}^{(1)}, r = 000) = p(X_2^{(1)} \mid X_{13}^{(1)}, r = 010) \\ (001) \to^2 (000) \;\; \textit{means} \;\; p(X_3^{(1)} \mid X_{12}^{(1)}, r = 000) = p(X_3^{(1)} \mid X_{12}^{(1)}, r = 001) \end{cases}
\end{aligned}
\tag{7}
$$

4 Full Law Identification With Positivity Violations

In the previous section we presented a new constructive identification algorithm that works under strict positivity of the missingness mechanism and target law—assumptions (P1) and (P2) respectively. However, in real-world missing data problems, positivity violations

are a common practical concern, especially when there are several variables that exhibit missingness. This is due to the exponential increase in the number of potential missingness patterns: with K missing variables, there are 2^K possible configurations of missingness indicators. As a result, it is common for many of these patterns to be entirely unobserved in the dataset. In such situations, it becomes unclear whether the full law is still identified and previous identification results for graphical models of missing data do not apply. In this section we extend PM-ID to handle certain kinds of positivity violations.

4.1 Formalizing The Relaxed Positivity Assumptions

We first formalize the relaxed positivity assumptions. Most often, data analysts are concerned about certain patterns of missingness not being available in their data. For example, in a three variable problem, an analyst might be concerned that the pattern 101 never appears in their data, which may correspond to the following positivity violation: $p(R_1 = 1, R_2 = 0, R_3 = 1) = 0$.

Such scenarios were not allowed in models satisfying (P1, P2) in the previous section. We relax this as follows. We still require positivity of the target law, but allow certain patterns to be unobserved. That is, we leave (P2) unchanged, and replace (P1) with a new positivity condition that requires positivity on the complete case pattern, but allows other patterns to be have zero support. That is, we introduce a new positivity condition,

(P3) The subset of patterns $\mathcal{R}^+ \subseteq \mathcal{R}$ that have positive support includes the complete case pattern, and $p(R = r) \geq \epsilon$ if $r \in \mathcal{R}^+$ and $p(R = r) = 0$ otherwise. Further, $\mathcal{R}^+$ is compatible with the factorization of $p(X^{(1)}, O, R, X)$ according to $\mathcal{G}$.

Before explaining the compatibility clause in (P3), we first confirm that (P1, P3) is a more relaxed set of positivity assumptions than (P1, P2) when $\mathcal{R}^+$ is a strict subset of $\mathcal{R}$. To see this, note that when $p(R = r) = 0$ then the joint distribution $p(X^{(1)}, O, R = r) = 0$ for all possible values of $X^{(1)}$ and O. This is because $p(R = r) = \sum_{X^{(1)},O} p(X^{(1)}, O, R = r)$ and since none of the terms in the summation can be negative, they must all be zero when $p(R = r) = 0$. Thus, when some patterns r do not have positive support, i.e., $\mathcal{R}^+ \subset \mathcal{R}$, the full law is clearly no longer strictly positive as was the case with (P1, P2). On the other hand, when $\mathcal{R}^+ = \mathcal{R}$, the two sets of assumptions are equivalent as expected.

We now explain the compatibility of patterns with the factorization of $\mathcal{G}$ through an example. Consider the m-graph $\mathcal{G}$ in Figure 2(a). By the factorization of the graph,

$$p(X^{(1)}, R = r) = p(X^{(1)}) \cdot p(R_1 = r_1 \mid X_2^{(1)}) \cdot p(R_2 = r_2 \mid X_1^{(1)}).$$

Suppose the pattern $R = 10$ does not have positive support, so $p(X^{(1)}, R = 10) = 0$. Since the target law $p(X^{(1)})$ is required to be positive by (P1), the above factorization then implies that at least one of $p(R_1 = 1 \mid X_2^{(1)})$ or $p(R_2 = 0 \mid X_1^{(1)})$ is equal to zero. If the former is true, then the pattern $R = 11$ should also have zero support since $p(R = 1 \mid X_2^{(1)})$ also appears in the factorization of $p(X^{(1)}, R = 11)$. If the latter is true, then the pattern $R = 00$ should also have zero support. Thus, the pattern $R = 10$ cannot have zero support in isolation if $p(X^{(1)}, O, R, X)$ factorizes according to the m-graph $\mathcal{G}$.

In the above example there are only two subsets of $\mathcal{R}$ that satisfy the compatibility requirement of (P3) while also including the complete case pattern: $\mathcal{R}^+ = \{11, 10\}$ and

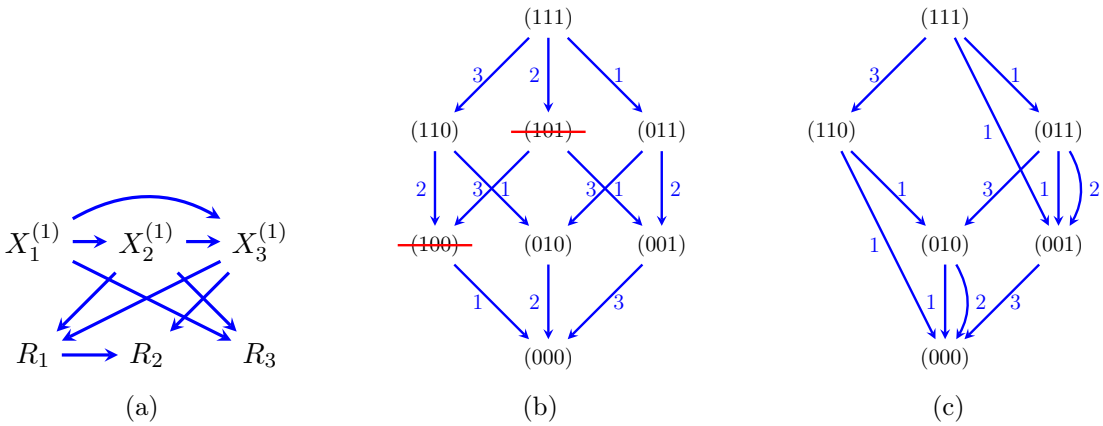


Figure 4: (a) An m-graph $\mathcal{G}$ used to illustrate pattern mixture identification with positivity violations on the patterns $R = 101$ and $R = 100$. (b) Pattern DAG obtained by applying the PM-ID construction algorithm. The red strikethrough indicates patterns with no support. (c) Pattern DAG obtained by applying the PM-ID+ construction algorithm.

$\mathcal{R}^+ = \{11, 01\}$. As negative examples, $\mathcal{R}^+ = \{10, 00\}$ is compatible but does not contain the complete case, while $\mathcal{R}^+ = \{11, 00\}$ is not compatible. Moving forward we assume that the observed set of patterns $\mathcal{R}^+$ is indeed compatible with the given m-graph $\mathcal{G}$, as in our work we assume the data are drawn from a missing data model that factorizes according to $\mathcal{G}$, and $\mathcal{G}$ is correctly specified. Designing tests of model correctness based on compatibility of the observed patterns is an interesting direction of future work; this would extend tests in Mohan and Pearl (2014), Nabi and Bhattacharya (2023), and Chen et al. (2023) that are based solely on conditional independence and generalized equality constraints.

4.2 PM-ID+ For Models With Positivity Violations

Under the new positivity assumptions (P1, P3), when $\mathcal{R}^+ = \mathcal{R}$ it suffices to use the PM-ID pattern DAG construction in Algorithm 1 along with Theorem 2 to identify the full law. However, in general, when some patterns do not have positive support, the identification methods presented in the previous section do not work. This is due to the fact that when $p(R = \widetilde{r}) = 0$ for a certain pattern $\widetilde{r} \in \mathcal{R}$, then the PM factorization and the Gibbs factors of $p(X^{(1)}, O, R = \widetilde{r})$ are not defined. Thus, any edge $\widetilde{r} \to^i r$ in the pattern DAG that suggests borrowing information from a pattern $\widetilde{r}$ with no support is also ill-defined.

Similar to our explanation of the PM-ID algorithm, we present a motivating example that provides concrete intuition for how we can modify the pattern DAG construction of PM-ID to handle positivity violations of the kind allowed by assumptions (P1, P3).

4.2.1 Motivating example for PM-ID+

Example 4 *Let $p(X^{(1)}, O, R, X)$ be a missing data model that factorizes according to the m-graph $\mathcal{G}$ in Figure 4(a) and has support for patterns $\mathcal{R}^+ = \{111, 011, 110, 010, 001, 000\}$.*

That is, the patterns $\{101, 100\}$ *have no support, so* $p(R = 101) = 0$ *and* $p(R = 100) = 0$. *Clearly this model satisfies assumptions (P1, P3)*[8] *but not (P1, P2).*

Say we naively apply the PM-ID pattern DAG construction algorithm to $\mathcal{G}$*; we would then obtain the pattern DAG shown in Figure 4(b). The red strikethroughs in the figure are used to denote the patterns that have no support in the model.*

1. *Based on this pattern DAG, we are still able to use the same arguments as before to identify the patterns* $111, 110, 011$, *and* 010, *as the identification of these patterns do not rely on borrowing information from the patterns with no support.*

2. *Identification for patterns with no support is also simple: since* $p(R = 101) = 0$ *and* $p(R = 100) = 0$, *we have that* $p(X^{(1)}, O, R = 101) = 0$ *and* $p(X^{(1)}, O, R = 100) = 0$.

3. *However, the patterns* 001 *and* 000, *whose identification arguments use information from patterns with no support, present a problem. Consider* $R = 001$ *for example: The edges* $(101) \to^1 (001)$ *and* $(011) \to^2 (001)$ *in Figure 4(b) suggest identification of the Gibbs factors of the extrapolation density of* $p(X^{(1)}, O, R = 001)$ *via the two equalities,*

$$p(X_1^{(1)} \mid X_{23}^{(1)}, R = 001) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 101), \text{ and}$$
$$p(X_2^{(1)} \mid X_{13}^{(1)}, R = 001) = p(X_2^{(1)} \mid X_{13}^{(1)}, R = 011).$$

   The problem lies in the fact that $p(X_1^{(1)} \mid X_{23}^{(1)}, R = 101)$ *is undefined, since the full law at* 101 *is zero, so the first equality cannot be used for identification. A similar problem occurs for* $R = 000$ *when attempting to use the edge* $(100) \to^1 (000)$ *for identification.*

4. *Recall that while the PM-ID pattern DAG construction only uses constraints of the form* $X_i^{(1)} \perp\!\!\!\perp R_i \mid \mathrm{mb}_{\mathcal{G}}(X_i^{(1)})$, *the m-graph* $\mathcal{G}$ *may imply additional constraints based on the local Markov property* (2). *Thus, in order to patch the issue of the edge* $(101) \to^1 (001)$ *being ill-defined due to a positivity violation in the pattern* 101, *we may employ additional constraints to tether the first Gibbs factor to a different well-defined conditional. By the local Markov property,* $\mathcal{G}$ *also implies* $X_1^{(1)} \perp\!\!\!\perp R_{12} \mid X_{23}^{(1)}, R_3$, *which leads to*

$$p(X_1^{(1)} \mid X_{23}^{(1)}, R = 001) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 111).$$

   The right hand side is a conditional of the complete case joint distribution, and hence is well-defined due to the positive support on 111. *Moreover, this conditional is also identified by consistency. This concludes that the full law at* 001 *is identified by replacing the ill-defined edge* $(101) \to^1 (001)$ *with* $(111) \to^1 (001)$ *instead. Note the local Markov property also allows us to add the edge* $(011) \to^1 (001)$, *offering multiple ways of identifying this Gibbs factor and possibly increasing data efficiency in estimation.*

5. *We can perform a similar patch for the pattern* $R = 000$. *By the local Markov property of* $\mathcal{G}$ *we have,*

$$p(X_1^{(1)} \mid X_{23}^{(1)}, R = 000) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 110), \text{ and}$$
$$p(X_1^{(1)} \mid X_{23}^{(1)}, R = 000) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 010).$$

8. We manually checked that this set of patterns is also compatible with the factorization of $\mathcal{G}$.

That is, the ill-defined edge $(100) \to^1 (000)$ *can be replaced by the edges* $(110) \to^1 (000)$ *and* $(010) \to^1 (000)$*, i.e., edges from patterns with positive support and that can be identified prior to* 000*. Thus, we conclude that* $p(X^{(1)}, O, R = 000)$ *is also identified.*

6. *The full pattern DAG obtained by replacing ill-defined edges due to positivity violations to edges that only use patterns with positive support is shown in Figure 4(c). The goal of PM-ID+ in the next subsection will be to produce such pattern DAGs.*

4.2.2 The PM-ID+ Algorithm

The example in the previous subsection provides a few key insights that we use to phrase an extension of the PM-ID algorithm for models with positivity violations.

1. Identification of patterns with no support is trivial as $p(X^{(1)}, O, R = r) = 0$ for any pattern $r \in \mathcal{R} \setminus \mathcal{R}^+$. So we only need to focus on identification of patterns in $\mathcal{R}^+$.

2. For patterns $r \in \mathcal{R}^+$ with positive support, the main recursive identification argument from Theorem 2 can be preserved if we are able to produce a PM-ID compatible pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$—a pattern DAG that satisfies (I1, I2)—such that $\widetilde{V} = \mathcal{R}^+$.

3. The construction of such a pattern DAG should use additional independences implied by the local Markov property of the m-graph $\mathcal{G}$. In particular, independences of the form $X_i^{(1)} \perp\!\!\!\perp R_j \mid \mathrm{mb}_{\mathcal{G}}(X_i^{(1)})$ for $j \neq i$, in addition to the independences $X_i^{(1)} \perp\!\!\!\perp R_i \mid \mathrm{mb}_{\mathcal{G}}(X_i^{(1)})$ used in the PM-ID construction algorithm in Algorithm 1.

Based on these insights, we propose the PM-ID+ pattern DAG construction strategy in Algorithm 2. We use $\mathrm{ID} \subseteq \mathcal{R}^+$ to denote the set of patterns without positive support that have been successfully identified at any given step of the algorithm. ID is initialized to be $\emptyset$, and at each iteration of the identification process, this set is potentially expanded to include newly identified patterns. By (P3), the complete case is in $\mathcal{R}^+$. Thus on the first iteration of the algorithm we are guaranteed to get identification of the complete case pattern by consistency, and so the complete case pattern is immediately added to ID. This also guarantees that if a pattern DAG is successfully produced by the PM-ID+ construction, it satisfies the condition (I1) that the complete case pattern is a root node of $\mathcal{H}$.

At each iteration, the algorithm loops through all patterns with positive support whose identifiability has not yet been determined, i.e., each $r \in \mathcal{R}^+ \setminus \mathrm{ID}$, and tries to establish their identifiability based on previously identified patterns with positive support. Concretely, for every index corresponding to actually missing variable $i \in \mathbb{M}(r)$, we list all other patterns $\widetilde{r} \in \mathrm{ID}$ such that the equality in (5) holds, implying that the i^{th} Gibbs factor of the extrapolation density of pattern r can be re-expressed as the i^{th} Gibbs factor of an already identified pattern $\widetilde{r}$. By the local Markov property, the set of all such patterns is computed in lines 9 and 10 of Algorithm 2 as,

$$\mathcal{Z}(r, i) = \{\widetilde{r} \in \mathrm{ID} \mid R_j \notin \mathrm{mb}_{\mathcal{G}}(X_i^{(1)}) \text{ for all } j \in \mathrm{diff}(r, \widetilde{r})\}, \tag{8}$$

where $\mathrm{diff}(r, \widetilde{r}) \coloneqq \{i \in [K] \mid r_i \neq \widetilde{r}_i\}$. If we are able to link every Gibbs factor of the extrapolation density of pattern r to at least one already identified pattern, then we know the joint $p(X^{(1)}, O, R = r)$ is identifiable and we can append edges of the form $\widetilde{r} \to^i r$ to the

Algorithm 2 PM-ID$^+$ Construction

Require: m-graph $\mathcal{G}(V, E)$, patterns with positive support $\mathcal{R}^+ = \{r \in \mathcal{R} \mid p(R = r) > 0\}$

1: Initialize a pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$ with vertex set $\widetilde{V} = \mathcal{R}^+$ and edge set $\widetilde{E} = \emptyset$
2: ▷ *Create variables to keep track of all patterns identified and new patterns identified in an iteration (initialize the latter with the complete case pattern)*
3: ID $\leftarrow \emptyset$ and $\Delta \leftarrow \{r = 1\}$
4: **while** $\Delta \neq \emptyset$ **do**
5: ID $\leftarrow$ ID $\cup\, \Delta$ and $\Delta \leftarrow \emptyset$
6: **for** $r \in \mathcal{R}^+ \setminus$ ID **do**
7: ▷ *Check if Gibbs factors $p(X_i^{(1)} \mid X_{-i}^{(1)}, O, R = r)$ of every unobserved variable can be expressed as a function of at least one pattern that has been identified so far*
8: **for** $i \in \mathbb{M}(r)$ **do**
9: Let $\text{diff}(r, \widetilde{r}) \coloneqq \{i \in [K] \mid r_i \neq \widetilde{r}_i\}$
10: $\mathcal{Z}(r, i) \leftarrow \{\widetilde{r} \in \text{ID} \mid R_j \notin \text{mb}_{\mathcal{G}}(X_i^{(1)})$ for all $j \in \text{diff}(r, \widetilde{r})\}$
11: **if** $\mathcal{Z}(r, i) = \emptyset$ **then** go to the next pattern in line 6
12: ▷ *Here we know that every Gibbs factor can be identified as a function of one or more identified patterns; add edges for this and mark the pattern as identified*
13: **for** all $i \in \mathbb{M}(r)$ and **for** all $\widetilde{r} \in \mathcal{Z}(r, i)$ add an edge $\widetilde{r} \to^i r$ to $\mathcal{H}$
14: $\Delta \leftarrow \Delta \cup \{r\}$
15: **if** $\Delta = \emptyset$ **then**
16: **return** "fail"
17: **return** $\mathcal{H}(\widetilde{V}, \widetilde{E})$

pattern DAG that satisfy the PM-ID compatibility criterion (I2) for pattern r by induction. This addition of edges occurs in line 13 of Algorithm 2.

Algorithm 2 iteratively updates the set of identified patterns in this manner, repeating the procedure until no further updates are possible. This can occur when ID $\subset \mathcal{R}^+$ is not identified, but no new pattern can be identified by our procedure. In this case the algorithm returns "fail". The other case is when ID $= \mathcal{R}^+$, so all patterns with positive support have been identified and the PM-ID+ construction returns a PM-ID compatible pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$ with $\widetilde{V} = \mathcal{R}^+$. As an example of success, the PM-ID+ construction algorithm yields the pattern DAG in Figure 4(c) for the m-graph in Figure 4(a) with positivity violations on patterns 101 and 100. The following result formalizes the soundness of PM-ID+.

Corollary 5 (Soundness of PM-ID+) *Suppose a missing data model $p(X^{(1)}, O, R, X)$ satisfies the positivity conditions (P1, P3) and factorizes according to an m-graph $\mathcal{G}(V, E)$. Then the full law $p(X^{(1)}, O, R)$ is identified if the PM-ID+ pattern DAG construction in Algorithm 2 succeeds in returning a pattern DAG $\mathcal{H}(\widetilde{V}, \widetilde{E})$, and is identified as follows.*

- *For patterns $r \in \mathcal{R} \setminus \mathcal{R}^+$, we have that $p(X^{(1)}, O, R = r) = 0$.*
- *For patterns $r \in \mathcal{R}^+$, we apply the PM-ID algorithm in Theorem 2 to $\mathcal{H}(\widetilde{V}, \widetilde{E})$.*

Proof Identification of patterns $r \in \mathcal{R} \setminus \mathcal{R}^+$ is trivial. Identification of all patterns $r \in \mathcal{R}^+$ follows from the fact that $\mathcal{H}$ is PM-ID compatible with $\widetilde{V} = \mathcal{R}^+$ and the assumptions (P1,

P2) can be swapped with (P1, P3) when all patterns in $\mathcal{H}$ have positive support. ■

Note that we make no claim of completeness for PM-ID+. That is, there may exist missing data models that satisfy (P1, P3) and factorize according to an m-graph, such that the PM-ID+ construction returns "fail", but the full law is in fact identified. Nonetheless, PM-ID+ covers a large set of MNAR models that may exhibit positivity violations, and that to our knowledge, have not been considered in prior work on m-graphs.

We provide one more example of the PM-ID+ pattern DAG construction algorithm that demonstrates an interesting aspect of the algorithm—while the PM-ID construction makes it so that a pattern r can only borrowing information from another pattern $\widetilde{r}$ with strictly fewer missing variables (i.e., $\mathbb{M}(r) \supset \mathbb{M}(\widetilde{r})$), the PM-ID+ construction only requires that $\widetilde{r}$ be identifiable prior to r, regardless of the relation between $\mathbb{M}(r)$ and $\mathbb{M}(\widetilde{r})$.

Example 5 *Let $p(X^{(1)}, O, R, X)$ be a missing data model that factorizes according to the m-graph in Figure 5(a) and $\mathcal{R}^+ = \{111, 110, 011, 100, 001\}$. Thus, $\mathcal{R} \setminus \mathcal{R}^+ = \{101, 010, 000\}$ have no support. We now list each iteration of the PM-ID+ construction in Algorithm 2 that leads to the pattern DAG in Figure 5(b).*

1. *$ID = \emptyset$ and the newly identified patterns $\Delta = \{111\}$*
2. *$ID = \{111\}$ and the newly identified patterns $\Delta = \{110, 011\}$, as $\mathcal{Z}(110, 3) = \{111\}$ and $\mathcal{Z}(011, 1) = \{111\}$. That is, borrowing information from just the complete pattern is sufficient for identifying these two patterns.*
3. *$ID = \{111, 110, 011\}$ and the newly identified patterns $\Delta = \{001\}$, as $\mathcal{Z}(001, 1) = \{111, 011\}$ and $\mathcal{Z}(001, 2) = \{111, 011\}$. That is, we can identify the Gibbs factors in the extrapolation density of $p(X^{(1)}, O, R = r)$ by borrowing information from both the 111 and 011 patterns.*
4. *$ID = \{111, 110, 011, 001\}$ and the newly identified patterns $\Delta = \{100\}$, as $\mathcal{Z}(100, 2) = \{110\}$ and $\mathcal{Z}(100, 3) = \{001\}$. This leads to the addition of an edge $(001) \to^3 (100)$ in $\mathcal{H}$ that represents an interesting case where, PM-ID+ borrows information from a pattern where the observed variables are not necessarily a strict superset of those in the current pattern.*
5. *Finally $ID = \mathcal{R}^+$ so the full law is identified, and the algorithm returns the pattern DAG $\mathcal{H}$ shown in Figure 5(b).*

4.3 Relative merits of PM-ID and PM-ID+

Before we present an imputation algorithm based on these identification results, we briefly comment on the relative tradeoffs between using PM-ID and PM-ID+ for the identification step preceding imputation of missing values.

The obvious advantage of PM-ID+ is that it can identify missing data models under a weaker set of positivity assumptions. That is, the set of missing data models that can be identified by PM-ID+ is strictly larger than those identifiable by PM-ID. Further, since PM-ID+ uses additional constraints encoded by the m-graph, it can often lead to pattern DAGs

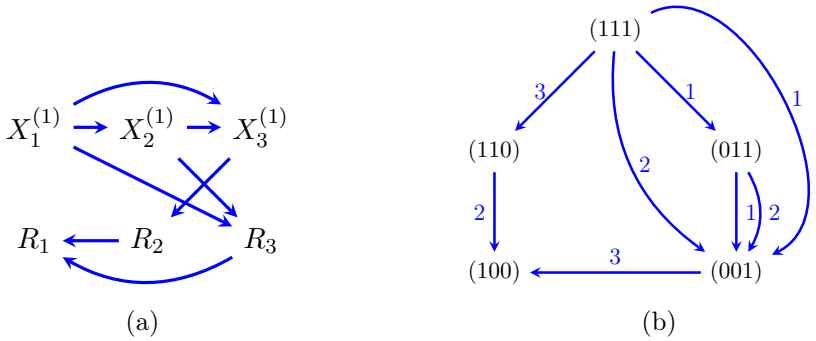


Figure 5: (a) An m-graph used to demonstrate that PM-ID+ can borrow information from patterns where the sets $\mathbb{M}(r)$ and $\mathbb{M}(\widetilde{r})$ are incomparable. (b) The pattern DAG produced by the PM-ID+ construction, where the edge $001 \to^3 100$ illustates this point.

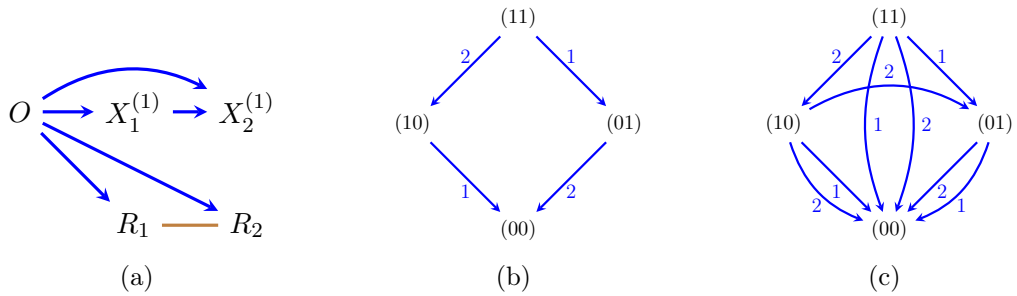


Figure 6: (a) A sparse m-graph used to demonstrate the relative data efficiency of PM-ID+ and the robustness to graph misspecification of PM-ID. (b) The pattern DAG obtained using the PM-ID construction. (c) The pattern DAG obtained using the PM-ID+ construction.

that make use of more rows of data than PM-ID, even when the full law is strictly positive. As an example, consider the m-graph $\mathcal{G}$ in Figure 6(a) and the pattern DAGs produced in Figures 6(b) and (c) produced by the PM-ID and PM-ID+ constructions respectively. The PM-ID+ pattern DAG makes use of more constraints encoded by the m-graph, e.g., when identifying the pattern 00, it uses information from all of the other patterns, while PM-ID does not. This may affect the efficiency of downstream imputation by dictating what rows of data can be used to fit each Gibbs factor in the extrapolation density. For example, for the pattern 00 with Gibbs factors $p(X_1^{(1)} \mid X_2^{(1)}, R = 00)$ and $p(X_2^{(1)} \mid X_1^{(1)}, R = 00)$, PM-ID+ proposes each of these Gibbs factors can be fit using data from any other row, while PM-ID proposes the first Gibbs factor can be fit only using rows of data with the pattern 10 and the second can be fit only using rows with the pattern 01. An interesting question for future is whether this also leads to greater precision in downstream estimates of target parameters.

Next, we provide a brief analysis on the computational complexity PM-ID and PM-ID+, as a function of the number of missing variables K and missing patterns with positive support $|\mathcal{R}|^+$. That is, for this analysis, we consider sets like the Markov blanket of variables

to have been pre-computed and stored, as these operations are fast and can be performed in linear time in number of edges and vertices of the graph using depth first search.

For PM-ID, we require all possible patterns to have positive support. In the outer loop we iterate over these patterns and in the inner loop, we iterate through every missing index in these patterns. That is, the outer loop executes $|\mathcal{R}|^+ = 2^K$ times and the inner loop executes at most K times (for the pattern with all zeros), where K is the number of potentially missing variables and $\mathcal{R}^+$ is the number of positivity patterns. This gives us a total computational complexity of $\mathcal{O}(K2^K)$ for PM-ID. The following lemma provides the computational complexity of PM-ID+.

Lemma 6 *PM-ID+ has worst case time complexity $\mathcal{O}(K|\mathcal{R}^+|^3)$, where K is the number of missing variables and $|\mathcal{R}|^+$ is the number of missingness patterns with support.*

Proof In the worst case, we identify only one new pattern with positive support on each iteration of PM-ID+. That is, the outermost while loop executes at most $|\mathcal{R}^+| - 1$ times. Similarly, the inner for loop on line 6 of the algorithm executes at most $|\mathcal{R}^+| - 1$ on the first iteration when only the complete case pattern is known to be identified. The block consisting of lines 8-13 and the loops in line 13 both perform at most $K(|\mathcal{R}^+| - 1)$ steps—when the pattern r is the pattern with all zeros and every other pattern with positive support has been identified before it. This gives us the overall complexity of $\mathcal{O}(K|\mathcal{R}^+|^3)$. ■

Note that when all patterns have positive support, i.e., $|\mathcal{R}^+| = 2^K$, Lemma 6 implies that PM-ID+ has worse computational complexity than PM-ID, as in this case $\mathcal{O}(K|\mathcal{R}^+|^3) = \mathcal{O}(K8^K)$. However, in most practical applications it is likely to be the case that $|\mathcal{R}^+| \ll 2^K$ and the number of positive patterns is, in fact, some function of the number of rows of data N. For example, in a problem with twenty missing variables (i.e., $K = 20$) there are over 1 million possible missingness patterns. However, most available datasets are likely to have number of rows N much smaller than that. Thus, the number of patterns with support $|\mathcal{R}^+|$ is typically an order of magnitude smaller than 2^K. For example, $|\mathcal{R}^+|$ may only be a polynomial (or perhaps even logarithmic) function of K and N. Therefore, in such cases, the time complexity of PM-ID+ is also polynomial, rather than exponential, with respect to the number of missing variables K and number of rows N. We also see this empirically in our simulations in Section 6, where PM-ID+ and the imputation algorithm MISPR can finish running relatively quickly when the number of patterns with positive support grows slowly with respect to the size of the dataset.

While it might seem like PM-ID+ is better than PM-ID in most respects, the simplicity of the PM-ID construction does have one significant benefit—it can protect against model misspecification when there is significant uncertainty about the exact structure of the m-graph $\mathcal{G}$. If, for example, the m-graph in Figure 6(a) is incorrect and there in fact exists an edge $X_2^{(1)} \to R_1$. Then the pattern DAG obtained from the PM-ID construction remains valid. However, the pattern DAG from PM-ID+ includes edges that are incorrect due to misspecification of the m-graph, e.g., the edge $(11) \to^2 (00)$ is incorrect as $p(X_2^{(1)} \mid X_1^{(2)}, R = 00) \neq p(X_2^{(1)} \mid X_1^{(2)}, R = 11)$ when $X_2^{(1)} \to R_1$ is present. Thus, in cases of model uncertainty, PM-ID might be preferrable to PM-ID+. In fact, the analyst need not even specify a full m-graph for PM-ID, but rather only justify the graphical condition (S1).

Note that neither PM-ID nor PM-ID+ support positivity violations on the complete case pattern. One way to handle this is to also incorporate independences of the form $X_i^{(1)} \perp\!\!\!\perp X_j^{(1)} \mid \text{mb}_{\mathcal{G}}(X_i^{(1)})$ into the pattern DAG construction process, i.e., use constraints on the target law $p(X^{(1)}, O)$ implied by the local Markov property of the m-graph $\mathcal{G}$. We provide an example of how to do this in Appendix A. However, we do not expand on such techniques further in this paper, as it is often desirable to assume the least restrictive model possible for the target law to prevent bias in downstream analysis.

5 MISPR: Multivariate Imputation via Supported Pattern Recursion

Having described our identification algorithms, we are finally ready to describe our imputation algorithm that uses either PM-ID or PM-ID+ as a pre-processing step to ensure the full law is identified and to direct the imputation procedure itself based on the pattern DAG.

Data analysts often only have access to a finite data table $\mathbb{D}_{N\times(2K+J)}$, a table with N rows of data where the n^{th} row of data is a vector $[x_1^n, \ldots, x_K^n, r_1^n, \ldots, r_K^n, o_1^n, \ldots, o_J^n]$, where the values o_j^n are always observed, while the values x_k^n could be missing (denoted as "?" or "NA"), as indicated by the value of the corresponding missingness indicator r_k^n. The goal of this section is to provide an imputation algorithm which fills in missing values in such a data table. Fortunately, since PM-ID and PM-ID+ are constructive algorithms, it is relatively easy to translate the identification strategy that operates on a distribution level to estimation and imputation algorithms with finite data.

To see this, it is insightful to note that knowing the extrapolation density for pattern r is sufficient to perform imputation for all data rows matching this missing pattern, independent of missing data model considered. Concretely, for each data row $D^n = (X_{\mathbb{M}(r^n)} =?, X_{\mathbb{O}(r^n)} = x^n_{\mathbb{O}(r^n)}, O = o^n, R = r^n)$ such that r^n matches a certain pattern r, the missing values for $X_{\mathbb{M}(r^n)}$ can be imputed by sampling values from the extrapolation density $x^{(1)}_{\mathbb{M}(r)} \sim p(X^{(1)}_{\mathbb{M}(r)} \mid X_{\mathbb{O}(r)} = x^n_{\mathbb{O}(r)}, O = o^n, R = r)$. Hence, knowing the extrapolation densities at all patterns present in the data table is sufficient for imputation of that table. Further, we can use PM-ID or PM-ID+, as appropriate, to obtain explicit expressions of the Gibbs factors of the extrapolation densities of all patterns with positive support. Hence, imputation for a data row with pattern r can proceed by Gibbs sampling using the Gibbs factors for the extrapolation density for this pattern.

This forms the basis for MISPR (Multivariate Imputation via Supported Pattern Recursion) defined in Algorithm 3. The algorithm starts by determining identifiability of the full law based on the given m-graph $\mathcal{G}$ and the patterns with positive support in the data table $\mathbb{D}$. When all patterns have positive support, we opt to use PM-ID as it provides some robustness to misspecification of the exact structure of the m-graph. Otherwise, we use PM-ID+.

If the identification step succeeds, we then process patterns according to any valid topological order of patterns $\prec_{\mathcal{H}}$, just as in Theorem 2. Suppose the algorithm is at some pattern r with missing indices $\mathbb{M}(r)$. To estimate the i^{th} Gibbs factor of the extrapolation density for $i \in \mathbb{M}(r)$, we find rows of data corresponding to patterns that we can borrow information from—this corresponds precisely to those patterns $\widetilde{r}$ such that $\widetilde{r} \to^i r$ exists in $\mathcal{H}$. By the induction hypothesis, all data rows corresponding to these patterns have been imputed in

Algorithm 3 Mutivariate Imputation via Supported Pattern Recursion (MISPR)

Require: m-graph $\mathcal{G}$, Data $\mathbb{D}_{N\times(2K+J)}$ where $D^n = [x_1^n, \ldots, x_K^n, r_1^n, \ldots, r_K^n, o_1^n, \ldots, o_J^n]$

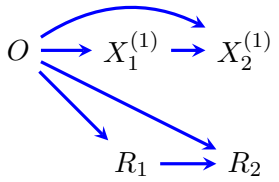
1: $\mathcal{R}^+ \leftarrow$ all unique missingness patterns r in $\mathbb{D}$
2: **if** $\mathcal{R} = \mathcal{R}^+$ **then**
3: $\quad \mathcal{H}(\widetilde{V}, \widetilde{E}) \leftarrow$ PM-ID Construction$(\mathcal{G}, \mathcal{R}^+)$
4: **else**
5: $\quad \mathcal{H}(\widetilde{V}, \widetilde{E}) \leftarrow$ PM-ID+ Construction$(\mathcal{G}, \mathcal{R}^+)$
6: **if** $\mathcal{H} =$ "fail" **then return** "fail"
7: **for** $r \in \prec_{\mathcal{H}}$ **do**
8: $\quad \triangleright$ *Fit Gibbs factors using data rows we can borrow information from according to* $\mathcal{H}$
9: $\quad$ **for** each $i \in \mathbb{M}(r)$ **do**
10: $\quad\quad$ Let $\text{pa}(r, i) := \{\widetilde{r} \in \mathcal{R}^+ \mid \widetilde{r} \rightarrow^i r \text{ in } \mathcal{H}\}$
11: $\quad\quad$ Fit $\widehat{p}(X_i^{(1)} \mid X_{-i}^{(1)}, O, R = r)$ using rows $\{D^n \mid r^n = \widetilde{r} \text{ for any } \widetilde{r} \in \text{pa}(r, i)\}$
12: $\quad \triangleright$ *Fill in missing values for all rows with pattern* r *with Gibbs sampling using the fitted Gibbs factors*
13: $\quad$ **for** $n = 1, \ldots, N$ **do**
14: $\quad\quad$ **if** $r^n = r$ **then**
15: $\quad\quad\quad$ Gibbs sample $x_{\mathbb{M}(r)}^{(1)} \sim \widehat{p}(X_{\mathbb{M}(r)}^{(1)} \mid X_{\mathbb{O}(r)} = x_{\mathbb{O}(r)}^n, O = o^n, R = r)$
$\quad\quad\quad$ using Gibbs factors $\{\widehat{p}(X_i^{(1)} \mid X_{-i}^{(1)}, O, R = r) \mid \forall i \in \mathbb{M}(r)\}$
16: $\quad\quad\quad$ Update $\mathbb{D}$ by assigning $x_{\mathbb{M}(r)}^n \leftarrow x_{\mathbb{M}(r)}^{(1)}$
return imputed data table $\mathbb{D}$

prior steps, and thus contain no missing values. Hence, the i^{th} Gibbs factors can be fit using all data rows matching the patterns $\text{pa}(r, i)$ computed in line 10 of the algorithm. Note these Gibbs factors can also be simplified according to the local Markov property, as $p(X_i^{(1)} \mid X_{-i}^{(1)}, O, R) = p(X_i^{(1)} \mid \text{mb}_{\mathcal{G}}(X_i^{(1)})$, which makes estimation more feasible in high dimensional settings. Estimation can proceed either parametrically or using flexible machine learning models, such as random forests.[9] The estimated Gibbs factors are then used to fill in the missing values for all data rows matching the current pattern r—this occurs in lines 13-16 of the algorithm. That is, in MISPR, imputation of data rows occurs according to a partial order on patterns with support, as outlined by the pattern DAG $\mathcal{H}$.

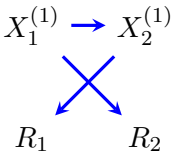
6 Numerical Experiments

In this section we evaluate the performance of MISPR in comparison with the MICE package available in the `R` programming language (Van Buuren and Groothuis-Oudshoorn, 2011), as it is currently the most popular off-the-shelf imputation software. To compare the asymptotic behavior of these methods, for each synthetic data generating process (DGP), we generate $N = 10^5$ samples for the smaller examples in Figures 7(a-e) and generate $N = 5 \times 10^5$ for the high-dimensional example in Figure 7(f). Details of each DGP are provided in Appendix B.

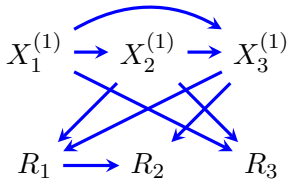
9. Though this can run the risk of leading to incongenial specifications of the joint extrapolation density.



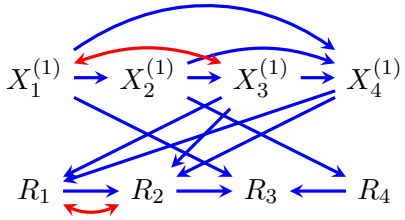
(a) A 3 variable MAR model



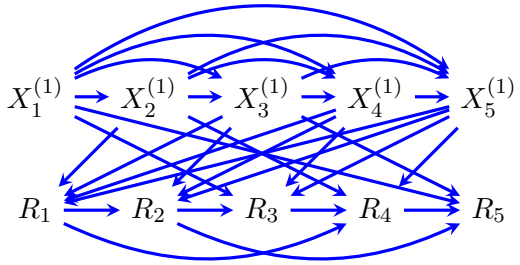
(b) A 2 variable MNAR model



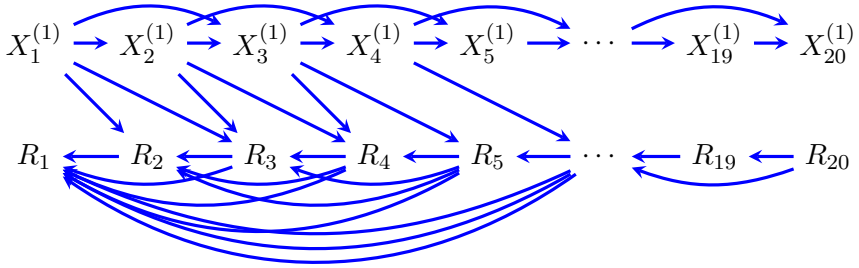
(c) The 3 variable m-DAG in Figure 4(a)



(d) A 4 variable m-ADMG



(e) A 5 variable MNAR model



(f) 20 variable MNAR model that is a submodel of the permutation model in Robins and Gill (1997)

Figure 7: Missing data models used in our numerical experiments.

For each experiment, we generate an observed dataset $\mathbb{D}$ with N samples by sampling according to the true full law $p_0(X^{(1)}, O, R)$, i.e., we draw N samples of $X^{(1)}, O, R \sim p_0$, and then set missing variables to be unobserved according to the rules of missing data consistency. The true full law p_0 for each experiment is selected randomly, so as not to pick any specific DGPs that may be favorable to one method or the other. Then, imputation is performed separately on the dataset with missing values $\mathbb{D}$ using MICE and MISPR with PM-ID+ and PM-ID in the identification step, when applicable. We use MICE with random forests to estimate each imputing distribution using the settings `mice(data, m = 7, method = 'rf', seed = 123, maxit=100)`. For MISPR, we also fit the Gibbs factors using random forests, and use a burn-in period of 500 steps for Gibbs sampling. We then use the aggregated datasets to obtain estimates of the target law $\widehat{p}(X^{(1)}, O)$ using maximum likelihood. We obtain 7 imputed datasets $\widehat{\mathbb{D}}_1, \ldots, \widehat{\mathbb{D}}_7$ from each method using different random seeds in this manner, and aggregate them as $\widehat{\mathbb{D}} = \bigcup_{i=1}^{7} \widehat{\mathbb{D}}_i$. We then compare the accuracy of imputation using MISPR and MICE by computing the L_2 and L_∞ distance between the fitted and ground truth target laws, i.e., $\|\hat{p}(X^{(1)}, O) - p_0(X^{(1)}, O)\|_q$ with $q = 2$ and $q = \infty$. We also report timings for each method in producing the seven imputed datasets. Table 4 reports these distances and timings for both MISPR and MICE across all DGPs. Entries marked with a '+' indicate the DGP contains a positivity violation. We briefly summarize the main findings below, where each bold subheading highlights the main finding from the experiment.

Expr 1: Ignoring positivity violations can lead to bias, even in MAR scenarios In this experiment, data are generated from two full data laws that factorize according to the m-graph in Figure 7(a), where one $X^{(1)} \perp\!\!\!\perp R \mid O$ so that the model is MAR. The first full data law we sample from is strictly positive, but the second is not. In the former case, MISPR (using either PM-ID+ or PM-ID) and MICE achieve fairly comparable results, albeit MICE is much slower and the bias from MISPR is also an order of magnitude lower (first row of Table 4). In the second case, however, the difference is much more noticeable, where despite the structural assumptions of the m-graph implying a MAR model, the correct handling of patterns with no support using PM-ID+ is essential. The second row of Table 4 shows how MISPR with PM-ID+ obtains much lower bias than MICE in this setting.

Expr 2: MISPR is able to handle both MNAR and sparse pattern support This experiment checks the performance of MISPR across a variety of MNAR full data laws, with and without positivity violations. This corresponds to full data laws drawn from Figures 7(b-d), with results reported in the corresponding rows of Table 4. MISPR outperforms MICE across the board in timing and bias, with the exception of one scenario where MICE obtains comparable results. This corresponds to a full data law with positivity violations that factorizes according to Figure 7(b). On inspection of this DGP, we realized that the particular positivity violations and the structural assumptions of this model actually reduce to a MAR model, which is an exceptionally rare occurrence in general.

Expr 3: Imputation with several missing variables and sparse pattern support In this experiment, we generate data from full data laws that factorize according to the m-graphs in Figures 7(e) and (f) containing 5 and 20 variables respectively. In such high-dimensions support on all possible patterns is exceedingly unlikely—in our DGPs the full data law for Figure 7(e) has 24 out of the 2^5 possible patterns, while the one for Figure 7(e) has only 476 out of the possible 2^{20} possible patterns (explicit details of these DGPs can be

DGP	MICE			MISPR w/ PM-ID+			MISPR w/ PM-ID		
	L_2	L_∞	Time	L_2	L_∞	Time	L_2	L_∞	Time
7a	0.0890	0.0568	6h26m	0.0060	0.0034	1h13m	0.0031	0.0020	1h18m
7a +	0.2329	0.1576	5h24m	0.0030	0.0016	0h29m	NA	NA	NA
7b	0.2580	0.1913	3h16m	0.0039	0.0030	0h50m	0.0038	0.0028	0h50m
7b +	0.0028	0.0023	9h17m	0.0026	0.0021	11 sec	NA	NA	NA
7c	0.0854	0.0568	18h	0.0075	0.0060	1h04m	0.0078	0.0061	1h23m
7c +	0.0197	0.0138	23h36m	0.0045	0.0034	0h44m	NA	NA	NA
7d	0.0678	0.0457	30h47m	0.0074	0.0034	1h28m	0.0057	0.0029	1h30m
7d +	0.0575	0.0406	42h53m	0.0099	0.0058	1h41m	NA	NA	NA
7e +	0.1205	0.0630	18h30m	0.0412	0.0324	3h	NA	NA	NA
7f +	-	-	-	2.4×10^{-5}	3.3×10^{-7}	28h38m	NA	NA	NA

Table 4: Comparison between MICE and MISPR on random full data laws that factorize according to m-graphs in Figure 7. Rows marked with '+' correspond to full data laws that have positivity violations. The L_2 and L_∞ distance of the true and estimated target laws are reported to measure the accuracy of the imputation. PM-ID is not applicable when there are positivity violations, hence we report "NA" for the corresponding rows. We were unable to obtain results for MICE for the full data law 7f + due to its long run times.

found in the Appendix B). For the 5 variable case we are able to produce comparisons to MICE that demonstrate that MISPR with PM-ID+ produces superior results. For the 20 variable case, the scaling of the MICE package did not allow us to obtain results for it—MICE took close to a day even for the 3-5 variable examples. Thus, we only run MISPR on the 20 variable example, and show that it produces estimates with very low bias, with reasonable running time on the 476 patterns. We suspect these running times can be improved even further with a better implementation of our method—our implementation is in Python, and does not use any parallelism to process patterns in the partial order that are incompatible (two patterns r and $\widetilde{r}$ are considered incompatible if neither $r \prec_{\mathcal{H}} \widetilde{r}$ nor $\widetilde{r} \prec_{\mathcal{H}} r$).

Summary MISPR outperforms MICE across a wide range of scenarios, including MAR models with positivity violations and MNAR models with and without positivity violations.

7 Discussion And Related Work

In this paper, we have added to the identification and imputation literature on missing data models, specifically those that factorize according to missing data graphs.

From an identification standpoint, our work is most similar to the identification methods proposed by Sadinle and Reiter (2017) and the pattern graph framework proposed by Chen (2022). Sadinle and Reiter (2017) provide a constructive argument for full law identification of the no self-censoring model (recall, a 3 variable version of this model is shown in Figure 1(c)) based on a log-linear parameterization of the full law. However, this constructive argument fails when there is positive support for only a subset of all missingness patterns, which we deal with in Section 4. Chen (2022) proposes a pattern graph representation of assumptions in missing data models, where the patterns with positive support are represented as vertices on a graph. The author then provides a factorization of the model with respect to the pattern graph itself. In contrast, the pattern graphs we propose here are not used to define the model, but rather encode identification assumptions implied by the factorization of an associated m-graph. That is, our use of pattern graphs still allows the analyst to specify assumptions in an intuitive manner based on missing edges in an m-graph. Another key difference is the restriction on edges in the two types of pattern graphs—in Chen (2022), if $\widetilde{r} \to r$ exists in $\mathcal{H}$, then it must be the case that $\mathbb{M}(r) \supset \mathbb{M}(\widetilde{r})$. This is not the case for us, as demonstrated in Figure 5(b), where the edge $(001) \to^3 (100)$ does not follow this rule.

From an imputation standpoint, our work is most similar to Kyono et al. (2021), Ren et al. (2023), and Karvanen and Tikka (2024). In Kyono et al. (2021) and Karvanen and Tikka (2024), the authors propose imputation methods for missing data models whose full laws are identified and can be represented as m-DAGs. Besides covering additional classes of m-graphs, our method is more general, even for m-DAGs. In Kyono et al. (2021), the missingness indicators in the m-DAG are not allowed to have any outgoing edges, while Karvanen and Tikka (2024) focus on m-DAGs that can be identified via a sequential procedure, i.e., models that can be identified via a total ordering. However, as we have seen in Section 3, partial orders play an important role for a sound and complete procedure, even for m-DAGs. Ren et al. (2023) propose a multiple imputation procedure for the no self-censoring model under certain parametric assumptions. Moreover, the identification step of their procedure requires full support on all possible missingness patterns, which we relax with PM-ID+ by leveraging other constraints that might be implied by the given m-graph. The authors do, however, provide more discussion on congenial specification of the imputing distributions in the no self-censoring model and ideas for sensitivity analysis to departures from the structural assumptions of the model. These contributions are complementary to our own work in this paper.

Finally we conclude with some brief remarks on how our algorithm differs from MICE (Raghunathan et al., 2001; Van Buuren, 2007). As we have seen in Section 6, MISPR outperforms MICE when the data are MNAR. The main reason for this is that MICE (and other similar multiple imputation algorithms) operates under a Missing At Random (MAR) assumption. As mentioned earlier, MICE also employs a Gibbs sampling algorithm when filling in missing values. Due to the MAR assumption, however, rather than borrowing information from different patterns according to a set of constraints encoded by a pattern DAG, MICE allows any pattern to borrow information from any other pattern when fitting the Gibbs factors. This would be valid for MAR models, but can fail for even simple

MNAR models, including some MNAR models where complete case analysis yields unbiased estimates of target parameters (Mathur and Shpitser, 2025).[10]

8 Conclusion

We have presented two constructive algorithms—PM-ID and PM-ID+—for identification of the full law in graphical models of missing data. PM-ID+ is able to identify full laws even when certain patterns of missingness have no support. An interesting open question is related to the completeness of PM-ID+: Is it complete for unrestricted target laws, and if not, what modifications are required for it be complete. Any improvements to these identification algorithms, including maximizing data efficiency when connecting patterns in the pattern DAG, would lead to immediate improvements for our imputation algorithm MISPR. Another interesting avenue for future research is to fully characterize the intersection between the pattern graph framework of Chen (2022) and the pattern graphs used in PM-ID and PM-ID+. This would lead to more intuitive representations of a wide class of pattern graphs as equivalent m-graphs.

Acknowledgements

Rohit Bhattacharya acknowledges support from the NSF CRII grant 234828. Ilya Shpitser acknowledges support from grants ONR N000142412701, NIH R01HS027819, NSF CAREER 1942239. The content of the information does not necessarily reflect the position or the policy of the Government, and no official endorsement should be inferred.

10. See also Mathur and Shpitser (2024) for graphical criteria for m-DAGs and m-ADMGs describing when MICE can work without modification.

References

M. J. Azur, E. A. Stuart, C. Frangakis, and P. J. Leaf. Multiple imputation by chained equations: What is it and how does it work? *International Journal of Methods in Psychiatric Research*, 20(1):40–49, 2011.

A. Berchtold. Treatment and reporting of item-level missing data in social science research. *International Journal of Social Research Methodology*, 22(5):431–439, 2019.

R. Bhattacharya, R. Nabi, I. Shpitser, and J. M. Robins. Identification in missing data models represented by directed acyclic graphs. In *Proceedings of the 35th Conference on Uncertainty in Artificial Intelligence*. AUAI Press, 2019.

R. Bhattacharya, D. Malinsky, and I. Shpitser. Causal inference under interference and network uncertainty. In *Uncertainty in Artificial Intelligence*, pages 1028–1038. PMLR, 2020.

H. Y. Chen. A semiparametric odds ratio model for measuring association. *Biometrics*, 63: 413–421, 2007.

J. M. Chen, D. Malinsky, and R. Bhattacharya. Causal inference with outcome-dependent missingness and self-censoring. In *Uncertainty in Artificial Intelligence*, pages 358–368. PMLR, 2023.

Y.-C. Chen. Pattern graphs: A graphical approach to nonmonotone missing data. *The Annals of Statistics*, 50(1):129–146, Feb. 2022. ISSN 0090-5364, 2168-8966. doi: 10.1214/21-AOS2094.

R. M. Daniel, M. G. Kenward, S. N. Cousens, and B. L. De Stavola. Using causal diagrams to guide analysis in missing data problems. *Statistical Methods in Medical Research*, 21 (3):243–256, 2012.

V. Didelez and M. J. Stensrud. On the logic of collapsibility for causal effect measures. *Biometrical Journal*, 64(2):235–242, 2022.

V. Didelez, S. Kreiner, and N. Keiding. Graphical models for inference under outcome-dependent sampling. 2010.

I. Eekhout, R. M. de Boer, J. W. Twisk, H. C. De Vet, and M. W. Heymans. Missing data: A systematic review of how they are reported and handled. *Epidemiology*, 23(5):729–732, 2012.

R. J. Evans. Margins of discrete Bayesian networks. *The Annals of Statistics*, 46(6A): 2623–2656, 2018.

A. Gain and I. Shpitser. Structure learning under missing data. In *International Conference on Probabilistic Graphical Models*, pages 121–132. Pmlr, 2018.

S. Geman and D. Geman. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Transactions on pattern analysis and machine intelligence*, (6):721–741, 1984.

J. Honaker, G. King, and M. Blackwell. Amelia II: A program for missing data. *Journal of Statistical Software*, 45:1–47, 2011.

J. Karvanen and S. Tikka. Multiple imputation and full law identifiability. *arXiv preprint arXiv:2410.18688*, 2024.

T. Kyono, Y. Zhang, A. Bellot, and M. van der Schaar. MIRACLE: Causally-aware imputation via learning missing data mechanisms. *Advances in Neural Information Processing Systems*, 34:23806–23817, 2021.

S. L. Lauritzen. *Graphical Models*, volume 17. Clarendon Press, 1996.

R. J. Little. Pattern-mixture models for multivariate incomplete data. *Journal of the American Statistical Association*, 88(421):125–134, 1993.

Y. Liu and A. C. Constantinou. Greedy structure learning from data that contain systematic missing values. *Machine Learning*, 111(10):3867–3896, 2022.

D. Malinsky, I. Shpitser, and E. J. Tchetgen Tchetgen. Semiparametric inference for non-monotone missing-not-at-random data: the no self-censoring model. *Journal of the American Statistical Association*, pages 1–9, 2021.

M. Mathur, T. Vanderweele, and I. Shpitser. A common-cause principle for eliminating selection bias in causal estimands through covariate adjustment. *Annals of Statistics*, 2025.

M. B. Mathur and I. Shpitser. Imputation without nightMARs: Graphical criteria for valid imputation of missing data. Technical report, Center for Open Science, 2024.

M. B. Mathur and I. Shpitser. Pitfalls of imputing using incomplete auxiliary variables. *American Journal of Epidemiology*, 194(6):1801–1802, 2025.

P. E. McKnight, K. M. McKnight, S. Sidani, and A. J. Figueredo. *Missing Data: A Gentle Introduction*. Guilford Press, 2007.

W. Miao, L. Liu, Y. Li, E. J. Tchetgen Tchetgen, and Z. Geng. Identification and semiparametric efficiency theory of nonignorable missing data with a shadow variable. *ACM/JMS Journal of Data Science*, 1(2):1–23, 2024.

K. Mohan and J. Pearl. On the testability of models with missing data. In *Artificial Intelligence and Statistics*, pages 643–650. PMLR, 2014.

K. Mohan and J. Pearl. Graphical models for processing missing data. *Journal of the American Statistical Association*, 116(534):1023–1037, 2021.

K. Mohan, J. Pearl, and J. Tian. Graphical models for inference with missing data. *Advances in Neural Information Processing Systems*, 26, 2013.

R. Nabi and R. Bhattacharya. On testability and goodness of fit tests in missing data models. In *Uncertainty in Artificial Intelligence*, pages 1467–1477. PMLR, 2023.

R. Nabi, R. Bhattacharya, and I. Shpitser. Full law identification in graphical models of missing data: Completeness results. In *Proceedings of the 37th International Conference on Machine Learning*, pages 7153–7163. PMLR, 2020.

R. Nabi, R. Bhattacharya, I. Shpitser, and J. Robins. Causal and counterfactual views of missing data models. *arXiv preprint arXiv:2210.05558*, 2022.

E. L. Ogburn, I. Shpitser, and Y. Lee. Causal inference, social networks and chain graphs. *Journal of the Royal Statistical Society Series A: Statistics in Society*, 183(4):1659–1676, 2020.

J. Pearl. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann, 1988.

J. Pearl. *Causality*. Cambridge University Press, 2009.

J. M. Peña. Unifying Gaussian LWF and AMP chain graphs to model interference. *Journal of Causal Inference*, 8(1):1–21, 2020.

N. J. Perkins, S. R. Cole, O. Harel, E. J. Tchetgen Tchetgen, B. Sun, E. M. Mitchell, and E. F. Schisterman. Principled approaches to missing data in epidemiologic studies. *American Journal of Epidemiology*, 187(3):568–575, 2018.

T. D. Pigott. A review of methods for missing data. *Educational Research and Evaluation*, 7(4):353–383, 2001.

T. E. Raghunathan, J. M. Lepkowski, J. Van Hoewyk, and P. Solenberger. A multivariate technique for multiply imputing missing values using a sequence of regression models. *Survey Methodology*, page 85, 2001.

B. Ren, S. R. Lipsitz, R. D. Weiss, and G. M. Fitzmaurice. Multiple imputation for non-monotone missing not at random data using the no self-censoring model. *Statistical Methods in Medical Research*, 32(10):1973–1993, 2023.

T. S. Richardson. Markov properties for acyclic directed mixed graphs. *Scandinavian Journal of Statistics*, 30(1):145–157, 2003.

T. S. Richardson, R. J. Evans, J. M. Robins, and I. Shpitser. Nested Markov properties for acyclic directed mixed graphs. *The Annals of Statistics*, 51(1):334–361, Feb. 2023. ISSN 0090-5364, 2168-8966. doi: 10.1214/22-AOS2253.

J. M. Robins and R. D. Gill. Non-response models for the analysis of non-monotone ignorable missing data. *Statistics in Medicine*, 16(1):39–56, 1997.

M. Saadati and J. Tian. Adjustment criteria for recovering causal effects from missing data. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 561–577. Springer, 2019.

M. Sadinle and J. P. Reiter. Itemwise conditionally independent nonresponse modelling for incomplete multivariate data. *Biometrika*, 104(1):207–220, 2017.

I. Shpitser. Consistent estimation of functions of data missing non-monotonically and not at random. *Advances in Neural Information Processing Systems*, 29:3144–3152, 2016.

I. Shpitser. The Lauritzen-Chen likelihood for graphical models. In *International Conference on Artificial Intelligence and Statistics*, pages 4181–4195. PMLR, 2023.

S. Sinharay, H. S. Stern, and D. Russell. The use of multiple imputation for the analysis of missing data. *Psychological methods*, 6(4):317, 2001.

P. L. Spirtes, C. N. Glymour, and R. Scheines. *Causation, Prediction, and Search*. MIT Press, 2000.

J. A. Sterne, I. R. White, J. B. Carlin, M. Spratt, P. Royston, M. G. Kenward, A. M. Wood, and J. R. Carpenter. Multiple imputation for missing data in epidemiological and clinical research: Potential and pitfalls. *Bmj*, 338, 2009.

Y. Sun, J. Li, Y. Xu, T. Zhang, and X. Wang. Deep learning versus conventional methods for missing data imputation: A review and comparative study. *Expert Systems with Applications*, 227:120201, 2023.

E. J. Tchetgen Tchetgen, I. R. Fulcher, and I. Shpitser. Auto-g-computation of causal effects on a network. *Journal of the American Statistical Association*, 116(534):833–844, 2021.

R. Tu, C. Zhang, P. Ackermann, K. Mohan, H. Kjellström, and K. Zhang. Causal discovery in the presence of missing data. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 1762–1770. PMLR, 2019.

S. Van Buuren. Multiple imputation of discrete and continuous data by fully conditional specification. *Statistical Methods in Medical Research*, 16(3):219–242, 2007.

S. Van Buuren and K. Groothuis-Oudshoorn. mice: Multivariate imputation by chained equations in R. *Journal of statistical software*, 45:1–67, 2011.

T. Verma and J. Pearl. Equivalence and synthesis of causal models. In *Proceedings of the Sixth Annual Conference on Uncertainty in Artificial Intelligence*, pages 255–270, 1990.

Appendix A. PM-ID When No Complete Cases Are Available

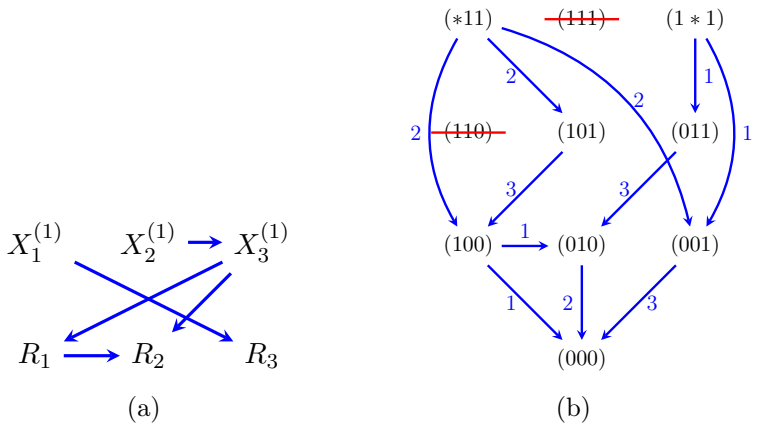


Figure 8: (a) An m-graph with a sparse structure for variables in $X^{(1)}$ used to demonstrate how identification can be achieved when there are no complete cases. For this example, we assume the patterns $\{111, 110\}$ have no support. (b) Pattern DAG used to identify the full law without using the complete case pattern. The marginals $p(X_{23}^{(1)}, R_2 = 1, R_3 = 1)$ and $p(X_{13}^{(1)}, R_1 = 1, R_3 = 1)$ are denoted as $(1 * 1)$ and $(*11)$, respectively.

Consider a missing data model that factorizes according to the m-graph in Figure 8(a) and has a positivity violation on one of the propensity scores: $p(R_2 = 1 \mid R_1 = 1, X_3^{(1)}) = 0$. This gives us a model that has no support for the complete case pattern 111 as well as the pattern 110. Thus, $\mathcal{R}^+ = \{011, 101, 001, 010, 100, 000\}$. Recall that PM-ID and PM-ID+ require the complete case pattern to be the root node of the pattern DAG, and serve as the base case for the identification argument in Theorem 2. Though the complete case pattern has no support here, we demonstrate that the recursive argument can still be applied by expanding the set of patterns considered to also include identified marginals of the full law.

In our problem, two marginals that are identified at the outset are,

$$p(X_{13}^{(1)}, R_1 = 1, R_3 = 1) \text{ and } p(X_{23}^{(1)}, R_2 = 1, R_3 = 1). \tag{9}$$

The marginal $p(X_{12}^{(1)}, R_1 = 1, R_2 = 1)$ involves a positivity violation as a direct consequence of there being no support on the patterns $\{111, 110\}$, and thus cannot be used. Nonetheless, we can extend the the recursive arguments of PM-ID and PM-ID+ by treating the identified marginals in (9) as two distinct root nodes of the pattern DAG.

Consider the pattern DAG in Figure 8(b), where the two marginals in (9) are set up as root nodes (the notation * is used to indicate variables that are marginalized.) Now, to achieve identification of other patterns, we use conditional independences in the target law implied by the m-graph shown in Figure 8(a). Recall that PM-ID used independences of the form $X_i^{(1)} \perp\!\!\!\perp R_i \mid \text{mb}_{\mathcal{G}}(X_i^{(1)})$, while PM-ID+ extended this to also use independences of the form $X_i^{(1)} \perp\!\!\!\perp R_j \mid \text{mb}_{\mathcal{G}}(X_i^{(1)})$ for $i \neq j$. An extension of these when there are no complete cases must also use independences of the form $X_i^{(1)} \perp\!\!\!\perp X_j^{(1)} \mid \text{mb}_{\mathcal{G}}(X_i^{(1)})$ for $i \neq j$ that arise due to missing edges among variables in $X^{(1)}$.

We identify the Gibbs factors of the extrapolation densities of the rest of the patterns by arguments encoded by the pattern DAG shown in Figure 8(b). Notably this pattern DAG uses the marginal $p(X_{23}^{(1)}, R_2 = 1, R_3 = 1)$ to identify Gibbs factors for the patterns $101, 100, 001$, and uses the marginal $p(X_{13}^{(1)}, R_1 = 1, R_3 = 1)$ to identify the Gibbs factors for the pattern $011, 001$. The equations in (10) make the arguments encoded in the pattern DAG more explicit, and also list the independences that lead to the equalities that permit identification. Such an algorithm might be labeled PM-ID*, by analogy with PM-ID+ due to its usage of margins of the full data law that have support as root nodes.[11]

$$
\begin{aligned}
&(011): \\
&\quad p(X_1^{(1)} \mid X_{23}^{(1)}, R = 011) = p(X_1^{(1)} \mid R_1 = 1, R_3 = 1) && (X_1^{(1)} \perp\!\!\!\perp R_{12}, X_{23}^{(1)} \mid R_3) \\
&(101): \\
&\quad p(X_2^{(1)} \mid X_{13}^{(1)}, R = 101) = p(X_2^{(1)} \mid X_3^{(1)}, R_2 = 1, R_3 = 1) && (X_2^{(1)} \perp\!\!\!\perp R, X_1^{(1)} \mid X_3^{(1)}) \\
&(001): \\
&\quad p(X_1^{(1)} \mid X_{23}^{(1)}, R = 001) = p(X_1^{(1)} \mid R_1 = 1, R_3 = 1) && (X_1^{(1)} \perp\!\!\!\perp R_{12}, X_{23}^{(1)} \mid R_3) \\
&\quad p(X_2^{(1)} \mid X_{13}^{(1)}, R = 001) = p(X_2^{(1)} \mid X_3^{(1)}, R_2 = 1, R_3 = 1) && (X_2^{(1)} \perp\!\!\!\perp R, X_1^{(1)} \mid X_3^{(1)}) \\
&(100): \\
&\quad p(X_2^{(1)} \mid X_{13}^{(1)}, R = 100) = p(X_2^{(1)} \mid X_3^{(1)}, R_2 = 1, R_3 = 1) && (X_2^{(1)} \perp\!\!\!\perp R, X_1^{(1)} \mid X_3^{(1)}) \\
&\quad p(X_3^{(1)} \mid X_{12}^{(1)}, R = 100) = p(X_3^{(1)} \mid X_{12}^{(1)}, R = 101) && (X_3^{(1)} \perp\!\!\!\perp R_3 \mid X_{12}^{(1)}, R_{12}) \\
&(010): \\
&\quad p(X_1^{(1)} \mid X_{23}^{(1)}, R = 010) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 100) && (X_1^{(1)} \perp\!\!\!\perp R_{12} \mid X_{23}^{(1)}, R_3) \\
&\quad p(X_3^{(1)} \mid X_{12}^{(1)}, R = 010) = p(X_3^{(1)} \mid X_{12}^{(1)}, R = 011) && (X_3^{(1)} \perp\!\!\!\perp R_3 \mid X_{12}^{(1)}, R_{12}) \\
&(000): \\
&\quad p(X_1^{(1)} \mid X_{23}^{(1)}, R = 000) = p(X_1^{(1)} \mid X_{23}^{(1)}, R = 100) && (X_1^{(1)} \perp\!\!\!\perp R_1 \mid X_{23}^{(1)}, R_{23}) \\
&\quad p(X_2^{(1)} \mid X_{13}^{(1)}, R = 000) = p(X_2^{(1)} \mid X_{13}^{(1)}, R = 010) && (X_2^{(1)} \perp\!\!\!\perp R_2 \mid X_{13}^{(1)}, R_{13}) \\
&\quad p(X_3^{(1)} \mid X_{12}^{(1)}, R = 000) = p(X_3^{(1)} \mid X_{12}^{(1)}, R = 001) && (X_3^{(1)} \perp\!\!\!\perp R_3 \mid X_{12}^{(1)}, R_{12})
\end{aligned}
\tag{10}
$$

In conclusion, the full law is identified.

11. In theory, PM-ID* exhausts the capability of the pattern mixture method employed in this paper. The other independence statements we have not used is the local Markov property relative to indicators R, $R_i \perp\!\!\!\perp V \setminus \mathrm{mb}_{\mathcal{G}}(R_i) \mid \mathrm{mb}_{\mathcal{G}}(R_i)$. We expect that a sound and complete identification theory for the class of models we explore under positivity violation will make use of both R_i and $X_i^{(1)}$ local Markov property.

Appendix B. Data Generating Processes For Experiments

In our experiment, all variables are binary. The following tables give the values of $p(V = 0 \mid W)$, where $V = 0$ is the name of the last column, and W are all the variables whose values are given in the other columns.

DGPs for graph (a)

- With positivity

$p(O = 0) = 0.660453$

	$X1 = 0$
O	
0	0.786893
1	0.637722

		$X2 = 0$
$X1$	O	
0	0	0.513861
	1	0.429143
1	0	0.664446
	1	0.149863

	$R1 = 0$
O	
0	0.883374
1	0.481486

		$R2 = 0$
O	$R1$	
0	0	0.891517
	1	0.310271
1	0	0.530085
	1	0.470104

- With positivity violation

$p(O = 0) = 0.512816$

	$X1 = 0$
O	
0	0.883374
1	0.481486

		$X2 = 0$
$X1$	O	
0	0	0.891517
	1	0.310271
1	0	0.530085
	1	0.470104

	$R1 = 0$
O	
0	0.970209
1	0.049576

		$R2 = 0$
O	$R1$	
0	0	0.430232
	1	0
1	0	0.713360
	1	0

DGPs for graph (b)

- With positivity

$p(X1 = 0) = 0.660453$

	$X2 = 0$
$X1$	
0	0.786893
1	0.637722

	$R1 = 0$
$X2$	
0	0.436589
1	0.920788

	$R2 = 0$
$X1$	
0	0.883374
1	0.481486

- With positivity violation

$p(X1 = 0) = 0.512816$

	$X2 = 0$
$X1$	
0	0.883374
1	0.481486

	$R1 = 0$
$X2$	
0	0.474077
1	0.386253

	$R2 = 0$
$X1$	
0	0.0
1	0.0

DGPs for graph (c)

- With positivity

$p(X_1^{(1)} = 0) = 0.512816$

	$X2 = 0$
$X1$	
0	0.883374
1	0.481486

		$X3 = 0$
$X1$	$X2$	
0	0	0.891517
	1	0.310271
1	0	0.530085
	1	0.470104

		$R1 = 0$
$X2$	$X3$	
0	0	0.526284
	1	0.163048
1	0	0.025951
	1	0.760911

		$R2 = 0$
$X3$	$R1$	
0	0	0.430232
	1	0.231407
1	0	0.713360
	1	0.741450

		$R3 = 0$
$X1$	$X2$	
0	0	0.529031
	1	0.404097
1	0	0.489594
	1	0.577123

- With positivity violation

$p(X_1^{(1)} = 0) = 0.937111$

	$X2 = 0$
$X1$	
0	0.231984
1	0.210310

		$X3 = 0$
$X1$	$X2$	
0	0	0.529031
	1	0.404097
1	0	0.489594
	1	0.577123

		$R1 = 0$
$X2$	$X3$	
0	0	0.492275
	1	0.476198
1	0	0.302251
	1	0.444998

		$R2 = 0$
$X3$	$R1$	
0	0	0.434144
	1	0
1	0	0.698981
	1	0

		$R3 = 0$
$X1$	$X2$	
0	0	0.372718
	1	0.408374
1	0	0.465549
	1	0.568108

DGPs for graph (d)

- With positivity

$p(U_1 = 0) = 0.412899$

$p(U_2 = 0) = 0.340442$

	$X_1^{(1)}$
U_1	
0	0.351195
1	0.818716

	$X_2^{(1)}$
$X_1^{(1)}$	
0	0.589324
1	0.652324

		$X_3^{(1)}$
$X_2^{(1)}$	U_1	
0	0	0.296391
	1	0.484488
1	0	0.620866
	1	0.559600

			$X_4^{(1)}$
$X_3^{(1)}$	$X_2^{(1)}$	$X_1^{(1)}$	
0	0	0	0.357485
		1	0.420410
	1	0	0.235693
		1	0.545312
1	0	0	0.316772
		1	0.514750
	1	0	0.859519
		1	0.659134

			R_1
$X_3^{(1)}$	$X_4^{(1)}$	U_2	
0	0	0	0.547038
		1	0.556173
	1	0	0.091318
		1	0.271319
1	0	0	0.711250
		1	0.343158
	1	0	0.261107
		1	0.789046

				R_2
$X_3^{(1)}$	$X_4^{(1)}$	R_1	U_2	
0	0	0	0	0.634742
			1	0.570906
		1	0	0.467930
			1	0.704905
	1	0	0	0.845492
			1	0.510379
		1	0	0.861527
			1	0.829096
1	0	0	0	0.465644
			1	0.356262
		1	0	0.197143
			1	0.285206
	1	0	0	0.437094
			1	0.256127
		1	0	0.476503
			1	0.590885

			R_3
$X_1^{(1)}$	R_2	R_4	
0	0	0	0.522999
		1	0.371357
	1	0	0.141733
		1	0.425316
1	0	0	0.445790
		1	0.538798
	1	0	0.424396
		1	0.462601

	R_4
$X_3^{(1)}$	
0	0.540869
1	0.181146

- With positivity violation

$p(U_1 = 0) = 0.743035$

$p(U_2 = 0) = 0.390206$

	$X_1^{(1)} = 0$
U_1	
0	0.821949
1	0.760629

	$X_2^{(1)} = 0$
$X_1^{(1)}$	
0	0.830070
1	0.952262

		$X_3^{(1)} = 0$
$X_2^{(1)}$	U_1	
0	0	0.493445
	1	0.756955
1	0	0.428295
	1	0.781199

			$X_4^{(1)} = 0$
$X_3^{(1)}$	$X_2^{(1)}$	$X_1^{(1)}$	
0	0	0	0.522999
		1	0.371357
	1	0	0.141733
		1	0.425316
1	0	0	0.445790
		1	0.538798
	1	0	0.424396
		1	0.462601

			$R_1 = 0$
$X_3^{(1)}$	$X_4^{(1)}$	U_2	
0	0	0	0.503617
		1	0.365114
	1	0	0.597744
		1	0.541078
1	0	0	0.822804
		1	0.352675
	1	0	0.464298
		1	0.079114

				$R_2 = 0$
$X_3^{(1)}$	$X_4^{(1)}$	R_1	U_2	
0	0	0	0	0.394719
			1	0.116755
		1	0	0.704038
			1	0.299178
	1	0	0	0.534339
			1	0.441878
		1	0	0.016644
			1	0.482661
1	0	0	0	0.639980
			1	0.507167
		1	0	0.647185
			1	0.853005
	1	0	0	0.889070
			1	0.545897
		1	0	0.547497
			1	0.511166

			$R_3 = 0$
$X_1^{(1)}$	R_2	R_4	
0	0	0	0.978111
		1	0.294879
	1	0	0.000000
		1	0.590751
1	0	0	0.345917
		1	0.018613
	1	0	0.000000
		1	0.724987

	$R_4 = 0$
$X_3^{(1)}$	
0	0.250863
1	0.925975

DGPs for graph (e)

- With positivity violation

$p(X_1^{(1)} = 0) = 0.638138$

	$X2 = 0$
$X1$	
0	0.970209
1	0.049576

		$X3 = 0$
$X1$	$X2$	
0	0	0.430232
	1	0.231407
1	0	0.713360
	1	0.741450

			$X4 = 0$
$X1$	$X2$	$X3$	
0	0	0	0.583809
		1	0.440495
	1	0	0.548947
		1	0.491406
1	0	0	0.413344
		1	0.595662
	1	0	0.452291
		1	0.559238

				$X5 = 0$
$X1$	$X2$	$X3$	$X4$	
0	0	0	0	0.931621
			1	0.423657
		1	0	0.411324
			1	0.656003
	1	0	0	0.787735
			1	0.091550
		1	0	0.988887
			1	0.810696
1	0	0	0	0.444835
			1	0.609901
		1	0	0.637284
			1	0.422973
	1	0	0	0.934693
			1	0.069927
		1	0	0.527842
			1	0.462012

				$R1 = 0$
$X2$	$X3$	$X4$	$X5$	
0	0	0	0	0.705047
			1	0.513365
		1	0	0.340205
			1	0.348838
	1	0	0	0.613077
			1	0.839110
		1	0	0.266611
			1	0.884113
1	0	0	0	0.404455
			1	0.158130
		1	0	0.442600
			1	0.523876
	1	0	0	0.431031
			1	0.541048
		1	0	0.526709
			1	0.598381

				$R2 = 0$
$X3$	$X4$	$X5$	$R1$	
0	0	0	0	0
			1	0.456267
		1	0	0
			1	0.694902
	1	0	0	0
			1	0.531102
		1	0	0
			1	0.532139
1	0	0	0	0
			1	0.229811
		1	0	0
			1	0.358095
	1	0	0	0
			1	0.519177
		1	0	0
			1	0.828434

				$R3 = 0$
$X1$	$X4$	$X5$	$R2$	
0	0	0	0	0.950850
			1	0.807773
		1	0	0.331640
			1	0.940146
	1	0	0	0.435004
			1	0.033379
		1	0	0.959593
			1	0.907998
1	0	0	0	0.377597
			1	0.161617
		1	0	0.506291
			1	0.479214
	1	0	0	0.306019
			1	0.745440
		1	0	0.944262
			1	0.499467

				$R4 = 0$
$X2$	$X5$	$R1$	$R3$	
0	0	0	0	0.576291
			1	0.483073
		1	0	0.973214
			1	0.618791
	1	0	0	0.360446
			1	0.633832
		1	0	0.084281
			1	0.900471
1	0	0	0	0.630489
			1	0.519740
		1	0	0.687443
			1	0.545025
	1	0	0	0.298542
			1	0.209205
		1	0	0.124970
			1	0.887454

				$R5 = 0$
$X1$	$X3$	$R2$	$R4$	
0	0	0	0	0.816639
			1	0.427999
		1	0	0.212742
			1	0.277295
	1	0	0	0.516575
			1	0.263809
		1	0	0.776928
			1	0.930711
1	0	0	0	0.816094
			1	0.577293
		1	0	0.908045
			1	0.823171
	1	0	0	0.240437
			1	0.678328
		1	0	0.417560
			1	0.742792

DGPs for graph (f) We randomly choose a DGP as follow

1. For each $i = 1, \ldots, 20$, sample random numbers in $(0, 1)$ and assign to $p(X_i^{(1)} = 0 \mid \text{pa}_{\mathcal{G}}(X_i^{(1)}))$.
2. For each $i = 1, \ldots, 20$, we assign values to $p(R_i = 0 \mid \text{pa}_{\mathcal{G}}(R_i))$ as follow. Note that $R_{i+k} \in \text{pa}_{\mathcal{G}}(R_i)$ for all $i = 1, \ldots, 20$ and $k = 0, \ldots, 10$ such that $i + k \leq 20$.
   (a) If $R_{i+k} = 0$ for any $k \geq 3$, then $p(R_i = 0 \mid \text{pa}_{\mathcal{G}}(R_i)) = 0$.
   (b) If $R_{i+1} = 0$ and/or $R_{i+2} = 0$, then sample random numbers in $(0, 0.5)$, then multiply by 1.8, and assign to the corresponding $p(R_i = 0 \mid \text{pa}_{\mathcal{G}}(R_i))$. The goal is to create values higher than 0.5 when $R_{i+1} = 0$ and/or $R_{i+2} = 0$.
   (c) Otherwise, sample random numbers in $(0, 1)$ and assign to $p(R_i = 0 \mid \text{pa}_{\mathcal{G}}(R_i))$.