

STROKE CLASSIFICATION USING VIRTUAL HYBRID EDGE DETECTION FROM *IN SILICO* ELECTRICAL IMPEDANCE TOMOGRAPHY DATA

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(Communicated by Faouzi Triki)

ABSTRACT. Electrical impedance tomography (EIT) combined with machine learning has shown promise for the classification of strokes. However, most previous works have used raw EIT voltage data as network inputs. We build upon a recent development which suggested the use of a special noise-robust virtual hybrid edge detection (VHED) functions computed from EIT data as network inputs, although that work used only simplified and mathematically ideal models. In this work, we design models with high detail and mathematical realism. Virtual patients are created using a physically detailed 2D head model which includes features known to create challenges in real-world imaging scenarios. Simulated noisy EIT electrode data, generated using the realistic complete electrode model as opposed to the mathematically ideal continuum model, is processed to obtain VHED functions. We compare the use of VHED functions as inputs against the alternative paradigm of using raw EIT voltages. Our results show that (i) stroke classification can be performed with high accuracy using 2D EIT data from physically detailed and mathematically realistic models, and (ii) in the presence of noise, VHED functions outperform raw data as network inputs.

1. Introduction. Stroke is a serious medical condition in which blood supply to part of the brain is blocked or significantly decreased. This prevents oxygen and other nutrients from reaching that part of the brain. Strokes are mainly classified into two groups: ischemic and hemorrhagic [15]. Ischemic stroke occurs when brain blood vessels become narrowed or blocked, impeding part of the brain from receiving blood. Hemorrhagic stroke occurs when a blood vessel ruptures or leaks, releasing blood inside the brain.

2020 *Mathematics Subject Classification.* Primary: 35R30; Secondary: 65N21, 68T07.

Key words and phrases. Electrical impedance tomography, complete electrode model, virtual hybrid edge detection, neural networks, stroke classification.

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Brain cells start to die in minutes without proper blood supply, and therefore stroke is an emergency situation that requires quick medical intervention to reduce brain damage and improve the prognosis of the patient [23, 33, 34]. The treatment depends on the type of stroke. Ischemic strokes are treated with tissue plasminogen activator medication to dissolve the clots, or, in some cases, thrombectomy to mechanically remove the clots from the affected blood vessel [6, 16, 24]. Hemorrhagic strokes, on the other hand, require medicine to reduce blood pressure and, if possible, a medical procedure to control the bleed, such as aneurysm clipping, coiling, or AVM endovascular embolization [13, 25, 31]. The medicine administered to treat one type of stroke causes damaging effects in the other type. That is, plasminogen activator medication (for ischemic treatment) causes reduction in coagulation capability, which is a serious issue for the hemorrhagic stroke patient, while medicine to reduce blood pressure (for hemorrhagic treatment) increases obstruction problems in the ischemic stroke patient.

Because of the antagonistic effects of the medicines, early diagnosis is key, as diagnosis must be made prior to beginning treatment. Unfortunately, ischemic and hemorrhagic strokes present very similar symptoms in visual clinical inspection [29]. At the time of this writing, X-ray computed tomography (CT) and magnetic resonance imaging (MRI) are the brain imaging techniques used to make a diagnosis, and therefore diagnosis can only be performed after transporting a patient to the hospital [10, 30, 32]. If a portable diagnostic method were developed, however, the necessary diagnostic equipment could be carried in the ambulance to permit earlier treatment, which could greatly improve patient outcomes.

Electrical impedance tomography (EIT) is a relatively new tomographic imaging technique which is a good candidate for this purpose, since EIT devices are portable and do not require radiation shielding or other bulky and expensive setups. EIT generates tomographic images of the electrical conductivity distribution on a target by measuring the electrical potential on its external surface when an electrical current is applied to the target [11].

Previous works. Several machine learning-based approaches have been proposed for the classification of strokes using EIT data, but these approaches have all considered learning directly from measured EIT voltage data.

In [27] a support vector machine (SVM) was applied directly to simulated EIT voltage measurements to detect the presence or absence of a hemorrhage in the brain. A 3D model was considered, and a ring of electrodes was positioned in the exterior of the model. The highly simplified head model consisted of only 2 layers: an inner brain layer, and an outer layer modeling tissues external to the brain. Therefore, the important effects produced by the highly resistive skull and the highly conductive cerebrospinal fluid (CSF) were not considered. The impact of noise, lesion location and size, electrode positioning, and different anatomies of the head were considered to test the methodology. A similar approach was presented in [14], but considered a more realistic 3D head model consisting of 4 layers. This model consists of a first layer that aggregates the scalp and skull layers, followed by separate layers representing the CSF, brain tissue (aggregating gray and white matter), and, finally, the ventricle layer. Additionally, the study also explored the impact of four different measurement frequencies on stroke identification and differentiation using SVM. Multi-frequency symmetry difference EIT combined with SVM was explored in [26]. This study employed a 3D head model comprising scalp,

skull, CSF, and brain layers. The methodology was applied to both simulated data and a limited set of human data (35 individuals).

In [35], a residual neural network was proposed for the classification of strokes in different regions of the head, again using minimally processed simulated EIT voltage measurements as inputs. A simplified 2D head model consisting of three layers representing the scalp, skull, and parenchyma (brain) was used. However, the highly conductive CSF and other anatomical tissues were not considered in this model. The performance of the method was evaluated when the data was corrupted by noise, when shape deformations occurred, and when the conductivity varied in the three layers of the model. However, these conditions were considered individually, while in real-world scenarios these variations would be simultaneous. Furthermore, in that study, difference EIT data was considered as an input to the network, so the proposed method requires healthy data that may not be available in practice. In [9], fully connected neural networks (FCNN) and convolutional neural networks (CNN) were employed for detection of brain hemorrhages from simulated absolute EIT data on a 3D head model, again using raw voltages as network inputs. The model in that work was also a simplified 3-layer version of the human head consisting of scalp, skull, and brain, again neglecting the conductive cerebrospinal fluid and other anatomical tissues. The performance of the two neural networks were tested using several sets of unseen data, and FCNN showed better generalization capabilities than CNN.

A novel methodology connecting EIT and X-ray tomography, based upon deep microlocal analysis, was introduced in [21] and recently analyzed in more detail in [2]. Using this new methodology, EIT measurements can be processed in a non-linear manner, and relevant geometric conductivity information concealed within these measurements can be extracted. Moreover, this information hidden within EIT data possesses the same linear geometry as data produced by parallel-beam computed tomography. These “virtual” X-ray attenuation profiles are called virtual hybrid edge detection (VHED) functions.

VHED was connected with neural network models in [1] for classifying strokes as ischemic or hemorrhagic. In that introductory work, VHED functions were computed from EIT data generated by applying the unrealistic mathematically ideal continuum model of EIT to highly simplified head phantoms on circular domains. The computational evidence presented there strongly suggests that (i) when applying machine learning to severely ill-posed problems, it pays to extract noise-robust features from the data to use as inputs for machine learning, as opposed to learning directly from measured data, and (ii) VHED profiles are useful features for classifying stroke, at least with data simulated from highly mathematically and physically unrealistic two-dimensional models. Although the results in that initial work demonstrated the potential advantages of using VHED functions in stroke classification, it was left to future work to determine whether the method would prove robust under more realistic conditions.

Contributions of the work. In this paper, we introduce several points of novelty in the computation of the VHED functions developed in [21], using more realistic domains and models. We also make a strengthened case for using VHED functions as inputs for the stroke classification problem. In particular, we continue the research line introduced in [1] and extend the results to a scenario which is much more mathematically realistic and physically detailed than was employed in that earlier work: (i) To simulate the virtual stroke patients, we consider phantoms which are

more general than the unit disc, specifically, two-dimensional anatomical models resembling a slice of a human head with detailed, physically realistic electrophysiology and a statistically realistic conductivity distribution, and (ii) instead of using the unrealistic continuum model for EIT measurements, our new method uses the realistic complete electrode model (CEM) measurements as its starting point.

Since the theory of VHED functions is based on measurements associated with the idealized continuum model, to compute the VHED functions from CEM measurements, we first need to mimic continuous measurements from data originated using CEM. For this step, we follow the theory presented in [20] and implement an algorithm that allows us to approximate unrealistic continuous measurements in two-dimensional EIT based on realistic electrode measurements. In the recent works which have developed the new field of VHED theory, the underlying analysis has been developed on the unit disk, and continuum EIT measurements have therefore been considered only on the unit circle. To handle the extension from the unit disk to more realistic non-circular domains and to electrode measurements, we again follow [20] and introduce a conformal mapping procedure between the unit circle and the true physical domain boundary. In this work, we consider the physically realistic scenario where a fixed helmet is placed upon the patient's head during data collection, so the domain boundary is taken to be an ellipse. This work represents the first-ever computation of VHED functions on a noncircular domain.

Structure of the work. This paper is organized as follows. In Section 2, we present the computation of the VHED functions from EIT data using the continuum model and then the extension in the case of the CEM. Also, we explain our simulation of virtual patients, introduce the neural network architecture, and describe the stroke classification problem. Then, in Section 3 the performance of the proposed methodology is evaluated using various case studies, and is compared against the case where raw EIT voltages are used as network inputs. Conclusions are given in Section 4. In Appendix, A we present a detailed explanation of how to compute a matrix approximation of the DN map associated with the idealized continuum mode but using measurements from the realistic complete electrode model.

2. Methods. In this section, we present how the VHED functions are computed from realistic EIT electrode measurements, the creation of virtual patients with enhanced physiological detail, and the stroke classification method.

2.1. From EIT data to VHED functions.

2.1.1. EIT conductivity equation. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, and simply connected domain (target), let $\sigma : \Omega \rightarrow (0, \infty)$ be an essentially bounded measurable function (conductivity) satisfying $\sigma(x) \geq c > 0$ for almost every $x \in \Omega$, let $u \in H^1(\Omega)$ be the voltage in the target, and let $\phi \in H^{1/2}(\partial\Omega)$ be the voltage at the boundary of the target. Then, the following conductivity equation holds [36]:

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega \quad (1)$$

$$u = \phi \quad \text{on } \partial\Omega, \quad (2)$$

and the associated Dirichlet-to-Neumann (DN) map $\Lambda_\sigma : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ is defined as

$$\Lambda_\sigma : \phi \mapsto \sigma \frac{\partial u}{\partial \nu}, \quad (3)$$

where ν denotes the unit outer normal to the boundary. EIT images are generated by solving the inverse conductivity problem: Find σ from the knowledge of Λ_σ [8, 5].

2.1.2. *From continuum data to VHED profiles.* There are many theoretical tools which have been developed to solve the conductivity equation (1), but here we focus on a special class of solutions called complex geometrical optics (CGO) solutions. These solutions are characterized by an exponential asymptotic behavior at infinity and provide the theoretical framework to introduce the VHED functions. From now on, we assume that $\sigma = 1$ outside of an open set Ω_0 satisfying $\overline{\Omega_0} \subset \Omega$.

In [5], the authors presented a way to construct the CGO solutions by reducing the conductivity equation (1) to the following Beltrami equation:

$$\overline{\partial}_z f_{\pm\mu} = \pm \mu \overline{\partial}_z f_{\pm\mu}, \quad (4)$$

with Beltrami coefficient $\mu := (1 - \sigma)/(1 + \sigma)$. The CGO solutions have the form

$$f_{\pm\mu}(z, k) = e^{ikz}(1 + \omega^\pm(z, k)),$$

with

$$\omega^\pm(z, k) = \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } |z| \rightarrow \infty,$$

where $ikz = ik(x_1 + ix_2)$, and where z is considered a spatial variable and k a spectral parameter. Existence, uniqueness, and properties of the CGO solutions are discussed in [5].

Note that, in order to compute $f_{\pm\mu}$, we need the coefficient μ which is computed from the conductivity σ . However, if we denote $W_\pm(z, k) = 1 + \omega^\pm(z, k)$, by [4], the following boundary integral equation (BIE) holds:

$$W_\pm(\cdot, k)|_{\partial\Omega} + 1 = (\mathcal{P}_\pm^k + \mathcal{P}_0)W_\pm(\cdot, k)|_{\partial\Omega}, \quad (5)$$

where $\mathcal{P}_\pm^k$ and $\mathcal{P}_0$ are certain projection operators that can be computed from Λ_σ . Therefore, we can compute the traces of the CGO functions $\omega^\pm$ directly from Λ_σ by solving (5) and without knowledge of μ .

In [21], a novel method connecting EIT and X-ray tomography was introduced. This method converts information contained in the ideal continuum DN map Λ_σ into virtual X-ray profiles known as VHED functions, which contain specific geometric information about the conductivity. In this methodology, the Beltrami equation (4) is treated as a scattering equation, and the CGO function $\omega^\pm$ can be expanded as follows:

$$\omega^\pm(z, k) = \sum_{n=1}^{\infty} \omega_n^\pm(z, k).$$

The computation of the VHED functions from the traces of the CGO functions $\omega^\pm$ consists of the following steps:

- (i) For any $\tau \in \mathbb{R}$ and $\varphi \in [0, 2\pi]$, write the spectral variable of CGO solutions as $k = \tau e^{i\varphi}$. This is similar to polar coordinates, but not quite so as τ can be negative.
- (ii) Compute the partial Fourier transform in τ ,

$$\widehat{\omega}_n^\pm(z, t, \varphi) := \mathcal{F}_{\tau \rightarrow t}(\omega_n^\pm(z, \tau e^{i\varphi})).$$

- (iii) Compute the averaged operators $T_n^\pm \mu$ given by the complex contour integrals

$$T_n^\pm \mu(t, \varphi) := \frac{1}{2\pi i} \int_{\partial\Omega} \widehat{\omega}_n^\pm(z, t, \varphi) dz. \quad (6)$$

Note that in step (ii), we have introduced a nonphysical artificial (i.e., virtual) variable t , known as the *pseudo-time*. Figure 1 depicts the resulting virtual X-ray

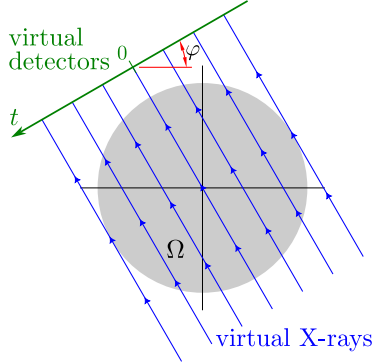


FIGURE 1. Parallel VHED X-rays and virtual detector geometry.

geometry. The angle φ represents the direction perpendicular to the virtual X-rays (blue) while t represents the position of a virtual X-ray detector, along the perpendicular direction (green).

This step permits that singularities from the interior of Ω propagate to the boundary $\partial\Omega$. Therefore, interior singularities of the unknown conductivity σ can be detected using boundary measurements, see Figure 2(a).

Based on [21], the leading-order term $\omega_1^\pm$ contains relevant information that allows detection of edges and singularities of the coefficient μ , and thus of the conductivity σ . In fact, in [21] it is shown that the following filtered back-projection formula holds:

$$(-\Delta)^{-1/2}(T_1^\pm)^*T_1^\pm\mu = \mu$$

and

$$T_1^\pm\mu(t, \varphi) = -2\pi i e^{i\varphi} \left(\frac{\partial}{\partial s} R\mu_\pm \right) \left(\frac{1}{2}t, e^{i\varphi} \right),$$

where R is the standard Radon transform.

Unfortunately, from the DN map, we can only compute the boundary trace of the full scattering series $\omega^\pm$. As we mention above, the first-order term $\omega_1^\pm$ constitutes the basis for an accurate reconstruction of the unknown conductivity σ , while the higher-order terms $\omega_n^\pm$ are related to scattering effects and these terms produce artifacts in the numerical simulations; see Figure 2(a). However, by a parity symmetry property described in [21], subtracting $\hat{\omega}^-$ from $\hat{\omega}^+$ eliminates the even terms, ω_{2n}^+ , so that their singularities can be suppressed; see Figure 2(b). In this study, we will work with the following VHED function:

$$T_{\text{odd}}\mu(t, \varphi) := \frac{T^+\mu(t, \varphi) - T^-\mu(t, \varphi)}{2},$$

where $T^\pm\mu$ denotes the average operator (6) applied to $\hat{\omega}^\pm$, respectively; see Figure 2(c).

2.1.3. From electrode data to VHED profiles. In order to compute the VHED functions, we need the DN map Λ_σ associated with the continuum model (1)-(2). However, in practical applications, a finite number of electrodes are attached at the boundary of the body. The resulting EIT measurement data comprise pairs of voltage and current measurements recorded at these electrodes. The most realistic

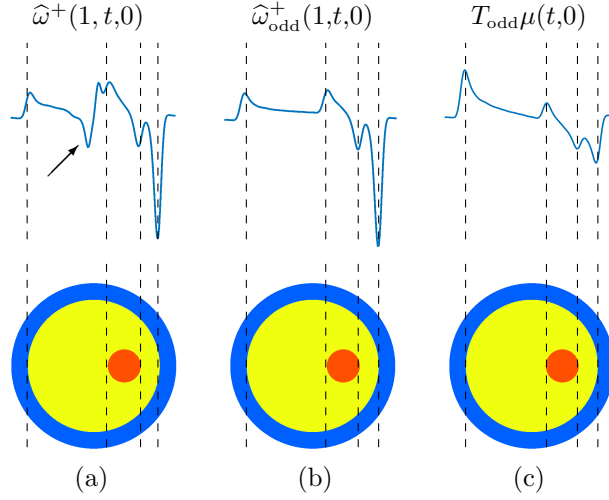


FIGURE 2. VHED functions revealing geometric information about the conductivity. Jump singularities in the conductivity are reflected in the VHED profiles in a similar manner to parallel beam X-ray tomography. The angle $\varphi = 0$ indicates the direction perpendicular to the virtual X-rays (dashed lines). (a) Function $\hat{\omega}^+(1, t, 0)$ propagates singularities from the interior of the domain to its boundary. Note that an artifact is visible at $t = 0$ (indicated by an arrow). (b) Function $\hat{\omega}_{\text{odd}}^+ := (\hat{\omega}^+ - \hat{\omega}^-)/2$. Subtracting $\hat{\omega}^-$ from $\hat{\omega}^+$ eliminates the even terms and suppresses the artifact at $t = 0$. Nevertheless, because of the election $z = 1$, singularities closer to the right edge have a larger amplitude than the singularities closer to the left edge. (c) After applying the averaging operator, the VHED profile $T_{\text{odd}}\mu$ shows equal-size (but opposite sign) singularities at the leftmost and rightmost vertical lines.

mathematical model for simulating electrode measurements is the *complete electrode model* (CEM) [36].

Let $e_m \subset \partial\Omega$, $m = 1, \dots, M$, be disjoint, open, and connected sets modeling the electrode patches. The CEM is given by the conductivity equation (1) and the following boundary conditions:

$$\begin{aligned} z_m \sigma \frac{\partial u}{\partial \nu} + u &= U_m, & \text{on } e_m, \quad m = 1, \dots, M \\ \sigma \frac{\partial u}{\partial \nu} &= 0, & \text{on } \partial\Omega \setminus \bigcup_{m=1}^M e_m \\ \int_{e_m} \sigma \frac{\partial u}{\partial \nu} ds &= I_m, & m = 1, \dots, M, \end{aligned}$$

where z_m denotes the contact impedance at electrode e_m , U_m represents the constant electric potential on electrode e_m , I_m is the net current through the electrodes, and the normal current density outside the electrodes is zero.

The *absolute* measurements of the CEM are modeled by the electrode current-to-voltage map $\mathcal{R}_\sigma^{\text{CEM}} : \mathbb{R}_\diamond^M \rightarrow \mathbb{R}_\diamond^M$ given by

$$\mathcal{R}_\sigma^{\text{CEM}} : I \mapsto U, \quad (7)$$

where $\mathbb{R}_\diamond^M$ denotes the mean-free subspace of $\mathbb{R}^M$. Imposing the current conservation law $\sum_{m=1}^M I_m = 0$ and fixing the ground level of the potentials $\sum_{m=1}^M U_m = 0$ guarantees the existence and uniqueness of the solution (u, U) of the CEM, where $u \in H^1(\Omega)$ and $U = (U_1, \dots, U_M)^T \in \mathbb{R}_\diamond^M$; for details see [36].

In [20], a method for approximating relative continuum EIT data based on measurement data from the CEM was presented. Given a bounded simply-connected domain $\Omega \subset \mathbb{R}^2$ and a set of continuum current patterns $\{\phi_k\}_{k=1}^{M-1}$ that one would like to apply along the boundary $\partial\Omega$, the proposed method tells us where the electrodes, e_m , should be located, based on the shape of Ω , and what the net electrode currents $\{I^{(k)}\}_{k=1}^{M-1}$ should be depending on the continuum current patterns.

The method is first developed for the unit disc $D \subset \mathbb{R}^2$, and is based on accurate interpolation and quadrature rules on the boundary ∂D for certain Sobolev spaces. These results can be then extended to more general domains Ω using a conformal bijective map $\Psi : D \rightarrow \partial\Omega$ whose restriction to ∂D defines a C^∞ -diffeomorphism between ∂D and $\partial\Omega$.

In Appendix A, we provide a detailed explanation of how to compute a matrix approximation $\mathbf{L}_\sigma$ of the DN map Λ_σ defined in (3), utilizing boundary measurements from the CEM model instead of the continuum model, and building upon the ideas developed in [20].

Once we have a matrix approximation $\mathbf{L}_\sigma$ of the DN map Λ_σ , we can use this to compute the traces of the CGO solutions by numerically solving the BIE (5). In practical applications, measurement noise in the DN matrix causes inaccuracy in the solution of (5). From simulations, one can observe that errors in the values of the CGO solutions $\omega^\pm(z, \tau e^{i\varphi})$ for $z \in \partial\Omega$ grow with $|\tau|$. Therefore, in order to mitigate inaccuracies when computing the VHED functions, we restrict the integration to the interval $[-\tau_m, \tau_m]$ (m for maximum value) and consider the following windowed Fourier transform

$$\tilde{\omega}^\pm(z, t, \varphi) := \int_{-\tau_m}^{\tau_m} W_{a_{\tau_m}}(\tau) e^{-i\tau t} \omega^\pm(z, \tau e^{i\varphi}) d\tau,$$

where $a_{\tau_m} > 0$ is a parameter such that $W_{a_{\tau_m}}(\tau) := e^{-a_{\tau_m}\tau^2}$ is very small outside the interval $[-\tau_m, \tau_m]$. Note that this Fourier windowing process in the frequency domain corresponds to convolution in the pseudo-time domain, that is,

$$\tilde{\omega}^\pm(z, t, \phi) = \widehat{W}_{a_{\tau_m}}(t) * \hat{\omega}^\pm(z, t, \phi).$$

Hence, due to the inevitable noise in the measurements, we are forced to work with blurred versions of the VHED functions as shown in Figure 3. It is clear from Figure 3 that if we could use unrealistic noise-free data, the classification of strokes would be an easy task, and could be performed without the aid of machine learning. However, when using realistic noisy data, this task becomes much more complex, and so we rely upon a neural network to accurately perform classification.

2.2. Simulation of stroke-EIT measurements. A statistical anatomical atlas of the electrical properties of the upper part of the human head was introduced in [28]. Based on this work, we consider a physiologically-detailed two-dimensional model of a slice of the human head, which we refer to as the reference head model. The head model, with dimensions 210 mm x 175 mm, comprises five different compartments of importance for the electrophysiology of the human head: grey matter, white matter, cerebrospinal fluid (CSF), skull, and other soft tissues representing the scalp: see Figure 4(a).

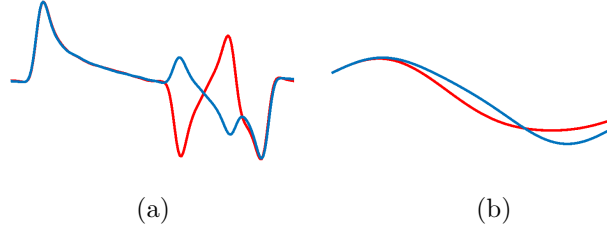


FIGURE 3. Effect of the Fourier windowing process applied to the VHED functions. The blue curve shows $T_{\text{odd}}\mu(t, 0)$ corresponding to conductivity in Figure 2 where the small disc inclusion has a conductivity value higher than background, while the red curve shows $T_{\text{odd}}\mu(t, 0)$ corresponding to the same conductivity but with the small disc inclusion having lower conductivity than the background. (a) Unrealistic VHED profiles with cutoff frequency $\tau_m = 50$, which would only be possible in a noise-free scenario. (b) Realistic VHED profiles considering a cutoff frequency $\tau_m = 5$, as would be appropriate in a real-world scenario with noise.

Recall that in order to compute the CGO solutions in an arbitrary domain $\Omega \subset \mathbb{R}^2$ using CEM measurements, the first step is to find a bijective conformal map Ψ from the unit disc D to Ω . If we took Ω to extend to the outer layer of the patient's skin, in practice this means that for each patient we would need to find a different conformal map, and this could be a difficult task. As a simple and physically realistic solution, we assume that a fixed elliptic belt or helmet with electrodes is placed on the head of the patient; see Figure 4(b). In this way, the boundary of the domain Ω is fixed, and only the inner compartments of the head model differ from patient to patient.

From the reference head model domain, we extracted a reference geometry for each compartment, and in order to simulate different patients, we randomly modified the reference geometry of the various compartments of the head model. Each compartment is randomly increased or reduced with respect to its reference size. This is achieved by multiplying the reference length, ℓ^i , of the i^{th} compartment by a scaling factor s^i drawn from a uniform distribution $\mathcal{U}(s_{\min}^i, s_{\max}^i)$. The corresponding values for each compartment are given in Table 1. Additionally, each region is randomly rotated around its reference axis by an angle α_i drawn from a uniform distribution $\mathcal{U}(-\alpha_{\max}^i, \alpha_{\max}^i)$, with the scalp being rotated first and its new central axis serving as a reference for the other compartments. The maximum rotation angle $\alpha_{\max}^i$ for each compartment is given in Table 1. Finally, the shape of each compartment also varies randomly. In the reference model, the boundary of the i^{th} region is defined by a set of points $\{(x_j^i, y_j^i)\}_{j=1}^{N_i}$. To generate a new patient model, a positive integer number F^i of Fourier basis functions is randomly selected from a uniform distribution over the interval $[f_{\min}^i, f_{\max}^i]$, where $f_{\min}^i$ and $f_{\max}^i$ are positive integers. Then, using these basis functions and the reference points $\{(x_j^i, y_j^i)\}_{j=1}^{N_i}$, a new set of points defining the modified boundary of the i^{th} compartment is generated. The number of terms chosen determines the level of detail captured in the curve defining the compartment boundary. Compare, for example, the inner compartments of Figure 4(b) and (c). The values $f_{\min}^i$ and $f_{\max}^i$ are given in Table 1. In this way, each sample has a unique head anatomy included in a fixed elliptic region.

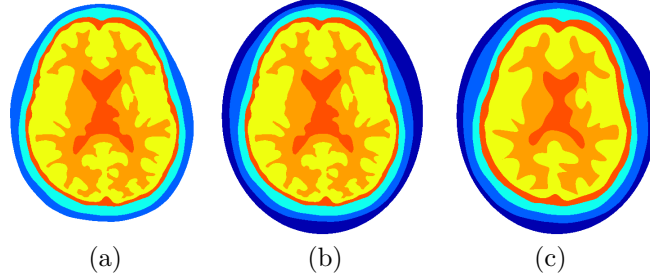


FIGURE 4. Two-dimensional model resembling a slice of a human head. (a) Reference head model containing five different compartments of importance for electrophysiology of the human head, arranged from outermost to innermost as follows: scalp, skull, cerebrospinal fluid, white matter, and grey matter. (b) Reference head model included in an elliptic shape. This dark blue region surrounding the soft tissue layer can be interpreted as a fixed helmet of constant conductivity with electrodes attached to its exterior. (c) Head anatomy of one simulated patient, generated by randomly modifying the shape, width, and location of each inner compartment of the reference head model.

For simplicity, we rescale the size of the head so that it is always contained within a fixed ellipse with semi-axes 0.85 and 1.

Additionally, in each of these head models, the conductivity value assigned to each component of the head is randomly sampled from a Gaussian distribution with mean and standard deviation in the ranges given in Table 1. These conductivity values are based on realistic estimates from [28, 17, 18, 19] at 1 kHz, and with an uncertainty of $\pm 10\%$ in the measurements. To model this uncertainty, we used normal distributions such that the $\pm 3\sigma$ interval corresponds to the measurement uncertainty. Prior to computation, all conductivity values are normalized by the scalp conductivity (0.32 S/m), which is necessary for calculating the CGO solutions. The region between the scalp and the boundary of the elliptic domain, represented in blue and acting as a fixed helmet, has a constant conductivity equal to 1 in all samples.

It is worth noting that the conductivity of the skull is very low and therefore is highly resistive to the electric currents. On the other hand, the CSF is highly conductive and acts as a shunt path for the current. These two regions form a double shield against the passage of electric current through the brain; any current that crosses the skull encounters an easy path via the CSF. This important effect is taken into consideration in this work. One of the goals of this work, therefore, was to demonstrate that VHED functions computed from EIT measurements can be used for stroke classification even in the presence of these physically realistic structures, which are frequently omitted from models presented in other works due to the challenges they create.

Finally, stroke inclusions consisting of discs or ellipses of various eccentricities and orientations are considered; see Figure 5. Exact parameters for the shapes and sizes of these inclusions, which differed for training and testing sets, are described in detail later in Section 2.3.2. The conductivity values for ischemic (low-conductivity) vs. hemorrhagic (high-conductivity) strokes were drawn from the distributions shown

TABLE 1. Parameters of the simulated two-dimensional head model: conductivity distribution, scaling factor, rotation angle, and number of basis functions range.

Region	Conductivity σ^i (S/m)	Scaling factor s^i	Rotation angle α^i	Range $f_{[min, f_{max}]}^i$
Scalp	$\mathcal{N}(0.32, 0.0107^2)$	$\mathcal{U}(0.98, 1.02)$	$\mathcal{U}(-3, 3)$	[80, 100]
Skull	$\mathcal{N}(0.020, 0.0007^2)$	$\mathcal{U}(0.98, 1.02)$	$\mathcal{U}(-2, 2)$	[80, 100]
Cerebrospinal fluid	$\mathcal{N}(2.0, 0.0667^2)$	$\mathcal{U}(0.97, 1.02)$	$\mathcal{U}(-2, 2)$	[20, 80]
Grey matter	$\mathcal{N}(0.098, 0.0033^2)$	$\mathcal{U}(0.97, 1.01)$	$\mathcal{U}(-2.5, 2.5)$	[50, 80]
White matter	$\mathcal{N}(0.062, 0.0021^2)$	$\mathcal{U}(0.98, 1.02)$	$\mathcal{U}(-2.5, 2.5)$	[80, 200]
Ischemic stroke	$\mathcal{N}(0.03, 0.0010^2)$			
Hemorrhagic stroke	$\mathcal{N}(0.70, 0.0233^2)$			

in Table 1. We assume that stroke inclusions are always on the right-hand side of the head model. That is because, in general, strokes are restricted to one half of the brain only, and the affected half is easily identifiable from the symptoms. Thus, there is no loss of generality in locating all the simulated strokes on the right-hand side of the brain, as the same process could be done for strokes located on the left-hand side of the brain.

To generate the synthetic data needed for the classification problem, we consider a large set of conductivity profiles with the properties described above, and for each sample, we solve the CEM model, using the finite element method [7], to obtain the corresponding EIT measurements. To recreate a more realistic scenario, we add noise to the computed EIT data. That is, for each sample, we collect all the electric potentials $U^{(k)}$, $k = 1, \dots, M - 1$ and form the single real-valued column vector

$$V = [U^{(1)}; U^{(2)}; \dots; U^{(M-1)}] \in \mathbb{R}^{M(M-1)}.$$

In practice, modern EIT systems tend to have 2^n voltage input channels, with some systems capable of handling measurements on up to 128 or 256 electrodes [37, 3, 12]. In this work, we followed the convention in [20] and used an odd number of electrodes, $M = 65$, although this number is chosen for convenience and the calculations could easily be formulated to work with arbitrary M .

After obtaining voltages, we add zero-mean random Gaussian noise to the vector V to obtain a noisy vector V^δ , with relative noise given by

$$\frac{\|V - V^\delta\|_2}{\|V\|_2} = \delta,$$

where $\|\cdot\|_2$ denotes the 2-norm. Following the techniques suggested in previous works, this noisy vector may be used directly as an input for the classification problem. In this present work, this method serves as a basis of comparison for our alternative strategy, wherein the voltages are further processed to obtain VHED functions. In this latter case, we reshape V^δ into a matrix and compute the noisy VHED profiles following the steps previously described in Section 2.1.3. In this

paper, we consider $\delta = 10^{-3}$ and $\delta = 10^{-2}$, and, as appropriate for these levels of noise, we take a cutoff frequency $\tau_m = 5$ in the computation of the VHED profiles.

Computation of the VHED profiles is not computationally intensive, and so does not add significant time to the stroke classification process as compared to the use of unprocessed voltages. In this work for example, using code which had not been optimized for efficiency, computation of VHED profiles from one set of EIT voltage data took only 28 seconds on a simple laptop with an Intel i7 processor and 16 GB of RAM, with the calculation of the CGO solutions being the most time-consuming part. However, the code could be optimized and run on a higher-performance machine to greatly reduce computation time, making the calculation and subsequent use of VHED functions a feasible diagnostic strategy for real-time applications.

2.3. Stroke classification using neural networks. The goal of this study is to explore the ability of machine learning to classify strokes as ischemic or hemorrhagic using EIT data simulated from mathematically realistic and physically detailed models. As mentioned in Section 1, most machine-learning-based approaches proposed for the classification of strokes have considered learning directly from measured EIT voltage data [27, 26, 35, 9, 14]. Here, we compare that method to the techniques which were introduced in [1], wherein noise-robust VHED functions computed from highly simplistic and mathematically ideal phantoms were used as network inputs. Next, we describe the neural networks we employed to compare these two different paradigms, as well as the details of the training and testing datasets.

2.3.1. Fully connected neural network (FCNN). In this work we consider a simple FCNN whose architecture consists of one input layer, one hidden layer, and one output layer. It should be noted that because we were able to obtain excellent and robust results using this simple network, we did not pursue the use of more sophisticated machine learning techniques for this study.

The size of the input layer depends on the type of input data. If raw voltage data is considered as input, then the potential measurements are stacked into a single real-valued column vector $V \in \mathbb{R}^{M(M-1)}$ which is used as the input. As mentioned in Section 2.2 in this work, we consider $M = 65$, and therefore each input $V \in \mathbb{R}^{4160}$. If a VHED function $T_{\text{odd}}\mu(t, \varphi)$ is instead considered as input data, this function is computed at angles $\varphi_i = i\frac{\pi}{8}$, for $i = 4, 5, 6, 12, 13, 14$, and the pseudo-time $t \in [-3, 3]$ is discretized over 256 points so that for each angle φ_i we have $T_{\text{odd}}\mu(t, \varphi_i) \in \mathbb{C}^{256}$. Then, the real and imaginary parts of the VHED profile at angles φ_i are concatenated to form a single real-valued column vector $T_{\text{odd}}\mu \in \mathbb{R}^{3072}$ that will be used as input. In order to unify notation, we denote both inputs as $x \in \mathbb{R}^{n_1}$.

The hidden layer has 30 neurons. This number was selected based on the results obtained in preliminary tests. The output layer consists of one neuron. The output, denoted by y , is a scalar that takes values in $\{0, 1\}$, where $y = 0$ represents an ischemic inclusion and $y = 1$ represents a hemorrhagic inclusion.

From a mathematical point of view, the FCNN can be represented as the function $\mathcal{F}_\theta : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$ given by

$$x \mapsto \mathcal{F}_\theta(x) := f(W_\theta^2 f(W_\theta^1 x + b_\theta^1) + b_\theta^2),$$

where $W_\theta^1 \in \mathbb{R}^{30 \times n_1}$ and $b_\theta^1 \in \mathbb{R}^{30}$ represent the weight matrix and the vector of biases at the hidden layer, respectively, $W_\theta^2 \in \mathbb{R}^{1 \times 30}$ and $b_\theta^2 \in \mathbb{R}$ represent the weight and the bias of the output layer, and f is the sigmoid activation function $f(t) = (1 + e^{-t})^{-1}$, applied element-wise. Using a more compact notation, the FCNN can be represented as

$$y = \mathcal{F}_\theta(x),$$

where $\theta \in \mathbb{R}^{30n_1+30+30+1}$ is a vector containing all the network parameters. In order to find the optimal parameters of the network, we consider the minimization of the binary cross entropy loss function

$$\mathcal{L}(\theta) = \sum_j -y_j \log(y_j^p) - (1 - y_j) \log(1 - y_j^p),$$

where y_j is the true value of the outcome, $y_j^p = \mathcal{F}_\theta(x_j)$ is the predicted value of the outcome, and the sum is over all samples in the training dataset. The scaled conjugate gradient algorithm was considered for the minimization of the loss function.

2.3.2. Training and testing datasets. In this study, we consider an elliptic belt or helmet with electrodes placed on the head of the patient and assume the measurements come from a two-dimensional cross-section of the human head. As described in Section 2.2, a cross-section containing five different compartments of importance for electrophysiology is considered as the head model, and an inclusion having low or high conductivity simulates an ischemic or hemorrhagic stroke, respectively.

In real-world scenarios, it is advantageous if a fairly simplistic dataset can be used to train a network which is then made to generalize to more diverse and complicated scenarios. For our training dataset, therefore, the inclusion is always simulated as a disc; see Figure 5(a). For each sample, the center of the disc and its radius are randomly generated. The radius is randomly sampled from a uniform distribution in the range $[0.1, 0.25]$, that is, $\mathcal{U}(0.1, 0.25)$. Additionally, the conductivity value of the inclusion is randomly sampled from a Gaussian distribution with mean and standard deviation in the ranges given in Table 1. The training dataset contains 4000 samples, and in order to have a balanced dataset, approximately 50% of the training data has a disc inclusion representing a hemorrhagic stroke and 50% a disc inclusion representing an ischemic stroke.

To test the performance of the proposed classification method, we consider three different and independent test sets where the stroke is modeled as: (i) one circular inclusion generated in the same way as the training data; see Figure 5(a); (ii) one elliptic inclusion with semi-major axis uniformly drawn from the distribution $\mathcal{U}(0.16, 0.26)$, and the semi-minor axis uniformly drawn from the distribution $\mathcal{U}(0.07, 0.16)$ and randomly rotated; see Figure 5(b); and (iii) two disjoint inclusions consisting of one disc inclusion and one elliptic inclusion (both representing hemorrhagic strokes or both representing ischemic strokes). In this last, case the radius of the circular inclusion is uniformly drawn from the distribution $\mathcal{U}(0.1, 0.15)$, and the semi-major and semi-minor axes of the elliptic inclusion are uniformly drawn from the distributions $\mathcal{U}(0.16, 0.26)$ and $\mathcal{U}(0.07, 0.16)$, respectively. The elliptic inclusion is also randomly rotated; see Figure 5(c).

Similar to the training data, the test data were generated by randomly sampling the shape, location, and conductivity value of each compartment of the head model, following the same process described in Section 2.2, and using the values listed in

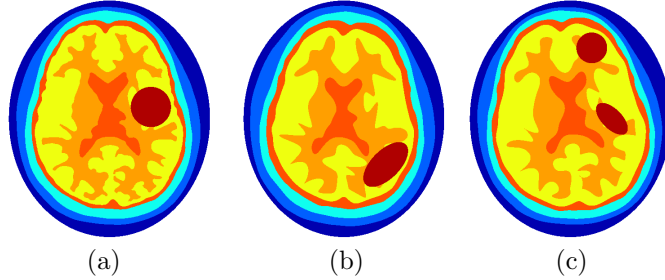


FIGURE 5. Samples of the training and test datasets. (a) The stroke is represented by circular inclusion. This type of inclusion is considered for the training dataset and for the first set of the test dataset. (b) A conductivity sample from the second test dataset where the stroke is modeled as an elliptic inclusion. (c) A conductivity sample from the third test dataset where the stroke is modeled as multiple inclusions containing one disc and one elliptic inclusion.

Table 1. Each test set consists of 1000 samples, and approximately 50% of the samples simulate a hemorrhagic stroke and 50% an ischemic stroke.

3. Results and discussion. We present the results of classifying strokes as ischemic or hemorrhagic using two different types of input data: raw EIT voltage measurements and VHED functions. As we mention in Section 2.3.2, to train the FCNN, we use a dataset having only circular inclusions. However, to determine whether the network can generalize to more diverse scenarios, we test the trained network with three different datasets: circular inclusions, elliptical inclusions, and multiple inclusions.

To measure the performance of the FCNN, we use the following metrics: (i) *sensitivity*, the proportion of actual positives (hemorrhagic cases) that are correctly classified as such, (ii) *specificity*, the proportion of actual negatives (ischemic cases) that are correctly identified as such, and (iii) *accuracy*, the fraction of correctly classified cases. For each of the 20 network trainings, these metrics were computed on the testing dataset, and the values reported correspond to the average values computed over the 20 trainings. Note that the accuracy values are presented along with their mean and standard deviation, which allows a better evaluation of the consistency and robustness of the results. During each training process, the data was randomly shuffled and the initial weights and biases were chosen randomly.

3.1. Classification performances. In Table 2, we present the performance metrics of the FCNN that was trained using a set of 4000 samples where the stroke is modeled as a disc inclusion, and tested using an independent set of 1000 samples also having a disc inclusion. We present the results obtained using noise-free data and from data corrupted by two different levels of noise, namely $\delta = 10^{-3}$ and $\delta = 10^{-2}$. The higher noise level of $\delta = 10^{-2}$ may be considered the most realistic scenario. We have highlighted in bold the highest accuracy value for each case.

Table 3 shows the performance metrics of the FCNN that was trained using the same set of 4000 samples having a disc inclusion, but was then tested using a set of 1000 samples where the stroke is modeled as an elliptic inclusion; see Figure 5(b). Since the elliptic inclusions differ in shape from the training dataset, this case study

TABLE 2. Performance metrics of FCNN evaluated on a test set of 1000 circular inclusions.

Noise level	Input	Accuracy	Sensitivity	Specificity
$\delta = 0$	V	0.987 $\pm$ 0.005	0.988	0.987
	$T_{\text{odd}}\mu$	0.985 $\pm$ 0.002	0.980	0.990
$\delta = 10^{-3}$	V^δ	0.971 $\pm$ 0.003	0.964	0.978
	$T_{\text{odd}}\mu^\delta$	0.984 $\pm$ 0.002	0.976	0.992
$\delta = 10^{-2}$	V^δ	0.937 $\pm$ 0.005	0.935	0.937
	$T_{\text{odd}}\mu^\delta$	0.961 $\pm$ 0.003	0.953	0.971

serves to evaluate the ability of neural networks to generalize to inputs completely different from those used during training.

TABLE 3. Performance metrics of FCNN evaluated on a test set of 1000 elliptic inclusions.

Noise level	Input	Accuracy	Sensitivity	Specificity
$\delta = 0$	V	0.969 $\pm$ 0.009	0.957	0.982
	$T_{\text{odd}}\mu$	0.965 $\pm$ 0.006	0.943	0.987
$\delta = 10^{-3}$	V^δ	0.947 $\pm$ 0.004	0.925	0.968
	$T_{\text{odd}}\mu^\delta$	0.962 $\pm$ 0.004	0.940	0.986
$\delta = 10^{-2}$	V^δ	0.885 $\pm$ 0.005	0.865	0.905
	$T_{\text{odd}}\mu^\delta$	0.917 $\pm$ 0.004	0.889	0.947

In Table 4, we report the results of the classification problem when the FCNN was trained using a set of 4000 samples having a disc inclusion and evaluated on a set of 1000 samples where the stroke is modeled as two disjoint inclusions shaped like a disc and ellipse, respectively; see Figure 5(c). Note that in this case the test dataset differs from the training dataset not only in shape, but also in the number of inclusions.

3.2. Discussion on the results. The numerical results presented in the previous section show that both raw EIT voltage data and VHED functions are excellent inputs for the stroke classification problem, even in the presence of the resistive skull layer and the shunting effects induced by the CSF. In the ideal noise-free case, use of raw voltages as inputs to the FCNN produced slightly better accuracy values as compared to the use of VHED functions. This is to be expected, given that the computations used to produce the VHED functions will introduce a small

TABLE 4. Performance metrics of FCNN evaluated on a test set of 1000 multiple inclusions.

Noise level	Input	Accuracy	Sensitivity	Specificity
$\delta = 0$	V	0.995 $\pm$ 0.003	0.996	0.994
	$T_{\text{odd}}\mu$	0.989 $\pm$ 0.001	0.984	0.993
$\delta = 10^{-3}$	V^δ	0.989 $\pm$ 0.002	0.990	0.988
	$T_{\text{odd}}\mu^\delta$	0.989 $\pm$ 0.001	0.982	0.995
$\delta = 10^{-2}$	V^δ	0.951 $\pm$ 0.004	0.954	0.947
	$T_{\text{odd}}\mu^\delta$	0.971 $\pm$ 0.002	0.962	0.979

amount of unavoidable numerical roundoff and truncation error, effectively adding some noise to the inputs. However, in the more realistic test scenarios where the simulated voltages were intentionally contaminated with added noise, the VHED functions performed much better than the raw voltages, and the difference is more evident when the noise level is higher (observe the more realistic case $\delta = 10^{-2}$). Despite the expected performance decrease for both methods in the presence of increased noise, the accuracy in the case of the VHED functions remains above 91% for all three different models of stroke inclusions, while in the case of raw data inputs, the accuracy drops to 88% for elliptic inclusions. These results reinforce the computational evidence presented in [1], which strongly suggests that VHED functions are more robust against noise corruption as compared to raw voltages.

Our preliminary results suggest that a simple FCNN trained using single circular-shaped strokes shows promising generalization capabilities to data generated from phantoms where stroke was modeled as an elliptic inclusion, and even in the case of multiple strokes modeled as a disc and an elliptic inclusion. This finding indicates that relatively simple training data may be sufficient for achieving good performance in more diverse scenarios, although further investigation is needed to confirm these results.

In Table 3, we can see that in the case of elliptic-shaped strokes there is a drop in accuracy for both methods as compared to the case of disc inclusions, which is to be expected given that the training phantoms differ from the test phantoms in this case. In contrast, in Table 4 we can see that in the case of multiple strokes there is an improvement in accuracy for both methods, which is likely explained by the large physical size of the total stroke region for this scenario since the larger total stroke size will correspond to a greater perturbation in boundary measurements. These results are consistent with those obtained in [1], where highly simplistic and mathematically unrealistic models were used. The consistency of these results adds credence to the conclusions of this present study, and suggests that the methods presented here are likely to be robust in a variety of different physical scenarios.

Observe that in all three cases the specificity for the networks using the VHED functions is over 94% for all noise levels. Furthermore, in all scenarios except the noise-free case for multiple strokes, VHED functions yielded higher specificity than raw voltages. Since specificity represents the proportion of actual ischemic

cases that are correctly identified, we therefore may conclude that VHED functions are capable of identifying ischemic strokes, especially with high precision. This is important because the majority of all strokes and stroke-related fatalities are ischemic. In fact, as reported in [22], in the United States it has been found that about 87% of all strokes and 70-80% of all stroke deaths are ischemic.

In this work we employed a very simple FCNN with excellent results. We consider it an advantage of the method that more complicated machine learning architecture did not prove necessary. One remaining simplifying assumption present in this work, however, was the use of a two-dimensional model. In practical real-world situations, where a belt or helmet with electrodes is placed on the head of a patient and the EIT data comes from a three-dimensional target, a simple approach like the one used here may not be sufficient. However, for that case we may extend the results presented in this work using more complex architectures or more sophisticated neural networks, and also employ a more comprehensive set of VHED profiles. This present work therefore serves as a springboard for future work toward clinical application of these methods.

4. Conclusions. In this article, a methodology for classification of stroke as either ischemic or hemorrhagic from *in silico* EIT data was presented and applied to physically detailed and mathematically realistic models which represented a 2D slice of a human head. The phantoms consisted of five regions relevant to electrophysiology of the human head. This included the highly resistive skull layer which impedes current flow, as well as the highly conductive cerebrospinal fluid which introduces shunting effects. Previous studies have frequently omitted one or both of these features from the models used, as the large contrast in the conductivity of these layers makes the classification problem much more difficult. Conductivity values for all tissues and stroke inclusions were furthermore drawn from statistically realistic distributions. Following the line of research introduced in [1], we extracted noise-robust geometric features known as VHED functions from EIT data, and used these features as inputs for the stroke classification problem. In this study we presented a more realistic scenario than in our previous work [1], which employed highly simplified disc-shaped phantoms and used the mathematically ideal continuum model of EIT. In this work, in addition to the use of a much more physically realistic and detailed noncircular phantom, we have considered electrode data simulated using the complete electrode model (CEM) instead of continuous data coming from the idealized continuum model.

We compared the use of VHED functions as network inputs to the technique employed in other works, which involves learning directly from the raw electrode data. We generated large datasets representing various stroke scenarios that were used to train and test fully connected neural networks using both methods under three different noise levels. The performance metrics of the classification problem yielded promising results and demonstrated that, in the presence of realistic noise levels, using the VHED functions is a better strategy than directly using EIT voltage measurements in terms of the accuracy and specificity metrics used to evaluate the results. This present work confirms and bolsters the conclusions made in the much more preliminary work [1].

Acknowledgments. J.P.A. acknowledges support from SeCyT-UNC Project Formar 2023-33820230100083CB. F.M. acknowledges support from São Paulo Research Foundation - (Process number: 2019/09154-7). S.R. and S.S. acknowledge support

from the Research Council of Finland (353097). M.A. acknowledges support from the Alphonse A. & Geraldine F. Arnold Endowment. M.L. acknowledges support from the FAME flagship, the Research Council of Finland (336786) and the European Research Council AdG project 101097198. Views and opinions expressed are those of the authors only and do not necessarily reflect those of the E.U.

Appendix A: details of computing the matrix $\mathbf{L}_\sigma$. In this appendix, we provide a detailed explanation of how to compute a matrix approximation $\mathbf{L}_\sigma$ of the DN map Λ_σ defined in (3), using the complete electrode model (CEM) and building upon the ideas developed in [20].

It is worth mentioning that in [20], instead of considering the Dirichlet condition (2), the Neumann condition

$$\frac{\partial u}{\partial \nu} = \phi \quad \text{on } \partial\Omega$$

satisfying $\int_{\partial\Omega} \phi \, ds = 0$ is considered. In this case, the boundary measurements are modeled by the Neumann-to-Dirichlet (ND) map $\mathcal{R}_\sigma : H_\diamond^{-1/2}(\partial\Omega) \rightarrow H_\diamond^{1/2}(\partial\Omega)$ given by

$$\mathcal{R}_\sigma : \phi \mapsto u|_{\partial\Omega},$$

where $H_\diamond^s(\partial\Omega)$ space consists of functions in the standard Sobolev space $H^s(\partial\Omega)$ having mean value zero.

The methodology presented in [20] permits to approximate relative continuum data associated to the map $\mathcal{R}_\sigma - \mathcal{R}_1$ based on measurement data from the CEM, where $\mathcal{R}_1$ is the ND map with $\sigma = 1$ everywhere. For a given bounded simply connected domain $\Omega \subset \mathbb{R}^2$ and a specified set of continuum current patterns $\{\phi_k\}_{k=1}^{M-1}$ that one would like to apply along the boundary $\partial\Omega$, the proposed method determines the optimal locations for the electrodes e_m based on the domain's geometry, as well as the corresponding net electrode currents $\{I^{(k)}\}_{k=1}^{M-1}$ that are tailored to the continuum current patterns.

The location of the electrodes e_m on the domain boundary $\partial\Omega$ is given by the following process. Let $\theta_m = m \frac{2\pi}{M}$ and $x_m = e^{i\theta_m}$ for $m = 1, \dots, M$. The points x_m can be interpreted as the electrodes' midpoints on the boundary of the unit disc D . Let $\Psi : D \rightarrow \Omega$ be a bijective conformal map whose restriction to ∂D defines a C^∞ -diffeomorphism between ∂D and $\partial\Omega$. Then, the electrodes' midpoints on the boundary of the domain Ω , denoted by y_m , are given by $y_m := \Psi(x_m)$ for $m = 1, \dots, M$.

Once we have defined the electrode locations, we define the set of input currents given the continuum current patterns. In this study we consider, as the continuum patterns, the following set of trigonometric basis functions

$$\phi_k(s) = \begin{cases} \sqrt{\frac{2}{L}} \cos\left(\frac{(k+1)}{2} \frac{2\pi s}{L}\right) & \text{odd } k, \\ \sqrt{\frac{2}{L}} \sin\left(\frac{k}{2} \frac{2\pi s}{L}\right) & \text{even } k, \end{cases} \quad (8)$$

for $k = 1, \dots, M-1$. Additionally, we assume that we have an arc length parametrization $\gamma : (0, L] \rightarrow \partial\Omega$, where L is the total arc length of $\partial\Omega$, and we denote by s_m the points such that $\gamma(s_m) = y_m = \Psi(x_m)$, for $m = 1, \dots, M$. Then, the corresponding

electrode current pattern for the CEM is defined as

$$I^{(k)} = \frac{2\pi}{M} \left(\begin{bmatrix} \phi_k(s_1) |\Psi'(x_1)| \\ \vdots \\ \phi_k(s_M) |\Psi'(x_M)| \end{bmatrix} - \frac{1}{M} \sum_{m=1}^M \phi_k(s_m) |\Psi'(x_m)| \mathbf{1} \right) \in \mathbb{R}_\diamond^M.$$

for $k = 1, \dots, M-1$, and where $\mathbf{1} = [1, \dots, 1]^T \in \mathbb{R}^M$.

After determining the location of the electrodes and the set of input current patterns, we utilize the corresponding CEM voltages to compute a matrix approximation $\mathbf{L}_\sigma$ of the DN map Λ_σ defined in (3). The constructive method consists of the following three steps:

- Step 1: Using the CEM, construct a bounded linear operator Υ_σ^M that approximates the relative ND map $\Upsilon_\sigma := \mathcal{R}_\sigma - \mathcal{R}_1$.
- Step 2: Compute a matrix approximation $\mathbf{Y}_\sigma^{\text{CEM}}$ of the operator Υ_σ^M in the subspace spanned by the basis (8).
- Step 3: Compute a matrix approximation $\mathbf{L}_\sigma$ of the operator Λ_σ from the matrix approximation $\mathbf{Y}_\sigma^{\text{CEM}}$.

Let us begin with Step 1, that is, the definition of the bounded operator Υ_σ^M .

Based on the approach of [20], given $U \in \mathbb{R}^M$ and considering the trigonometric functions given by

$$\varphi_m(\theta) = \frac{1}{M} \sum_{n=0}^{M-1} e^{in(\theta - \theta_m)}, \quad m = 1, \dots, M,$$

we can build a trigonometric interpolation operator $Q^M : \mathbb{R}^M \rightarrow H^t(0, 2\pi)$, $t \in \mathbb{R}$, defined as

$$Q^M U = \sum_{m=1}^M U_m \varphi_m.$$

Therefore, given $U \in \mathbb{R}^M$, $Q^M U \circ \beta^{-1}$ defines a function in $H^t(\partial D)$; see [20] for more details.

Bearing all the above in mind, given a continuum current pattern ϕ with zero mean, the approach presented in [20] for approximating the smooth output $\Upsilon_\sigma \phi$ using electrode measurements is as follows:

- Step 1.1: Define

$$I^{(\phi)} = \frac{2\pi}{M} \left(\begin{bmatrix} \phi(y_1) |\Psi'(x_1)| \\ \vdots \\ \phi(y_M) |\Psi'(x_M)| \end{bmatrix} - \frac{1}{M} \sum_{m=1}^M |\Psi'(x_m)| \phi(y_m) \mathbf{1} \right) \in \mathbb{R}_\diamond^M \quad (9)$$

as the electrode current pattern for the CEM.

- Step 1.2: Perform the measurements of the CEM to get

$$U^{(\phi)} = (\mathcal{R}_\sigma^{\text{CEM}} - \mathcal{R}_1^{\text{CEM}}) I^{(\phi)} \in \mathbb{R}_\diamond^M.$$

- Step 1.3: Apply trigonometric interpolation and define

$$\tilde{g}^{(\phi)} = Q^M U^{(\phi)} \circ \beta^{-1} \in H_\diamond^t(\partial D).$$

- Step 1.4: Form an approximation for the continuum boundary potential as

$$g = \tilde{g}^{(\phi)} \circ \Psi^{-1} \in H_\diamond^t(\partial \Omega).$$

Then, the operator $\Upsilon_\sigma^M : H_\diamond^r(\partial\Omega) \rightarrow H_\diamond^t(\partial\Omega)$ given by

$$\Upsilon_\sigma^M : \phi \mapsto g \quad (10)$$

defines a bounded operator for $r > \frac{1}{2}$ and $t \in \mathbb{R}$; see [20] for more details.

For Step 2 of the process, given a set of continuum basis functions $\{\phi_k\}_{k=1}^{M-1}$, a discrete matrix approximation $\mathbf{Y}_\sigma^{\text{CEM}}$ of the map Υ_σ^M can be constructed by first computing $\Upsilon_\sigma^M \phi_k$ and then evaluating the inner products $\langle \Upsilon_\sigma^M \phi_k, \phi_\ell \rangle =: (\mathbf{Y}_\sigma^{\text{CEM}})_{\ell,k}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in L^2 . In particular, if we take $\{\phi_k\}_{k=1}^{M-1}$ to be the trigonometric basis functions given by (8), and for simplicity denoting $U^{(k)} = U^{(\phi_k)}$, the matrix elements of $\mathbf{Y}_\sigma^{\text{CEM}}$ can be computed as follows:

$$\begin{aligned} (\mathbf{Y}_\sigma^{\text{CEM}})_{\ell,k} &= \langle \Upsilon_\sigma^M \phi_k, \phi_\ell \rangle \\ &= \int_0^L (Q^M U^{(k)} \circ \beta^{-1} \circ \Psi^{-1} \circ \gamma)(s) \phi_\ell(s) ds \\ &= \sum_{m=1}^M U_m^{(k)} \int_0^L \varphi_m(\beta^{-1}(\Psi^{-1}(\gamma(s)))) \phi_\ell(s) ds \\ &\approx \frac{2\pi}{M} \sum_{m=1}^M U_m^{(k)} \phi_\ell(y_m) |\Psi'(x_m)| \end{aligned}$$

where in the last line we have used the fact that

$$\begin{aligned} &\int_0^L \varphi_m(\beta^{-1}(\Psi^{-1}(\gamma(s)))) \phi_\ell(s) ds \\ &= \int_{\partial\Omega} \varphi_m(\beta^{-1}(\Psi^{-1}(y))) \phi_\ell(\gamma^{-1}(y)) dy \\ &= \int_{\partial D} \varphi_m(\beta^{-1}(x)) \phi_\ell(\gamma^{-1}(\Psi(x))) |\Psi'(x)| dx \\ &= \int_0^{2\pi} \varphi_m(\theta) \phi_\ell(\gamma^{-1}(\Psi(\beta(\theta)))) |\Psi'(\beta(\theta))| d\theta \\ &\approx \frac{2\pi}{M} \sum_{j=1}^M \varphi_m(\theta_j) \phi_\ell(\gamma^{-1}(\Psi(\beta(\theta_j)))) |\Psi'(\beta(\theta_j))| \\ &= \frac{2\pi}{M} \sum_{j=1}^M \delta_{m,j} \phi_\ell(s_j) |\Psi'(x_j)| \\ &= \frac{2\pi}{M} \phi_\ell(s_m) |\Psi'(x_m)|, \end{aligned}$$

where in the fourth line we have applied the trapezoidal quadrature rule for equian-gular points, and in the fifth line we have used the fact that $\varphi_m(\theta_j) = \delta_{m,j}$; see [20] for more details.

In summary, the process of building a numerical matrix approximation $\mathbf{Y}_\sigma^{\text{CEM}}$ of the operator Υ_σ^M given by (10) and acting in the subspace spanned by the basis (8) consists of:

- Step 2.1: For $k = 1, \dots, M-1$, define the electrode current pattern $I^{(k)} = I^{(\phi_k)}$ according to (9).
- Step 2.2: For $k = 1, \dots, M-1$, perform the measurements of CEM with $I^{(k)}$ as input current to get $U^{(k)} = (\mathcal{R}_\sigma^{\text{CEM}} - \mathcal{R}_1^{\text{CEM}}) I^{(k)}$, and then form the

matrix

$$\mathbf{U} = [U^{(1)} U^{(2)} \dots U^{(M-1)}].$$

- Step 2.3: Construct the matrix

$$\mathbf{I} = [I^{(1)} I^{(2)} \dots I^{(M-1)}].$$

- Step 2.4: Compute a numerical matrix approximation of the map Υ_σ^M as

$$\mathbf{Y}_\sigma^{\text{CEM}} = \mathbf{U}^T \mathbf{I}.$$

Finally, let us focus on Step 3 of the process, namely the computation of a matrix approximation $\mathbf{L}_\sigma$ of the DN map Λ_σ from the matrix $\mathbf{Y}_\sigma^{\text{CEM}}$.

Recall that $\mathbf{Y}_\sigma^{\text{CEM}}$ is a matrix approximation of the operator Υ_σ^M that approximates the relative ND map $\Upsilon_\sigma = \mathcal{R}_\sigma - \mathcal{R}_1$ using the CEM model. Therefore, from a numerical perspective, we have a matrix approximation of Υ_σ , but using electrode measurement instead of continuum measurements.

Let $\mathbf{R}_1$ be a matrix with elements given by $(\mathbf{R}_1)_{i,j} := \langle \mathcal{R}_1 \phi_j, \phi_i \rangle$. This matrix is an approximation of the ND map $\mathcal{R}_1$ in the basis (8), and, consequently, $\mathbf{Y}_\sigma^{\text{CEM}} + \mathbf{R}_1$ is a matrix approximation of the ND map $\mathcal{R}_\sigma$ in the same basis.

Taking the above into account, a matrix approximation of the DN map Λ_σ , in the basis (8), is given by $(\mathbf{Y}_\sigma^{\text{CEM}} + \mathbf{R}_1)^{-1}$. However, the DN map Λ_σ acts also on constant functions. Therefore, we need to extend the matrix $(\mathbf{Y}_\sigma^{\text{CEM}} + \mathbf{R}_1)^{-1}$ matrix onto the full space $\text{span}\{\phi_0, \phi_1, \dots, \phi_{M-1}\}$, where $\phi_0(s) = |\partial\Omega|^{-1/2}$.

In summary, the process of building a numerical matrix approximation $\mathbf{L}_\sigma$ of the DN map Λ_σ from the matrix $\mathbf{Y}_\sigma^{\text{CEM}}$, and acting in the subspace spanned by the basis (8) augmented by the constant basis function $\phi_0(s) = |\partial\Omega|^{-1/2}$, consists of:

- Step 3.1: Compute a matrix approximation $\mathbf{R}_1$ of $\mathcal{R}_1$ in the basis (8).
- Step 3.2: Compute a matrix approximation $\tilde{\mathbf{L}}_\sigma$ of Λ_σ in the basis (8) as

$$\tilde{\mathbf{L}}_\sigma = (\mathbf{Y}_\sigma^{\text{CEM}} + \mathbf{R}_1)^{-1}.$$

- Step 3.3: Extend the matrix $\tilde{\mathbf{L}}_\sigma$ and define

$$\mathbf{L}_\sigma = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathbf{L}}_\sigma \end{bmatrix}.$$

REFERENCES

- [1] J. P. Agnelli, A. Çöl, M. Lassas, R. Murthy, M. Santacesaria and S. Siltanen, [Classification of stroke using neural networks in electrical impedance tomography](#), *Inverse Problems*, **36** (2020), 115008, 26 pp.
- [2] M. Alsaker, S. Rautio, F. Moura, J. P. Agnelli, R. Murthy, M. Lassas, J. L. Mueller and S. Siltanen, [CT scans without X-rays: Parallel-beam imaging from nonlinear current flows](#), *Applied Mathematics for Modern Challenges*, **5** (2025), 1-35.
- [3] K. Y. Aristovich, G. S. D. Santos, B. C. Packham and D. S. Holder, [A method for reconstructing tomographic images of evoked neural activity with electrical impedance tomography using intracranial planar arrays](#), *Physiological Measurement*, **35** (2014), 1095.
- [4] K. Astala and L. Päivärinta, A boundary integral equation for Calderón's inverse conductivity problem, *Collectanea Mathematica*, (2006) 127-139.
- [5] K. Astala and L. Päivärinta, [Calderón's inverse conductivity problem in the plane](#), *Annals of Mathematics*, **163** (2006), 265-299.
- [6] D. Barthels and H. Das, [Current advances in ischemic stroke research and therapies](#), *Biochimica et Biophysica Acta (BBA) - Molecular Basis of Disease*, **1866** (2020), 165260.
- [7] S. C. Brenner and L. Ridgway Scott, *The Mathematical Theory of Finite Element Methods*, Texts in Applied Mathematics. Springer, 2008.

- [8] A. P. Calderón, On an inverse boundary value problem, *Seminar on Numerical Analysis and its Applications to Continuum Physics*, (1980), 65-73.
- [9] V. Candiani and M. Santacesaria, [Neural networks for classification of strokes in electrical impedance tomography on a 3D head model](#), *Mathematics in Engineering*, **4** (2022), 1-22.
- [10] J. A. Chalela et al., [Magnetic resonance imaging and computed tomography in emergency assessment of patients with suspected acute stroke: A prospective comparison](#), *The Lancet*, **369** (2007), 293-298.
- [11] M. Cheney, D. Isaacson and J. C. Newell, [Electrical impedance tomography](#), *SIAM Review*, **41** (1999), 85-101.
- [12] V. A. Cherepenin et al., [Three-dimensional EIT imaging of breast tissues: system design and clinical testing](#), *IEEE Transactions on Medical Imaging*, **21** (2002), 662-667.
- [13] C. Cordonnier, A. Demchuk, W. Ziai and C. S. Anderson, [Intracerebral haemorrhage: Current approaches to acute management](#), *The Lancet*, **392** (2018), 1257-1268.
- [14] J. Culpepper, H. Lee, A. Santorelli and E. Porter, [Applied machine learning for stroke differentiation by electrical impedance tomography with realistic numerical models](#), *Biomedical Physics & Engineering Express*, **10** (2023), 015012.
- [15] G. A. Donnan, M. Fisher, M. Macleod and S. M. Davis, [Stroke](#), *The Lancet*, **371** (2008), 1612-1623.
- [16] M. R. B. Evans, P. White, P. Cowley and D. J. Werring, [Revolution in acute ischaemic stroke care: A practical guide to mechanical thrombectomy](#), *Practical Neurology*, **17** (2017), 252-265.
- [17] S. Gabriel, R. W. Lau and C. Gabriel, [The dielectric properties of biological tissues: I. literature survey](#), *Physics In Medicine And Biology*, **41** (1996), 2231-2249.
- [18] S. Gabriel, R. W. Lau and C. Gabriel, [The dielectric properties of biological tissues: II. measurements in the frequency range 10 Hz to 20 GHz](#), *Physics In Medicine And Biology*, **41** (1996), 2251-2269.
- [19] S. Gabriel, R. W. Lau and C. Gabriel, [The dielectric properties of biological tissues: III. parametric models for the dielectric spectrum of tissues](#), *Physics In Medicine And Biology*, **41** (1996), 2271-2293.
- [20] H. Garde and N. Hyvönen, [Mimicking relative continuum measurements by electrode data in two-dimensional electrical impedance tomography](#), *Numerische Mathematik*, **147** (2021), 579-609.
- [21] A. Greenleaf, M. Lassas, M. Santacesaria, S. Siltanen and G. Uhlmann, [Propagation and recovery of singularities in the inverse conductivity problem](#), *Analysis & PDE*, **11** (2018), 1901-1943.
- [22] R. A. Grysiwicz, K. Thomas and D. K. Pandey, [Epidemiology of ischemic and hemorrhagic stroke: Incidence, prevalence, mortality and risk factors](#), *Neurologic Clinics*, **26** (2008), 871-895.
- [23] W. Hacke et al., [Association of outcome with early stroke treatment: Pooled analysis of ATLANTIS, ECASS and NINDS rt-PA stroke trials](#), *Lancet*, **363** (2004), 768-774.
- [24] W. Hacke et al., [Thrombolysis with alteplase 3 to 4.5 hours after acute ischemic stroke](#), *New England Journal of Medicine*, **359** (2008), 1317-1329.
- [25] J. C. Hemphill et al., [Guidelines for the management of spontaneous intracerebral hemorrhage](#), *Stroke*, **46** (2018), 2032-2060.
- [26] B. McDermott, A. Elahi, A. Santorelli, M. O'Halloran, J. Avery and E. Porter, [Multi-frequency symmetry difference electrical impedance tomography with machine learning for human stroke diagnosis](#), *Physiological Measurement*, **41** (2020), 075010,.
- [27] B. McDermott, M. O'Halloran, P. Emily and A. Santorelli, [Brain haemorrhage detection using a SVM classifier with electrical impedance tomography measurement frames](#), *PLoS ONE*, **13** (2018), e0200469.
- [28] F. S. Moura, R. G. Beraldo, L. A. Ferreira and S. Siltanen, [Anatomical atlas of the upper part of the human head for electroencephalography and bioimpedance applications](#), *Physiological Measurement*, **42** (2021), 105015.
- [29] S. Ojaghhihaghghi, S. S. Vahdati, A. Mikaeilpour and A. Ramouz, [Comparison of neurological clinical manifestation in patients with hemorrhagic and ischemic stroke](#), *World Journal of Emergency Medicine*, **8** (2017), 34.
- [30] C. Provost et al., [Magnetic resonance imaging or computed tomography before treatment in acute ischemic stroke](#), *Stroke*, **50** (2019), 659-664.

- [31] A. I. Qureshi, A. D. Mendelow and D. F. Hanley, [Intracerebral haemorrhage](#), *The Lancet*, **373** (2009), 1632-1644.
- [32] C. M. Rapillo et al., [Moving from CT to MRI paradigm in acute ischemic stroke: Feasibility, effects on stroke diagnosis and long-term outcomes](#), *Stroke*, **55** (2024), 1329-1338.
- [33] J. L. Saver, [Time is brain-quantified](#), *Stroke*, **37** (2006), 263-266.
- [34] J. L. Saver et al., [The “golden hour” and acute brain ischemia](#), *Stroke*, **41** (2010), 1431-1439.
- [35] Y. Shi, Z. Tian, M. Wang, Y. Wu, B. Yang and F. Fu, [Residual convolutional neural network-based stroke classification with electrical impedance tomography](#), *IEEE Transactions on Instrumentation and Measurement*, **71** (2022), 1-11.
- [36] E. Somersalo, M. Cheney and D. Isaacson, [Existence and uniqueness for electrode models for electric current computed tomography](#), *SIAM Journal on Applied Mathematics*, **52** (1992), 1023-1040.
- [37] G. Xu et al., A 128-electrode three dimensional electrical impedance tomography system, In *2007 29th Annual International Conference of the IEEE Engineering in Medicine and Biology Society*, (2007), 4386-4389.

Received May 2025; revised September 2025; early access November 2025.