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The (l, k) Facility Location Duel

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Motivated by applications in clustering and information retrieval, we extend the classical Hotelling setting to deal with multi-facility location duels. In the classical Hotelling setting, customers' locations are taken from the uniform distribution on the $[0, 1]$ segment, and there are two facility owners, each needs to decide on the location of her (single) facility, aiming to maximize the proportion of customers closer to it. We extend this duel to competition among the owner of k facilities to the owner of l facilities, for arbitrary k, l, where $l \leq k$. Our main message is quite striking: in no equilibrium a facility will materialize in a location which is not part of the social welfare maximizing locations of the player who has k facilities, if she were to locate her facilities under no competition. This is obtained despite the lack of pure strategy equilibrium in many of these settings. We also study two sets of other natural families of distributions; for one of the families we show the above findings extend beyond the uniform distribution; for the other we show that selecting only among the stronger-player social welfare maximizing locations will not materialize in equilibrium.

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1. INTRODUCTION

Facility location has been a topic of major interest in the OR and CS literature for many years. It has been also of major interest to the data sciences, as the problem of clustering is closely related to the facility location problem, where centroids are interpreted as facility locations, and algorithms for finding them, such as k-means, are the standard practice [Shalev-Shwartz and Ben-David 2014]. On the other hand, in economics, facility location games have been introduced. The classical work by Hotelling [Hotelling 1929] considers a duel between two parties who compete over consumers distributed uniformly over the interval $[0, 1]$; each party locates its facility on that interval, and grab the proportion of the population closer to it. As it turns out the only equilibrium in that setting is for both parties to locate their facility at 0.5. Interestingly, while overwhelming many extensions of that basic setting exist (see related work below), the economic studies refer to competition between single-facility owners only, which make that work non-applicable to many applications, as the ones mentioned above (e.g. in clustering we are typically after selecting several centroids/clusters). Our aim is to expand the study of the Hotelling facility location duel to the general case where the players hold arbitrary k and l facilities, respectively.

To further motivate the above foundational study, consider strategic behaviors of publishers in an information retrieval [IR] context (see [Ben-Basat et al. 2015] for some formal definitions). Indeed, the publishers' competition game can be viewed as a problem of facility location in competitive settings, as originally introduced by Hotelling. If one views each customer as a reader whose interests are a trade-off between the amount of news versus the amount of articles in a newspaper, and sellers as publishers who control the amount of news versus articles to be published in their newspapers, we get an IR publishing game that coincides with the basic Hotelling setting. Needles to say, the IR setting is much broader, as customers and publishers' possibilities may be defined by quite arbitrary spaces, and various constraints on that setting may hold, but IR games are natural extensions of the above basic setting.

Assume a "strong" publisher who controls several outlets, which it can maintain, e.g. two different Internet versions of its newspaper. This publisher can be viewed as being able to locate two "facilities" in the space of published data rather than only one; however, a "weak" publisher who can not maintain two such versions will need and be able to locate only one "facility". How would these different powers effect the behav-

ior of the publishers? What would be optimal strategies for the different publishers? This is a novel challenge and question, which illustrates how valuable and deep the understanding of the above games may be for theory and practice. [1]

1.1. The Basic Model

Consider the following setting:

(1) Player 1 has l facilities, and player 2 has k facilities. Wlog we assume $l \leq k$. The players need to decide where on the interval $[0, 1]$ they would locate their facilities. We refer to the facilities' location of player i as the strategy of i, denoted by x_i.
(2) Customers are distributed uniformly[2] along the segment $[0, 1]$.
(3) Each customer selects his nearest facility, where we break ties by unbiased coin flips. We refer to the proportion of customers selecting a player i's facility as player i's expected payoff / utility. Given the strategy profile (x_1, x_2) we denote the expected utility of player i by $U^i(x_1, x_2)$. Note that this is a fixed sum game.
(4) Each player wants to maximize her expected payoff.

Throughout this paper, we address this setting as the "l vs k setting". This model can be generalized from several different angles: The number of players, the metric space they lay in, differentiating between the products (e.g by setting different prices), the payoff function, etc. Unless stated otherwise, we consider the above foundational model; later we consider some other customer distributions.

1.2. Related Work

The model we adopt in service of initiating this study is the original model of Hotelling phrased as a facility location duel in the spirit of work on duelling algorithms [Immorlica et al. 2011]. [Anderson et al. 1992] contains a nice survey of the literature on facility location games up to 1992. For a more recent survey the reader is referred to [Brenner 2011]. While many extensions of the above basic setting have been introduced, in most of the existing literature the set of possible actions (i.e. possible facility locations) is very simple, e.g, a segment or a circle. For a recent discussion of general graph structures in that setting see [Fournier and Scarsini 2014]. None of that literature refers to players, each controlling several facilities to be located as their actions. In the algorithmic game theory literature the case we have only one entity that dictates the place of a facility (or several facilities), while the locations of customers are their private information and are strategically reported, emerged as a main source of interest [Procaccia and Tennenholtz 2009; Schummer and Vohra 2007]. In this context the players are the customers, while our work extends facility location games where the players are the facilities' owners.

1.3. Results

Our findings for the foundational problem are quite striking. We characterize the equilibria for customers arriving from the uniform distribution; we show that in no equilibrium a facility will be located in place which is not selected in the optimal facility locations of the player who has the larger number of facilities, if he were alone and maximizing social welfare of the customers. Notice that this result is obtained although in many of the above settings (e.g. in all cases where one player controls one facility, and the other controls $k \geq 2$ facilities) there is no pure strategy equilibrium.

More technically, we show the existence of equilibrium for the l vs k problem defined in sub-section 1.1, in which player 1 does not contribute to the social welfare. In this

[1] See [Izsak et al. 2014] for a study of other IR duels.
[2] This assumption will be relaxed in section 5.

equilibrium, it holds that if at time 0 player 2 (with k facilities) was the only server, and her goal was to maximize the social welfare, and at time 1 the other player would arrive - player 2 would stick with her original locations, and player 1 will locate her facilities in places that are already occupied with player 2's facilities. Then we show that these properties hold in any equilibrium. Namely, there is no equilibrium where more than k facilities are visible to the customers, and as a by product - the player that poses less facilities can be removed from the game without hurting the social welfare.

Beyond this main result, we also study two sets of other natural families of distributions. One of these families captures situations where the customers generally have a strong preference towards one of the sides of the segment. The other, in a sense complementary family, captures situations where the majority of people does not hold a strong preference towards the sides of the segment, but rather enjoys being close to the median. In the context of a duel between a single facility owner and the owner of two facilities we show, for any distribution in the first family, the existence of an equilibrium where the optimal (single player) 2-facility locations are selected by the 2-facility owner, while the single facility owner chooses uniformly among them, exposing a phenomenon as observed in the uniform distribution case. In the same context, we show that for the second family, no such equilibrium exist for any corresponding distribution.

2. WARMUP - 1 VS 2

We begin with a warmup, illustrating our setting. Imagine a world where only one player (denoted player 2) with 2 facilities exists, and she wishes to maximize the social welfare. The optimal facility locations are $\frac{1}{4}, \frac{3}{4}$.[3] At this point in time, another player, which is denoted as player 1 (with 1 facility) is entering the market. What will she do? Notice that already this extension of the classical setting does not possess a pure strategy equilibrium. Any deterministic tuple of strategies will imply a beneficial deviation by at least one of he players.

2.1. Equilibrium Protocol

We define the following protocol for this setting:

PROTOCOL 1.

(*1*) *Player 1 chooses her facility's location using*

$$x_1^{p_1} = \begin{cases} \frac{1}{4} & w.p. \frac{1}{2} \\ \frac{3}{4} & w.p. \frac{1}{2} \end{cases}$$

(*2*) *Player 2 has deterministic locations:* $x_2^{p_1} = (\frac{1}{4}, \frac{3}{4})$

Note that according to this protocol, player 1 does not have any effect on the social welfare. In other words, all customers are indifferent to player 1 being in the game.

THEOREM 2.1 (1 VS 2 EQUILIBRIUM). *Protocol 1 is an equilibrium for the* 1 *vs* 2 *case, with expected payoffs of* $U^1(x_1^{p_1}, x_2^{p_1}) = \frac{1}{4}, U^2(x_1^{p_1}, x_2^{p_1}) = \frac{3}{4}$.

The proof is in the appendix. Next, we aim to show the uniqueness of Protocol 1 as an equilibrium. Remember that in Protocol 1 player 1 does not contribute to the social welfare. We will use the following.

LEMMA 2.2. *Assume* (x_1, x_2) *is an equilibrium profile. It holds that:*

(*1*) $U^1(x_1, x_2) = \frac{1}{4},\ U^2(x_1, x_2) = \frac{3}{4}$

[3] See Theorem 3.1 for a proof of the more general case.

(2) $P(\frac{1}{4} \leq x_1 \leq \frac{3}{4}) = 1$
(3) $P(x_1 < \frac{1}{2}) > 0$
(4) $P(x_1 > \frac{1}{2}) > 0$

PROOF.

(1) Suppose that $U^1(x_1, x_2) < \frac{1}{4}$.We saw that when player 1 plays according to Protocol 1, there is no deterministic strategy for player 2 that can yield more than $\frac{3}{4}$. Therefore, by changing x_1 to the strategy suggested in Protocol 1, player 1 improves her reward, in contradiction to x_1, x_2 being an equilibrium. The same argument hold for the case when $U^2(x_1, x_2) < \frac{3}{4}$.
(2) From 1 we know that $U^2(x_1, x_2) = \frac{3}{4}$. Assume $P(\frac{1}{4} \leq x_1 \leq \frac{3}{4}) < 1$, hence $P(x_1 > \frac{3}{4}) > 0$ or $P(x_1 < \frac{1}{4}) > 0$. We saw that if player 2 plays according to Protocol 1, any deterministic strategy $y \in Support(x_1)$ such that $y < \frac{1}{4}$ or $y > \frac{3}{4}$ yields player 1 less than $\frac{1}{4}$. Therefore, if player 2 will change x_2 to Protocol 1 he would get $U^2(x_1, x_2^{p_1}) > \frac{3}{4}$, In contradiction to x_1, x_2 being an equilibrium.
(3) Assume $P(x_1 < \frac{1}{2}) = 0$. From (2) it follows that $P(\frac{1}{2} \leq x_1 \leq \frac{3}{4}) = 1$. If player 2 changes her strategy to $x_2' = (\frac{7}{16}, \frac{13}{16})$, the reward of player 1 would be:

$$U^1(x_1, x_2') = \underbrace{P(\frac{1}{2} \leq x_1 \leq \frac{3}{4})}_{=1} \left(\frac{\frac{13}{16} - \frac{7}{16}}{2}\right) = \frac{3}{16} < \frac{1}{4}$$

But from (1) we know that any equilibrium gives player 1 reward of $\frac{1}{4}$.
(4) Similar to (3)

□

2.2. Uniqueness of Protocol 1

At this point, it is clear that according to Protocol 1 the presence of player 1 in the game does not serve the public (i.e. the customers) in terms of social welfare. Is there another equilibrium for players 1,2 such that an arriving customer will be able to select between 3 different facilities (different in terms of location)?

THEOREM 2.3. *Protocol 1 is the only equilibrium for the 1 vs 2 facility location game.*

PROOF. Assume there is another equilibrium profile, (x_1, x_2). From Lemma 2.2 it follows that $P(\frac{1}{4} \leq x_1 \leq \frac{3}{4}) = 1$, and since (x_1, x_2) is an equilibrium, x_2 is a best response to x_1. In addition, we know from Lemma 2.2 part (1) that $U^1(x_1, x_2) = \frac{1}{4}$. Because player 1 is not playing according to Protocol 1 (since if she was, player 2 would use her only best response[4] - which is also playing according to Protocol 1), it holds that:

$$\min\left\{P(x_1 = \frac{1}{4}), P(x_1 = \frac{3}{4})\right\} < \frac{1}{2}$$

Next, we suggest an alternative strategy for player 2, x_2', and prove that $U^1(x_1, x_2') < U^1(x_1, x_2) = \frac{1}{4}$.

[4]See the case analysis of Theorem 2.1

In case $P(x_1 = \frac{1}{4}) = \min\Big\{P(x_1 = \frac{1}{4}), P(x_1 = \frac{3}{4})\Big\} < \frac{1}{2}$:
Denote[5]:

$$\epsilon \in \Big\{t : 0 \leq P\big(\frac{1}{4} < x_1 < \frac{1}{4} + t\big) < \frac{1 - 2P(x_1 = \frac{1}{4})}{6},\ t > 0\Big\}, \quad \delta = \epsilon \cdot \frac{\frac{1}{2} - P(x_1 = \frac{1}{4})}{1 - P(x_1 = \frac{1}{4})}$$

From Lemma 2.2 we know that $P(\frac{1}{4} \leq x_1 \leq \frac{3}{4}) = 1$, thus if player 2 chooses to play $x_2' = (\frac{1}{4} + \epsilon, \frac{3}{4} + \delta)$, the utility of player 1 is:

$$\begin{aligned} U^1(x_1, x_2') \leq P(x_1 = \frac{1}{4}) \cdot (\frac{1}{4} + \frac{\epsilon}{2}) + P(\frac{1}{4} < x_1 < \frac{1}{4} + \epsilon) \overbrace{(\frac{1}{4} + \epsilon)}^{\text{upper bound}} \\ + P(\frac{1}{4} + \epsilon < x_1 \leq \frac{3}{4}) \cdot (\frac{1}{4} + \frac{\delta}{2} - \frac{\epsilon}{2}) \end{aligned}$$

We denote $p_1 = P(x_1 = \frac{1}{4})$ and $p_2 = P(\frac{1}{4} < x_1 < \frac{1}{4} + \epsilon)$. Therefore -

$$\begin{aligned} &= \frac{1}{4} - \frac{\epsilon}{2}\big[1 - 2p_1 - 3p_2\big] + \frac{\delta}{2}\big[1 - p_1 - p_2\big] \\ &< \frac{1}{4} - \frac{\epsilon}{2}\big[1 - 2p_1 - 3 \cdot \frac{1 - 2p_1}{6}\big] + \frac{\delta}{2}\big[1 - p_1 - p_2\big] \\ &= \frac{1}{4} - \frac{\epsilon}{2}\big(\frac{1}{2} - p_1\big) + \frac{\delta}{2}(1 - p_1) - \frac{\delta \cdot p_2}{2} \\ &\overset{\text{plugging } \delta}{=} \frac{1}{4} - \frac{\epsilon}{2}\big(\frac{1}{2} - p_1\big) + \frac{\epsilon}{2} \cdot \frac{\frac{1}{2} - p_1}{1 - p_1}(1 - p_1) - \frac{\delta \cdot p_2}{2} = \frac{1}{4} - \frac{\overbrace{\delta \cdot p_2}^{>0}}{2} < \frac{1}{4} \end{aligned}$$

Thus $U^1(x_1, x_2') < \frac{1}{4}$, so x_2' is a better response to x_1 than x_2, which means that x_1, x_2 cannot be an equilibrium.

Similarly for $P(x_1 = \frac{3}{4}) = \min\Big\{P(x_1 = \frac{1}{4}), P(x_1 = \frac{3}{4})\Big\} < \frac{1}{2}$.

□

3. L VS K EXISTENCE

We are now back to the general case - player 1 has l facilities and player 2 has k facilities ($l \leq k$).

First, for completeness, consider a scenario where only player 2 is in the game. It is quite straightforward to show the following: [6]

THEOREM 3.1. *The optimal facility locations for player 2 are :*

$$x_2 = \left(\frac{1}{2k}, ..., \frac{i}{2k}, ... \frac{2k - 1}{2k}\right)$$

Our main aim is to see the ramifications of having the above duel.

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[5] ϵ is well-defined since $0 < \frac{1 - 2P(x_1 = \frac{1}{4})}{6} < 1$ when $P(x_1 = \frac{1}{4}) < \frac{1}{2}$, and therefore $\delta > 0$.
[6] See the appendix for the full proof.

3.1. l vs k - Equilibrium Protocol

PROTOCOL 2.

(*1*) *Player 1 chooses a tuple of size* l *uniformly from* $\{\frac{1}{2k}, \frac{3}{2k}, ...\frac{2k-1}{2k}\}$. *Denote this strategy* $x_1^{p_{lk}}$.
(*2*) *Player 2 chooses deterministic locations:* $x_2^{p_{lk}} = (\frac{1}{2k}, \frac{3}{2k}, ...\frac{2k-1}{2k})$

Remark: if $l = k$, they both play the same strategy. Denote the strategies in this special case $x_1^{p_{kk}}, x_2^{p_{kk}}$.

In the rest of this section we will prove:

THEOREM 3.2. *Protocol 2 is an equilibrium for the* l *vs* k *facility location game on* $[0, 1]$ *with uniform customers' distribution.*

To prove this theorem, we show that none of the players has greater payoff in case she deviates; this will be obtained using a series of lemmas. We first show:

LEMMA 3.3. *In the* l *vs* k *game where* $1 \leq l < k$, *in case player 2 plays according to Protocol 2, any deterministic strategy played by player 1 with* $l < k$ *facilities can lead to maximum expected payoff of* $\frac{l}{2k}$ *(for player 1).*

PROOF. For any deterministic strategy x_1, any facility $a_i \in x_1$ is either:

(1) located on a facility of player 2. In this case, a_i's contribution to the payoff of player 1 is $\frac{1}{2k}$ or less.
(2) located between two facilities of player 2. In this case, a_i's contribute exactly $\frac{1}{2k}$ to player 1's payoff.
(3) located between one facility of player 1 and one facility of player 2. In this case, it contributes less than $\frac{1}{2k}$.
(4) located between two facilities of player 1. In this case, a_i has no contribution to player 1's payoff (namely, player 1 will get the same payoff with this facility).
(5) located between 0 or 1, and facility of player 2. In this case, it contributes less than $\frac{1}{2k}$.

In the first two cases, the contribution of a single facility to player 1's payoff is at most $\frac{1}{2k}$, whereas in the other three cases it is less. Overall, player 1 cannot get more than $\frac{l}{2k}$. □

The following lemma deals with the k vs k game, for arbitrary k. In addition to being a basic building block for the general result, it also implies a unique deterministic equilibrium for the symmetric case.

LEMMA 3.4. *In the* k *vs* k *game, if player 1 plays according to Protocol 2, any deterministic strategy played by player 2 can lead to maximum profit of* $\frac{1}{2}$ *(for player 2). Moreover, any deterministic strategy* $x_2 \neq x_2^{p_{kk}}$ *will yield less than* $\frac{1}{2}$.

PROOF.

Denote y_j as the locations of player 1 (as in Protocol 2) and b_j as the locations of player 2 (for $j \in \{1, ...k\}$).

We define a set of internal intervals:

$$I_i = \left[y_i, y_{i+1}\right],\ i \in \{1, 2, ..., k-1\}$$

Note that according to $x_1^{p_{kk}}$ each endpoint of such interval holds a facility (exactly k facilities on k different endpoints). We also define the external intervals $I_0 = [0, y_1), I_k = (y_k, 1]$. Note that external interval can only contain one facility of player 2 (since a second facility in these intervals is redundant), and in case it does

- moving the facility towards the open part of the interval (not necessarily on it) can only increase the payoff of player 2, hence there is no facility on $0, 1$.

Let ϵ_0 be:

$$\epsilon_0 = \min\left\{|b_j - y_i| : b_j \in x_2, b_j \neq y_i\right\}$$

In general, ϵ_0 is the distance between the closest facilities of the two players which are not placed on the same spot. Since $x_2^{p_{kk}} \neq x_2$, we know that there is at least one facility of player 2 which lays in the interior of some interval (not on its endpoint, and hence $\epsilon_0 \neq 0$). Denote this facility b_m, and its corresponding intervals I_j.

At each step we modify x_2, while not decreasing player 2's payoff.

Step 1: if I_j has three or more facilities (and therefore it is internal),we look at facilities b_l, b_r such that:

$$b_l = \min\{b_i : b_i \in x_2, b_i \in I_j\}, b_r = \max\{b_i : b_i \in x_2, b_i \in I_j\}$$

And w.l.o.g $b_m \in (b_l, b_r)$. We tempurarly remove b_m from I_j. Note that the damage (in terms of payoff) from this move is upper bounded by $\frac{1}{4k}$ (in case there are exactly 3 facilities in I_j and b_l, b_r lays on its endpoints y_{j-1}, y_j). Next, if player 2 does not have facility in $I_0 \cup \{y_1\}$ (or $I_{k+1} \cup \{y_k\}$), she can place her m-th facility on y_1 (y_k) and increase her payoff by at least $\frac{1}{4k}$, so this transition results in increasing her payoff. Otherwise, we look at the sum of distances between every 2 facilities of player 2 (after removing b_m):

$$\sum_{\substack{i=2 \\ i \neq j, j+1}}^{k} (b_i - b_{i-1}) \geq \frac{k-1}{k} \quad \Rightarrow \quad \frac{1}{k-2} \sum_{\substack{i=2 \\ i \neq j, j+1}}^{k} (b_i - b_{i-1}) \geq \frac{k-1}{k-2}\frac{1}{k} > \frac{1}{k}$$

Thus the average distance between any two neighbor facilities of player 2 is greater than k, and therefore there exist open interval J with length greater than $\frac{1}{k}$ with no facilities of player 2 inside it. If J contains two facilities of player 1, we place b_m between them, and as a result player 2 gains $\frac{1}{2k}$. Otherwise, player 1 have only one facility inside J, and by placing b_m in the same point player 2 gains at least $\frac{1}{4k}$.

Overall, player 2 can only increase her payoff from this transition.

Step 2: If I_j contains two facilities, there are 3 cases (remember that b_m is not on an endpoint):

(1) they are both placed in the interior of I_j (not on the endpoints). In this case, we assume that they are located on $\frac{2j-1}{2k} + \epsilon_0, \frac{2j+1}{2k} - \epsilon_0$, since any other locations in the interior translates to lower payoff.

(2) one facility is located on the left endpoint and b_m is in the interior. In this case, we can assume that $b_m = y_{j+1} - \epsilon_0$. Otherwise, moving b_m to $b_m < b'_m < y_{j+1}$ will have the following effect on player 2's payoff:

$$[\frac{3}{4}(b'_m - b_{m-1}) + \frac{1}{2}(y_{j+1} - b'_m)] - [\frac{3}{4}(b_m - b_{m-1}) + \frac{1}{2}(y_{j+1} - b_m)] = \frac{1}{4}(b'_m - b_m) > 0$$

Therefore by setting $b_m = y_{j+1} - \epsilon_0$ player 2's payoff can only increase.

(3) one facility is located on the right endpoint and the other is in the interior. Completely symmetrical to case (2), hence we omit it.

Step 3: b_m is the only facility contained in I_j. If b_m is located on an endpoint of I_j or if it is located ϵ_0 closed to an endpoint, we skip this step (Note that if this is not the case, I_j must be an internal interval). Otherwise, moving b_m inside the interior of I_j does not have any effect on player 2's payoff, hence we place b_m on $y_j + \epsilon_0$ (or $y_1 - \epsilon_0$ if $j = 1$).

Step 4: We define a sequence of strategies x_2^n to be x_2 with $\epsilon = \frac{1}{n}$ for $n \geq n_0 = \lceil \frac{1}{\epsilon_0} \rceil$. Obviously, the corresponding payoff sequence $U^2(x_1^{p_{kk}}, x_2^n)$ is strictly increasing in n, and thus $U^1(x_1^{p_{kk}}, x_2^n)$ is strictly decreasing in n.

We define -

$$\mathcal{V}_{b_j}(x_1^{p_{kk}}, x_2^n)$$

as the contribution of single facility y_j of player 1 to her payoff when she plays $x_1^{p_{kk}}$ and player 2 plays x_2^n. As a result of steps 1-3, for each y_i exactly one of the following holds:
(1) player 2 has facility in y_i
(2) player 2 has facility in $y_i - \epsilon$ (but not in $y_i + \epsilon$)
(3) player 2 has facility in $y_i + \epsilon$ (but not in $y_i - \epsilon$)
(4) player 2 has facilities in $y_i + \epsilon, y_i - \epsilon$
(5) player 2 has no facilities in $(y_{i-1} + \epsilon, y_{i+1} - \epsilon)$ (denote $y_{-1} = -\epsilon, y_{k+1} = 1 + \epsilon$ in order to cover the case where $i = 1, k$)

We define a partition of $\{y_1, ...y_k\}$, C_p for $p = 1, ..5$, by placing y_i in the corresponding case. Since each player has k facilities, any $y_i \in C_4$ suggest that there exist $y_{i'} \in C_5$, hence $|C_4| = |C_5|$. Therefore:

$$\lim_{n\to\infty} U^1(x_1^{p_{kk}}, x_2^n) = \lim_{n\to\infty} \sum_{j=1}^{k} \mathcal{V}_{b_j}(x_1^{p_{kk}}, x_2^n) = \frac{|C_1| + |C_2| + |C_3|}{2k} + 0 \cdot |C_4| + \frac{|C_5|}{k}$$

$$\overset{|C_4|=|C_5|}{=} \frac{|C_1| + |C_2| + |C_3|}{2k} + \frac{|C_4| + |C_5|}{2k} = \frac{k}{2k} = \frac{1}{2}$$

Thus $U^1(x_1^{p_{kk}}, x_2^{p_{kk}}) = \frac{1}{2}$ only when $\epsilon = 0$ (as in Protocol 2), and otherwise $U^1(x_1^{p_{kk}}, x_2)$ is greater than $\frac{1}{2}$. and , so overall we conclude that player 1's expected payoff is at least $\frac{1}{2}$. $\square$

In order to analyze how well can player 2 respond to player 1's strategy, we present the following notation:

$$\mu\big([a,b]\big) - \text{the expected number of player 1's facilities in the interval } [a,b]$$

In general, μ is used to quantify the probability of seeing a facility of player 1 in some sub segment. μ is also a measure, and we shall use some of its properties.[7]. Note that if player 1 has l facilities then $\mu\big([0,1]\big) = l$.

The following lemma uses the previous one; its proof exploits basic connections between k vs k duels and general l vs k duels.

LEMMA 3.5. *In l vs k game, if the strategy used by player 1 satisfies:*

$$\forall i, j \in \{1, 2...k\} : \ \mu\big(y_i\big) = \mu\big(y_j\big) = \frac{l}{k} \quad \left(y_i = \frac{2i-1}{2k}\right)$$

[7]See the appendix for a more formal treatment

any deterministic strategy played by player 2 can lead to minimum profit of $\frac{l}{2k}$ (for player 1)

Remark: Note that $x_1^{p_{lk}}$ satisfies the above condition.

PROOF. Denote such strategy of player 1 as f_1. We define

$$\mathcal{V}_{a_i}(f_1, x_2)$$

as the payoff that facility a_i contributes to player 1 in case she is playing f_1 and player 2 plays x_2. Notice that for $i \in \{1, 2...l\}$ it holds that adding one more facility can only decrease each single facility's contribution, Namely $\mathcal{V}_{a_i}(f_1, x_2) \geq \mathcal{V}_{a_i}(f_1 \cup a_{l+1}, x_2)$. Let x_2 be a deterministic strategy of player 2, and let X_1 be the random variable that represents player 1's locations. Therefore -

$$U^1(X_1, x_2) = \sum_{y_i \in X_1} \mathcal{V}_{y_i}(X_1, x_2) = \sum_{i=1}^{k} \mathcal{V}_{y_i}(X_1, x_2)\mathbb{1}[y_i \in X_1]$$

$$\geq \sum_{i=1}^{k} \mathcal{V}_{y_i}(X_1 \cup x_1^{p_{kk}}, x_2)\mathbb{1}[y_i \in X_1] \overset{X_1 \subseteq x_1^{p_{kk}}}{=} \sum_{i=1}^{k} \mathcal{V}_{y_i}(x_1^{p_{kk}}, x_2)\mathbb{1}[y_i \in X_1]$$

Taking the expectation of the last argument:

$$U^1(f_1, x_2) = E\big[U^1(X_1, x_2)\big] = E\big[\sum_{i=1}^{k} \mathcal{V}_{y_i}(x_1^{p_{kk}}, x_2)\mathbb{1}[y_i \in X_1]\big] = \sum_{i=1}^{k} \mathcal{V}_{y_i}(x_1^{p_{kk}}, x_2)\mu\big(y_i\big)$$

$$= \frac{l}{k} \underbrace{\sum_{i=1}^{k} \mathcal{V}_{y_i}(x_1^{p_{kk}}, x_2)}_{U^1(x_1^{p_{kk}}, x_2)}$$

We saw that $U^1(x_1^{p_{kk}}, x_2) \geq \frac{1}{2}$, and this inequality holds as equality only when $x_2 = x_2^{p_{kk}}$, therefore:

$$U^1(f_1, x_2) \geq \frac{l}{2k}$$

□

Proof of Theorem 3.2. Lemmas 3.3 and 3.5 imply that any change in player i's strategy will lead to decrease in her expected reward, yielding the desired result.

4. L VS K QUASI-UNIQUENESS

In this section we complete, when augmented with the results of the previous section, the main result of the paper. Namely, we will show that any equilibrium of the above duels have the property that player 2 always uses the previously mentioned equilibrium protocol, while player 1 uses some distribution over the set of locations selected by player 2. Moreover, based on the results of the previous section we characterize how this distribution may look like. While this does not imply uniqueness as in the case 1 vs 2, it implies that player 1 does not contribute in any equilibrium any new location, beyond what player 2 contributes, which are the locations that correspond to the single-player k-facility social welfare maximizing assignment. We term this phenomenon quasi-uniqueness, as the location of the facilities is uniquely determined.

LEMMA 4.1. *Properties of any l vs k equilibrium –*
If (x_1, x_2) is an equilibrium profile, then the following holds:

(*1*) *Player 1 gets an expected payoff of* $\frac{l}{2k}$
(*2*) $\mu\big([0,\frac{1}{2k})\big)=\mu\big((\frac{2k-1}{2k},1]\big)=0$
(*3*) $U^1(x_1,x_2)=U^1(x_1,x_2^{p_{lk}})$

PROOF.

(1) In case player 1 gets less - she can deviate to Protocol 2 and get $\frac{l}{2k}$, and in case she gets more - player 2 can deviate to Protocol 2 and make her get $\frac{l}{2k}$.
(2) Any facility which is located in $[0,\frac{1}{2k})$ will give payoff of less than $\frac{1}{2k}$ to player 1, hence contradiction to 1. Similarly for $(\frac{2k-1}{2k},1]$.
(3) If $U^1(x_1,x_2)>U^1(x_1,x_2^{p_{lk}})\Rightarrow U^2(x_1,x_2)<U^2(x_1,x_2^{p_{lk}})$, in contradiction to x_1,x_2 being an equilibrium. On the other hand, we saw that $\forall x_1' : U^1(x_1',x_2^{p_{lk}})\leq\frac{l}{2k}$, hence $\frac{l}{2k}=U^1(x_1,x_2)<U^1(x_1,x_2^{p_{lk}})\leq\frac{l}{2k}$ is also a contradiction.

□

THEOREM 4.2. *Quasi-Uniqueness of equilibrium in the* l *vs* k *game*

In the l *vs* k *facility location duel on the* $[0,1]$ *segment with uniform customers' distribution, the strategies* (x_1,x_2) *are in equilibrium iff:*

(*1*) x_1 *satisfies that* $\forall i,j\in\{1,2...k\}:\ \mu\big(y_i\big)=\mu\big(y_j\big)=\frac{l}{k}\quad(y_i=\frac{2i-1}{2k})$
(*2*) $x_2=x_2^{p_{lk}}$

PROOF. We begin with the easier side - if $(1),(2)$ are satisfied, since player 2 has no benefit in deviating according to Lemma 3.5, and player 1 has the same inclination according to Lemma 3.3, we conclude that the profile (x_1,x_2) is in equilibrium.

On the other direction: Assume (x_1,x_2) is an equilibrium strategy profile for players 1 and 2 . In case player 1 plays according to Protocol 2, the (only) best response for player 2 is to play according to the protocol as well. Thus if $x_1=x_1^{p_{lk}}$ it follows that $x_2=x_2^{p_{lk}}$, and since Protocol 2 satisfies (1) and (2) we are done.

If this not the case, then $x_1\neq x_1^{p_{lk}}$. Next, we try to find strategy for player 2, x_2', such that $U^1(x_1,x_2)>U^1(x_1,x_2')$, and using Lemma 4.1 it is equivalent to show that $U^1(x_1,x_2')<U^1(x_1,x_2^{p_{lk}})=\frac{l}{2k}$. By doing so, we show that player 2 will deviate in order to increase her payoff (and decrease player 1's). Using the fact that (x_1,x_2) is an equilibrium profile, and hence x_2' should not exist - we can characterize the possible strategies x_1.

step 1: $\mu\big(y_1\big)\geq\mu\big((y_1,y_2]\big)$
Assume by contradiction that $\mu\big((y_1,y_2]\big)-\mu\big(y_1\big)=\delta>0$. Consider

$$x_2'=(y_1+\epsilon_1,y_2+\epsilon_0,y_3+\epsilon_0...,y_k+\epsilon_0)$$

for $\epsilon_1,\epsilon_0>0$ such that [8] :

$$\mu\big((y_1,y_1+\epsilon_1)\big)<\frac{\delta}{6},\quad\epsilon_0<\frac{\delta\cdot\epsilon_1}{4}$$

[8] We can assume that $\mu(y_i+\epsilon_j)=0$ for $i\in\{1,...k\},j\in\{0,1\}$ because of Lemma A.3 in the appendix, and that such ϵ_0,ϵ_1 exist. We should also state that $\epsilon_0,\epsilon_1<\frac{1}{2k}$.

Namely, x_2' is similar to $x_2^{p_{1k}}$, but every facility is shifted right by ϵ_0, except from the facility on y_1 which is shifted right by ϵ_1. By Lemma 4.1 part (2) we are guaranteed that there are no facilities of player 1 in $[0, \frac{1}{2k}), (\frac{2k-1}{2k}, 1]$. Therefore, the payoff of player 1:

$$\begin{aligned}
U^1(x_1, x_2') &\leq U^1(x_1, x_2^{p_{1k}}) + \mu\big(y_1\big) \cdot (\frac{\epsilon_1}{2}) + \mu\big((y_1, y_1+\epsilon_1)\big) \cdot (\epsilon_1) + \mu\big([y_1+\epsilon_1, y_2+\epsilon_0]\big) \cdot (\frac{\epsilon_0 - \epsilon_1}{2}) \\
&\leq \frac{l}{2k} - \frac{\epsilon_1}{2} \underbrace{\Big(\mu\big([y_1+\epsilon_1, y_2+\epsilon_0]\big) - \mu\big(y_1\big) - 2 \cdot \mu\big((y_1, y_1+\epsilon_1)\big)\Big)}_{* \geq \frac{\delta}{2}} + \epsilon_0 \\
&\leq \frac{l}{2k} - \frac{\delta \cdot \epsilon_1}{4} + \epsilon_0 < \frac{l}{2k}
\end{aligned}$$

And $*$ is due to:

$$\begin{aligned}
&\mu\big([y_1+\epsilon_1, y_2+\epsilon_0]\big) - \mu\big(y_1\big) - 2 \cdot \mu\big((y_1, y_1+\epsilon_1)\big) \\
&\geq \mu\big([y_1+\epsilon_1, y_2]\big) - \mu\big(y_1\big) - 2 \cdot \mu\big((y_1, y_1+\epsilon_1)\big) \\
&= \mu\big((y_1, y_2]\big) - \mu\big((y_1, y_1+\epsilon_1)\big) - \mu\big(y_1\big) - 2 \cdot \mu\big((y_1, y_1+\epsilon_1)\big) \\
&= \mu\big((y_1, y_2]\big) - \mu\big(y_1\big) - 3 \cdot \mu\big((y_1, y_1+\epsilon_1)\big) \\
&> \delta - \frac{3 \cdot \delta}{6} = \frac{\delta}{2}
\end{aligned}$$

Hence we got a contradiction and $\mu\big(y_1\big) \geq \mu\big((y_1, y_2]\big)$. By symmetry we can also conclude that $\mu\big(y_k\big) \geq \mu\big([y_{k-1}, y_k)\big)$.
step 2: $\forall 1 \leq i \leq k-2: \ \mu\big((y_i, y_{i+1}]\big) \geq \mu\big((y_{i+1}, y_{i+2}]\big)$
Assume by contradiction that $\mu\big((y_{i+1}, y_{i+2}]\big) - \mu\big((y_i, y_{i+1}]\big) = \delta > 0$ holds for some i. Consider

$$x_2' = (y_1 + \epsilon_0, ...y_i + \epsilon_0, y_{i+1} + \epsilon_1, y_{i+2} + \epsilon_0, .., y_k + \epsilon_0)$$

for $\epsilon_1, \epsilon_0 > 0$ such that:

$$\mu\big((y_{i+1}, y_{i+1} + \epsilon_1)\big) < \frac{\delta}{4}, \quad \epsilon_0 < \frac{\delta \cdot \epsilon_1}{4l}$$

Namely, x_2' is similar to $x_2^{p_{1k}}$, but every facility is shifted right by ϵ_0, except from the facility on y_{i+1} which is shifted right by ϵ_1. The payoff of player 1:

$$\begin{aligned}
U^1(x_1, x_2') &\leq U^1(x_1, x_2^{p_{1k}}) + \mu\big([y_1, y_1+\epsilon_0)\big) \cdot \epsilon_0 + \mu\big((y_i+\epsilon_0, y_{i+1}+\epsilon_1)\big) \cdot (\frac{\epsilon_1 - \epsilon_0}{2}) \\
&\quad + \mu\big([y_{i+1}+\epsilon_1, y_{i+2}+\epsilon_0]\big) \cdot (\frac{\epsilon_0 - \epsilon_1}{2}) \\
&\leq \frac{l}{2k} + l \cdot \epsilon_0 - \frac{\epsilon_1}{2} \cdot \underbrace{\Big(\mu\big([y_{i+1}+\epsilon_1, y_{i+2}+\epsilon_0]\big) - \mu\big((y_i+\epsilon_0, y_{i+1}+\epsilon_1)\big)\Big)}_{* \geq \frac{\delta}{2}} \\
&\leq \frac{l}{2k} + l \cdot \epsilon_0 - \frac{\delta \cdot \epsilon_1}{4} < \frac{l}{2k}
\end{aligned}$$

And $*$ is due to:

$$\mu\big([y_{i+1}+\epsilon_1, y_{i+2}+\epsilon_0]\big) - \mu\big((y_i+\epsilon_0, y_{i+1}+\epsilon_1)\big) \geq \mu\big([y_{i+1}+\epsilon_1, y_{i+2}]\big) - \mu\big((y_i, y_{i+1}+\epsilon_1)\big)$$
$$= \mu\big((y_{i+1}, y_{i+2}]\big) - \mu\big((y_{i+1}, y_{i+1}+\epsilon_1\big) - \mu\big((y_i, y_{i+1}]\big) - \mu\big((y_{i+1}, y_{i+1}+\epsilon_1)\big)$$
$$= \mu\big((y_{i+1}, y_{i+2}]\big) - \mu\big((y_i, y_{i+1}]\big) - 2\mu\big((y_{i+1}, y_{i+1}+\epsilon_1)\big) > \delta - 2\cdot\frac{\delta}{4} = \frac{\delta}{2}$$

From symmetry we can also conclude that

$$\forall 1 \leq i \leq k-2: \ \mu\big([y_i, y_{i+1})\big) \leq \mu\big([y_{i+1}, y_{i+2})\big)$$

step 3: From steps 1,2 and monotonicity of μ:

$$\mu\big(y_1\big) \geq \mu\big((y_1, y_2]\big) \geq \mu\big((y_2, y_3]\big) ..... \geq \mu\big((y_{k-2}, y_{k-1}]\big) \geq \mu\big((y_{k-1}, y_k]\big) \geq \mu\big(y_k\big)$$
$$\mu\big(y_1\big) \leq \mu\big([y_1, y_2)\big) \leq \mu\big([y_2, y_3)\big) ..... \leq \mu\big([y_{k-2}, y_{k-1})\big) \leq \mu\big([y_{k-1}, y_k)\big) \leq \mu\big(y_k\big)$$

Therefore all inequalities holds as equalities, and it holds that [9]:

(1) $\forall i,j \in \{1...k\}: \ \mu\big(y_i\big) = \mu\big(y_j\big) = \frac{\mu\big([0,1]\big)}{k} = \frac{l}{k}$
(2) $\forall i,j \in \{1...k\}: \mu\big((y_i, y_{i+1})\big) = \mu\big((y_j, y_{j+1})\big) = 0$

So (1) is satisfied. From Lemma 3.5 we conclude that in case player 2 plays strategy $x_2 \neq x_2^{p_{1k}}$ her payoff is less than $1 - \frac{l}{2k}$, hence (2) is satisfied as well.

□

5. EXTENSIONS TO NON-UNIFORM DISTRIBUTIONS

In this section we relax the assumption of uniform distribution over the customers. We will assume customers' preferences are distributed along $[-\infty, \infty]$ according to some (known) distribution function $f_c(t)$. We will denote the cumulative distribution function as $F_c(t)$. We restrict the scope of this section to continuous functions. Some discrete settings will be briefly mentioned in the discussion section. Our focus is on the extension of the foundational 1 vs 2 setting.

Definition 5.1. Assume there are 2 facilities a, b located on the segment $[0,1]$. We say that a point c is an *indifference point* if a customer located at c is indifferent between going to a or going to b. This is equivalent to:

$$|c-a| = |c-b|$$

LEMMA 5.2. *If one player with two facilities aims to maximize the social welfare, she should locate her facilities on* a, b $(a \leq b)$ *such that -*

$$\begin{cases} P(t \leq a) = P(a \leq t \leq \frac{a+b}{2}) \\ P(\frac{a+b}{2} \leq t \leq b) = P(t \geq b) \end{cases}$$

We omit the proof of this lemma. A simple explanation to this result is that the optimal locations for a, b turns $c = \frac{a+b}{2}$ into an indifference point, and a, b are the medians of $[0, c], [c, 1]$ respectively.

[9] Since $\forall i : 1 \leq i \leq k-1$ it follows that $\mu\big([y_i, y_{i+1})\big) = \mu\big((y_i, y_{i+1}]\big) \Rightarrow \mu\big(y_i\big) = \mu\big(y_{i+1}\big)$, $\mu\big(y_1\big) = \mu\big((y_1, y_2]\big) \Rightarrow \mu\big((y_1, y_2)\big) = 0$ and also $\mu\big([y_i, y_{i+1})\big) = \mu\big([y_{i+1}, y_{i+2})\big) \Rightarrow \mu\big((y_i, y_{i+1})\big) = \mu\big((y_{i+1}, y_{i+2})\big) = 0$

5.1. Equilibrium in the Non-Uniform Case: LM distributions

Denote q_a^c the a-th quantile of $F_c(t)$, namely $q_a^c = \inf\{t \in [-\infty, \infty] : a \leq F_c(t)\}$. A simple generalization of the equilibrium protocol introduced earlier is as follows:

PROTOCOL 3.

(*1*) *Player 1 chooses her facility's location using*

$$x_1^{p_3} = \begin{cases} q^c_{1/4} & w.p. \frac{1}{2} \\ q^c_{3/4} & w.p. \frac{1}{2} \end{cases}$$

(*2*) *Player 2 locates her facilities deterministically* $x_2^{p_3} = (q^c_{1/4}, q^c_{3/4})$.

However, Protocol 3 is not guaranteed to hold as an equilibrium for all possible customer preferences distributions. For example, as will be implied by our general result in the following sub-section it does not hold as equilibrium for $Beta(\alpha, \alpha)$. Nevertheless, we can characterize a large family of natural distributions for which Protocol 3 is in fact equilibrium.

Definition 5.3. Light-Middle distribution (or simply LM)
A distribution $f_c(t)$ is *LM* if it satisfies:

(1) $\frac{q^c_{1/4}+q^c_{3/4}}{2} = q^c_{1/2}$

(2) $\forall x \in (q^c_{1/4}, q^c_{3/4}) : \int_{\frac{q^c_{1/4}+x}{2}}^{\frac{q^c_{3/4}+x}{2}} f_c(t)dt \leq \frac{1}{4}$

An example of LM distribution is provided in figure 1 (a) .On first sight, the second condition may appear to be quite technical. The goal behind the next lemma is to point to a simple sufficient condition on when a given customers' preferences distribution is LM, pointing to its generality.

LEMMA 5.4. *If* $f_c(t)$ *satisfies:*

(*1*) *First condition of LM distribution*
(*2*) $f_c(t)$ *is monotonically non increasing in* $[q^c_{1/4}, q^c_{1/2}]$
(*3*) $f_c(t)$ *is monotonically non decreasing in* $[q^c_{1/2}, q^c_{3/4}]$

Then $f_c(t)$ *is a LM distribution.*[10]

The meaning of LM is that customers generally have a strong preferences towards one of the "sides" of the segment. More generally, there is no dense subset of the customers hidden between the 1/4 quantile and the 3/4 quantile. This general set of distributions do allow for the following:

THEOREM 5.5 (1 VS 2 EQUILIBRIUM WITH LM DISTRIBUTION). *Protocol 3 is an equilibrium for* 1 *vs* 2 *facility location with any LM customers' distribution.*

The proof of this theorem relies on the next two lemmas:

LEMMA 5.6. *If the customers distribution is LM, and player 2 plays according to Protocol 3, any deterministic strategy played by player 1 can lead to maximum expected reward of* $\frac{1}{4}$.

[10] See proof of this lemma in the appendix

PROOF. Denote x_1 as the strategy of player 1. For all possible values of x_1, we look at $U^1(x_1, x_2^{p_3})$:

$$(1)\ x_1 < q^c_{1/4} : \int_{-\infty}^{\frac{x+q^c_{1/4}}{2}} f_c(t)dt < \int_{-\infty}^{q^c_{1/4}} f_c(t)dt = \frac{1}{4}$$

$$(2)\ x_1 = q^c_{1/4} : \frac{1}{2}\int_{-\infty}^{\frac{q^c_{1/4}+q^c_{3/4}}{2}} f_c(t)dt \overset{\text{property 1}}{=} \frac{1}{2}\int_{-\infty}^{q^c_{1/2}} f_c(t)dt = \frac{1}{4}$$

$$(3)\ q^c_{1/4} < x_1 \le q^c_{1/2} : \underbrace{\int_{\frac{q^c_{1/4}+x_1}{2}}^{\frac{q^c_{3/4}+x_1}{2}} f_c(t)dt \le \frac{1}{4}}_{\text{property 2}}$$

And similarly for the cases we omitted. □

LEMMA 5.7. *If the customers preferences' distribution is LM , and player 1 plays according to Protocol 3, any deterministic strategy played by player 2 can lead to maximum expected reward of* $\frac{3}{4}$.

PROOF. We know that player 1 plays according to $x_1^{p_3}$. We denote $x_2 = (a, b)$ as the deterministic strategy where player 2 locates one facility on a and one facility on b (w.l.o.g. $a \le b$). We create a partition of $[-\infty, \infty]$, and consider all possible cases. Denote

$$I_1 = (-\infty, q^c_{1/4}), I_2 = \{q^c_{1/4}\}, I_3 = (q^c_{1/4}, q^c_{3/4}), I_4 = \{q^c_{3/4}\}, I_5 = (q^c_{3/4}, \infty)$$

and let $L(l) = i$ such that $l \in I_i$. The possible cases for $(L(a), L(b))$ are:

$$\underbrace{(1,1),(5,5)}_{1},\underbrace{(1,2),(4,5)}_{2},\underbrace{(3,3)}_{3},\underbrace{(2,3),(3,4)}_{4},\underbrace{(1,3),(3,5)}_{5},\underbrace{(1,4),(2,5)}_{6},\underbrace{(1,5)}_{7},\underbrace{(2,2),(4,4)}_{8}\underbrace{(2,4)}_{9}$$

Two cases marked with the same digit are completely symmetrical, so we show only one of them (the left one). For each case, we look at $U^1(x_1^{p_3}, x_2)$:

$$(1)\ > \frac{1}{2}\int_{q^c_{1/4}}^{\infty} f_c(t)dt + \frac{1}{2}\int_{q^c_{1/2}}^{\infty} f_c(t)dt = \frac{1}{2}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{1}{2} = \frac{5}{8}$$

$$(2)\ = \frac{1}{2}\cdot\frac{1}{2}\int_{q^c_{1/4}}^{\infty} f_c(t)dt + \frac{1}{2}\int_{q^c_{1/2}}^{\infty} f_c(t)dt = \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{1}{2} = \frac{7}{16}$$

$$(3)\ > \frac{1}{2}\int_{-\infty}^{q^c_{1/4}} f_c(t)dt + \frac{1}{2}\int_{q^c_{3/4}}^{\infty} f_c(t)dt = \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{4}$$

$$(4)\ = \frac{1}{2}\cdot\frac{1}{2}\int_{-\infty}^{\frac{b+q^c_{1/4}}{2}} f_c(t)dt + \frac{1}{2}\int_{\frac{b+q^c_{3/4}}{2}}^{\infty} f_c(t)dt$$

$$= \frac{1}{2}\cdot\Big(\frac{1}{2}\int_{-\infty}^{\frac{b+q^c_{1/4}}{2}} f_c(t)dt + \int_{\frac{b+q^c_{1/4}}{2}}^{\infty} f_c(t)dt - \int_{\frac{b+q^c_{1/4}}{2}}^{\frac{b+q^c_{1/4}}{2}} f_c(t)dt\Big)$$

$$\overset{\text{property 2}}{\ge} \frac{1}{2}\cdot\Big(\int_{-\infty}^{\infty} f_c(t)dt - \frac{1}{2}\int_{-\infty}^{\frac{b+q^c_{1/4}}{2}} f_c(t)dt - \frac{1}{4}\Big)$$

$$= \frac{1}{2}\cdot\Big(\frac{3}{4} - \frac{1}{2}\int_{-\infty}^{\frac{b+q^c_{1/4}}{2}} f_c(t)dt\Big) = \frac{3}{8} - \frac{1}{4}\int_{-\infty}^{\frac{b+q^c_{1/4}}{2}} f_c(t)dt$$

Recall that $b < q^c_{3/4} \Rightarrow \dfrac{b+q^c_{1/4}}{2} < \dfrac{q^c_{3/4}+q^c_{1/4}}{2} = q^c_{1/2}$:

$$= \frac{3}{8} - \frac{1}{4}\int_{-\infty}^{\frac{b+q^c_{1/4}}{2}} f_c(t)dt \geq \frac{3}{8} - \frac{1}{4}\int_{-\infty}^{q^c_{1/2}} f_c(t)dt = \frac{1}{4}$$

$$(5)\ = \frac{1}{2}\int_{\frac{a+q^c_{1/4}}{2}}^{\frac{b+q^c_{1/4}}{2}} f_c(t)dt + \frac{1}{2}\int_{\frac{b+q^c_{3/4}}{2}}^{\infty} f_c(t)dt = \frac{1}{2}\cdot\Big(\int_{\frac{a+q^c_{1/4}}{2}}^{\infty} f_c(t)dt - \int_{\frac{b+q^c_{1/4}}{2}}^{\frac{b+q^c_{3/4}}{2}} f_c(t)dt\Big)$$

$$\geq \frac{1}{2}\cdot\Big(\frac{3}{4} - \int_{\frac{b+q^c_{1/4}}{2}}^{\frac{b+q^c_{3/4}}{2}} f_c(t)dt\Big) \overset{\text{property 2}}{\geq} \frac{1}{2}\cdot\Big(\frac{3}{4}-\frac{1}{4}\Big) = \frac{1}{4}$$

$$(6)\ = \frac{1}{2}\int_{\frac{a+q^c_{1/4}}{2}}^{\frac{q^c_{1/4}+q^c_{3/4}}{2}} f_c(t)dt + \frac{1}{2}\cdot\frac{1}{2}\int_{\frac{a+q^c_{3/4}}{2}}^{\infty} f_c(t)dt \overset{\text{property 1}}{=} \frac{1}{2}\int_{\frac{a+q^c_{1/4}}{2}}^{q^c_{1/2}} f_c(t)dt + \frac{1}{2}\cdot\frac{1}{2}\int_{\frac{x+q^c_{3/4}}{2}}^{\infty} f_c(t)dt$$

$$\geq \frac{1}{2}\int_{q^c_{1/4}}^{q^c_{1/2}} f_c(t)dt + \frac{1}{2}\cdot\frac{1}{2}\int_{q^c_{1/2}}^{\infty} f_c(t)dt \geq \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$(7)\ > \frac{1}{2}\int_{q^c_{1/4}}^{\frac{q^c_{1/4}+b}{2}} f_c(t)dt + \frac{1}{2}\int_{\frac{q^c_{3/4}+a}{2}}^{q^c_{3/4}} f_c(t)dt \geq \frac{1}{2}\int_{q^c_{1/4}}^{q^c_{1/2}} f_c(t)dt + \frac{1}{2}\int_{q^c_{1/2}}^{q^c_{3/4}} f_c(t)dt$$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$(8)\ = \frac{1}{2}\cdot\frac{1}{2}\int_{-\infty}^{\infty} f_c(t)dt + \frac{1}{2}\int_{q^c_{1/2}}^{\infty} f_c(t)dt = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

$$(9) = \frac{1}{2}\cdot\frac{1}{2}\int_{-\infty}^{q^c_{1/2}} f_c(t)dt + \frac{1}{2}\cdot\frac{1}{2}\int_{q^c_{1/2}}^{\infty} f_c(t)dt = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}$$

Therefore it holds that for all possible strategies x_2 $U^1(x_1^{p_3}, x_2) \geq \frac{1}{4}$, and because this is a 1-sum game we know that $U^2(x_1^{p_3}, x_2) \leq \frac{3}{4}$.

□

Proof of Theorem 5.5. Lemmas 5.6 and 5.7 imply that any change in player i's strategy will lead to decrease in her expected reward, and therefore it is an equilibrium by definition.

5.2. The MHM distributions

Next, we shall analyze a family of natural distributions where Protocol 3 does not hold as equilibrium. Our result will show that selecting among the optimal single-player facility locations will not hold as equilibrium.

Definition 5.8. A distribution is *Monotone-Heavy-Middle* (or simply MHM) if it satisfies:

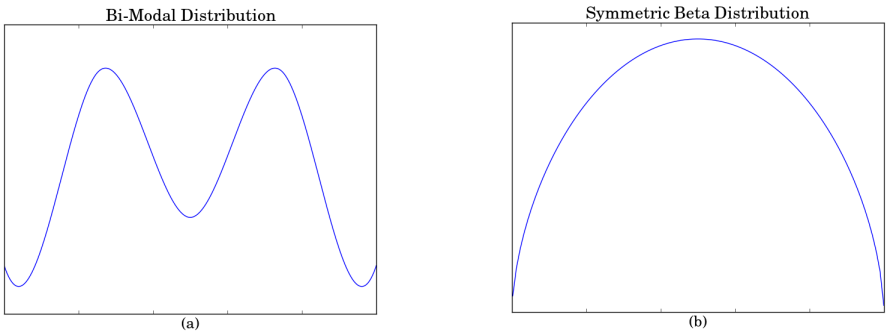


Fig. 1. Examples of a LM (a) and a MHM (b) distributions

(1) $\frac{q^c_{1/4}+q^c_{3/4}}{2} = q^c_{1/2}$
(2) $f_c(t)$ is monotonically increasing in $[q^c_{1/4}, q^c_{1/2}]$
(3) $f_c(t)$ is monotonically decreasing in $[q^c_{1/2}, q^c_{3/4}]$

A wide range of classical preferences distributions are MHM - Normal, Beta etc (see figure 1 (b) for an illustration). The situations MHM captures are where the majority of people does not hold a strong preference towards the sides, but rather enjoys being the median. In a sense, such distributions are the complement of the LM distributions.

One may wonder if there are any locations, not necessarily the single-player optimal locations as in Protocol 3, where player 2 can locate her facilities and player 1 act in a similar manner (i.e. select between player 2's facilities). The following lemma shows that this situation is impossible:

LEMMA 5.9. *There are no non-strictly-mixed equilibria for any MHM distribution.*

PROOF. Since there is neither a pure equilibrium nor an equilibrium where player 1 plays pure strategy and player 2 play mixed strategy, we are left with the case where player 2 plays a deterministic strategy (a, b), and player 1 selects between these locations with some probabilities p_a, p_b. Assume by contradiction that a profile of such strategies (x_1, x_2) is an equilibrium. Using the fact that $U^1(a,(a,b)) = U^1(b,(a,b))$ we conclude that:

$$\frac{1}{2}F_c(\frac{a+b}{2}) = \frac{1}{2}\cdot\left(1 - F_c(\frac{a+b}{2})\right) \Rightarrow \frac{a+b}{2} = q^c_{1/2}$$

Hence $U^1(x_1, x_2) = \frac{1}{4}$. Furthermore, Since (x_1, x_2) is an equilibrium profile, player 1 does not deviate, hence $\forall \epsilon > 0$ it holds that:

$$F_c(a) - \epsilon < \frac{1}{2}F_c(\frac{a+b}{2}) \Rightarrow a \leq q^c_{1/4}$$

And using similar argument we conclude that $b \geq q^c_{3/4}$. By definition of MHM $f_c(t)$ is monotonically decreasing in $[q^c_{1/2}, q^c_{3/4}]$, thus:

$$\forall t \in [0, \frac{q^c_{3/4} - q^c_{1/2}}{2}] : f_c(q^c_{1/2} + t) > f_c(\frac{q^c_{1/2} + q^c_{3/4}}{2} + t)$$
$$\Rightarrow \int_0^{\frac{q^c_{3/4}-q^c_{1/2}}{2}} f_c(q^c_{1/2} + t)dt > \int_0^{\frac{q^c_{3/4}-q^c_{1/2}}{2}} f_c(\frac{q^c_{1/2} + q^c_{3/4}}{2} + t)dt$$

$$\Rightarrow F_c(\frac{q^c_{1/2}+q^c_{3/4}}{2}) - F_c(q^c_{1/2}) > F_c(q^c_{3/4}) - F_c(\frac{q^c_{1/2}+q^c_{3/4}}{2}) \Rightarrow F_c(\frac{q^c_{1/2}+q^c_{3/4}}{2}) > \frac{5}{8}$$

And similarly we show that $F_c(\frac{q^c_{1/2}+q^c_{1/4}}{2}) < \frac{3}{8}$. Now, if player 1 deviates and plays $x'_1 = q^c_{1/2}$, her payoff will be:

$$F_c(\frac{b+q^c_{1/2}}{2}) - F_c(\frac{a+q^c_{1/2}}{2}) \geq F_c(\frac{q^c_{1/2}+q^c_{3/4}}{2}) - F_c(\frac{q^c_{1/2}+q^c_{1/4}}{2}) > \frac{1}{4}$$

In contradiction to (x_1, x_2) being an equilibrium profile. □

As an immediate result it follows that Protocol 3 does not hold as equilibrium in MHM distributions. The question of whether a given MHM distribution has any equilibrium or not remains open.

6. DISCUSSION AND FUTURE WORK

This work presents a foundational extension to the classical facility location duel, motivated by IR and clustering settings. Following it there are many possible extensions of the l vs k setting introduced, which one may pursue. Naturally, one type of possible extensions is to consider more general networks or multi-dimensional spaces. Other interesting types of extensions is by considering natural discrete versions, as explained below.

Consider for example a space of ordered categories, 1 to 6, where each customer belongs uniformly to one category. In this case it is easy to see that if there are two players, with 1 and 2 facilities respectively, then the profile (x_1, x_2) where -

$$x_2 = (2,5) \quad , x_1 = \begin{cases} 2 & w.p. \frac{1}{2} \\ 5 & w.p. \frac{1}{2} \end{cases}$$

is in equilibrium. However, in this setting other equilibria exist, for example -

$$x_2 = (2,5) \quad , x'_1 = \begin{cases} 2 & w.p. \frac{1}{2} \\ 4 & w.p. \frac{1}{2} \end{cases}$$

And obviously (x'_1, x_2) has a better social welfare, since with positive probability a bystander will observe 3 facilities instead of 2. This result is very sensitive to the number of ordered categories and the distribution of customers among them.

The above observation may imply that a mediator which controls the information communicated to the players about the customers' preferences, may segment the space in a way that will increase the overall social welfare (in the spirit of [Kamenica and Gentzkow 2011] and [Emek et al. 2014]).

Another option is to consider a finite domain - imagine that instead of known probability distribution each publisher has access to the history of N customers' preferences. In this case, each player might consider clustering of these customers in order to place her facilities, or estimating the real customers' distribution function based on these samples prior to placing them.

A different aspect is the online case: imagine that the two players can move their facilities in a repeated game, where customers arrive dynamically, and in each iteration they get a signal. The signal may be the real preferences of the customers which the player served, or only the identity of the facilities that served them. The study of players' strategies in that setting would be closely related to variants of the multi-armed bandit problem [Cesa-Bianchi and Lugosi 2006] as well as to research in multi-agent learning [Brafman and Tennenholtz 2004], and may deserve a close study.

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A. APPENDIX

For section 2:

The proof of Theorem 2.1 relies on the next two lemmas:

LEMMA A.1. *In case player 2 plays according to Protocol 1, any deterministic strategy played by player 1 can lead to maximum expected payoff of* $\frac{1}{4}$.

PROOF. we know that player 2 locate her two facilities on $(\frac{1}{4}, \frac{3}{4})$. Assume player 1 locates her (only) facility on x_1. His payoff $U^1(x_1, x_2^{p_1})$ is given by:

$$\begin{aligned}
&(1)\ x_1 < \frac{1}{4}: && \frac{x_1 + \frac{1}{4}}{2} = \frac{x_1}{2} + \frac{1}{8} < \frac{1}{4} \\
&(2)\ x_1 = \frac{1}{4}: && \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\
&(3)\ \frac{1}{4} < x_1 < \frac{1}{2}: && \frac{x_1 - \frac{1}{4}}{2} + \frac{\frac{3}{4} - x_1}{2} = \frac{1}{4}
\end{aligned}$$

The other cases are symmetrical. We get that no deterministic strategy for player 1 can give her expected payoff greater than $\frac{1}{4}$. □

LEMMA A.2. *In case player 1 plays according to Protocol 1, any deterministic strategy played by player 1 can lead to maximum expected reward of* $\frac{3}{4}$.

PROOF. We know that player 1 plays according to $x_1^{p_1}$. Denote $x_2 = (a, b)$ as the deterministic strategy where player 2 locates one facility on a and one facility on b (w.l.o.g. $a \leq b$). We create a partition of $[0, 1]$, and analyze the payoff of player 1 for all possible cases of $x_2 = (a, b)$. Denote

$$I_1 = \left[0, \frac{1}{4}\right], I_2 = \left\{\frac{1}{4}\right\}, I_3 = \left(\frac{1}{4}, \frac{3}{4}\right), I_4 = \left\{\frac{3}{4}\right\}, I_5 = \left(\frac{3}{4}, 1\right)$$

And let $L(l) = i$ such that $l \in I_i$. The possible cases for $(L(a), L(b))$ are:

$$\underbrace{(1,1),(5,5)}_{1}, \underbrace{(1,2),(4,5)}_{2}, \underbrace{(3,3)}_{3}, \underbrace{(2,3),(3,4)}_{4}, \underbrace{(1,3),(3,5)}_{5}, \underbrace{(1,4),(2,5)}_{6}, \underbrace{(1,5)}_{7}, \underbrace{(2,2),(4,4)}_{8} \underbrace{(2,4)}_{9}$$

Two cases marked with the same digit are completely symmetrical, so we show only one of them. For each case, we look at $U^1(x_1^{p_1}, x_2)$:

$$\begin{aligned}
&(1)\ \frac{1}{2}\left(\frac{b + \frac{1}{4}}{2}\right) + \frac{1}{2}\left(\frac{b + \frac{3}{4}}{2}\right) = \frac{1}{4} + \frac{b}{2} < \frac{3}{4} \\
&(2)\ \frac{1}{2}\left[\frac{a + \frac{1}{4}}{2} + \frac{1}{2}\left(1 - \frac{a + \frac{1}{4}}{2}\right)\right] + \frac{1}{2}\left(\frac{b + \frac{3}{4}}{2}\right) = \frac{a}{8} + \frac{17}{32} < \frac{3}{4} \\
&(3)\ \frac{1}{2}\left(1 - \frac{a + \frac{1}{4}}{2}\right) + \frac{1}{2}\left(\frac{\frac{3}{4} + b}{2}\right) = \frac{5}{8} - \frac{a}{4} + \frac{b}{4} < \frac{3}{4} \\
&(4)\ \frac{1}{2}\left[\frac{a+b}{2} \cdot \frac{1}{2} + 1 - \frac{a+b}{2}\right] + \frac{1}{2}\left(\frac{b + \frac{3}{4}}{2}\right) = \frac{11}{16} - \frac{a}{8} + \frac{b}{8} = \frac{21}{32} + \frac{b}{8} < \frac{3}{4} \\
&(5)\ \frac{1}{2}\left(1 - \frac{b-a}{2}\right) + \frac{1}{2}\left(\frac{b + \frac{3}{4}}{2}\right) = \frac{11}{16} + \frac{a}{4} < \frac{3}{4} \\
&(6)\ \frac{1}{2}\left(1 - \frac{b-a}{2}\right) + \frac{1}{2}\left[\left(\frac{a + \frac{3}{4}}{2} + \frac{1}{2}\left(1 - \frac{a + \frac{3}{4}}{2}\right)\right]\right. = \frac{21}{32} + \frac{3a}{8} < \frac{3}{4} \\
&(7)\ \frac{1}{2}\left(1 - \frac{b-a}{2}\right) + \frac{1}{2}\left(1 - \frac{b-a}{2}\right) = 1 - \frac{b}{2} + \frac{a}{2} < 1 - \frac{3}{8} + \frac{1}{8} = \frac{3}{4} \\
&(8)\ \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} < \frac{3}{4} \\
&(9)\ \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{3}{4}
\end{aligned}$$

It follows that if player 1 plays according to Protocol 1, all strategies reward player 2 with $\frac{3}{4}$ or less. Note that the only case where player 2 receives payoff of exactly $\frac{3}{4}$ is (9),

which is the strategy we proposed her in Protocol 1, hence playing according to $x_2^{p_1}$ is the best response for $x_1^{p_1}$. $\square$

Proof of Theorem 2.1. : Lemmas A.1 and A.2 implies that any change in player i's strategy will lead to decrease in her expected reward, and therefore it is an equilibrium by definition.

For section 3:
Proof of Theorem 3.1 -

PROOF. Let $x_2 = (a_1, a_2 ... a_k)$ be the (sorted) locations player 2 chose. We also note $a_0 = 0, a_{k+1} = 1$. For $1 \leq i \leq k+1$ we define

$$u_i = a_i - a_{i-1} \tag{1}$$

Instead of analyzing X, we analyze the corresponding $U = (u_1, u_2 ... u_{k+1})$. Note that $\sum_{j=1}^{k+1} u_j = 1$ must hold, so we use Lagrange multipliers for finding the local minima of the distance function $D(U)$ subject to these equality constraints. For $i \in \{2, 3, ...k-1\}$, the expected distance of a customer to her nearest facility:

$$\int_{a_{i-1}}^{a_i} \min\{t - a_{i-1}, a_i - t\} \cdot 1 dt = \int_0^{\frac{u_i}{2}} 2t dt = [t^2]\Big|_0^{\frac{u_1}{2}} = \frac{u^2}{4}$$

and for u_1 (same result for u_{k+1}):

$$\int_0^{u_1} t dt = [\frac{t^2}{2}]\Big|_0^{u_1} = \frac{u_1^2}{2}$$

In total, The expected distance from a random customer to the closest facility is:

$$D(U, \lambda) = \frac{u_1^2}{2} + \sum_{i=2}^{k} \frac{u_i^2}{4} + \frac{u_{k+1}^2}{2} - \lambda(\sum_{j=1}^{k+1} u_j - 1)$$

Taking the partial derivative we get -

$$\begin{cases} \frac{\partial D(U,\lambda)}{\partial u_i} = 0 \Rightarrow & u_i = 2\lambda, \\ \frac{\partial D(U,\lambda)}{\partial u_{k+1}} = 0 \Rightarrow & u_{k+1} = \lambda \\ \frac{\partial D(U,\lambda)}{\partial u_1} = 0 \Rightarrow & u_1 = \lambda \\ \frac{\partial D(U,\lambda)}{\partial \lambda} = 0 \Rightarrow & \sum_{j=1}^{k+1} u_j = 1 \end{cases}$$

When we solve these equations we get that the optimal intervals are:

$$u_i = \begin{cases} \frac{1}{2k} & i = 1, k+1 \\ \frac{1}{k} & i \in \{2, 3..k\} \end{cases}$$

Changing the form of equation 1, we get the recurrence relation $a_i = u_i + a_{i-1}$ were u_i is known and $a_0 = 0$. The solution vector is exactly $\left(\frac{1}{2k}, ..., \frac{i}{2k}, ... \frac{2k-1}{2k}\right)$ $\square$

For section 4:
First, we aim to describe what μ is. A mixed strategy x_1 of player 1 can be thought of as a probability measure P_{x_1} over $\left([0,1]^n, \mathcal{B}([0,1]^n)\right)$. Over the same space, we define a measure μ as follows

$$\mu\big([a,b]\big) = \sum_{i=1}^{l} P_{x_1}\big(Proj_i(x_1) \in [a,b]\big)$$

Where $Proj_i$ is the projection of x_1 on the i-th axis. In another words:

$$\mu\big([a,b]\big) - \text{the expected number of player 1's facilities in } [a,b]$$

The terms $\mu\big((a,b)\big), \mu\big((a,b]\big)$ and $\mu\big([a,b)\big)$ are also valid , and for simplicity we use $\mu\big(z\big)$ to denote the probability of seeing a facility of player 1 under x_1 in z. Obviously, P_{x_1} is induced by x_1, and therefore μ is also induced by x_1, so we should have denoted it as μ_{x_1}. However, since the context x_1 is clear whenever we use it, we simply denote this induced measure as μ.

LEMMA A.3. *For any strategy x_1 of player 1 the following holds:*

(*1*) *μ is a measure.*
(*2*) *For any $x_1, a \in [0,1)$ and $\epsilon > 0$ there exists $\delta > 0$ such that:*

$$\mu\big((a, a+\delta)\big) < \epsilon$$

(*3*) *For any $0 \le a < b \le 1$ there exists $\delta > 0$ such that $a + \delta < b$ and $\mu\big(a+\delta\big) = 0$*

PROOF.

(1) In order to show that μ is a measure we need to show:
   (a) Non-negativity: $\forall A \in \mathcal{B}([0,1])$

$$\mu\big(A\big) = \sum_{i=1}^{l} P_{x_1}\big(Proj_i(x_1) \in A\big) \overset{\text{P is a measure}}{\geq} 0$$

   (b) Null empty set:

$$\mu\big(\phi\big) = \sum_{i=1}^{l} P_{x_1}\big(Proj_i(x_1) \in \phi\big) = \sum_{i=1}^{l} P_{x_1}\big(\phi\big) \overset{P \text{ is a measure}}{=} 0$$

   (c) Countable additivity: for disjoint set $A_1, A_2 ... \in \mathcal{B}([0,1])$:

$$\mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{l} P\big(proj_i(x_1) \in \bigcup_{i=1}^{\infty} A_i\big) \overset{P \text{ is a measure}}{=} \sum_{i=1}^{l}\sum_{j=1}^{\infty} P\big(proj_i(x_1) \in A_i\big)$$
$$= \sum_{j=1}^{\infty} \mu\big(A_i\big)$$

(2) We define a decreasing sequence of sets $B_i \in \mathcal{B}([0,1])$:

$$B_i = \Big(a, a + \frac{1}{n_0 + i}\Big)$$

   Where n_0 is the minimal integer satisfying $a + \frac{1}{n_0} \le 1$. Since $B_i \downarrow \phi$ and μ is a measure, we know that $\mu\big(B_i\big) \downarrow \mu\big(\phi\big) = 0$ also holds, and therefore there exists j such that:

$$\mu\big(B_j\big) = \mu\big((a, a + \frac{1}{n_0 + j})\big) < \epsilon$$

   Hence we denote $\delta = \frac{1}{n_0 + j}$ and the claim holds.

(3) Assume by contradiction that for any δ such that $0 < \delta < b-a$ it holds that $\mu\big(a+\delta\big) > 0$. We look at the closed interval:

$$\big[\frac{b-a}{4}, \frac{b-a}{2}\big] \subset \big(0, b-a\big)$$

Since it is closed and μ is bounded, μ gets minimum inside it, hence we denote:

$$c = \min_{y \in \left[\frac{b-a}{4}, \frac{b-a}{2}\right]} \mu\big(a+y\big)$$

But now:

$$\mu\big(\big[\frac{b-a}{4}, \frac{b-a}{2}\big]\big) \geq \sum_{n=1}^{\infty} \mu\big(\frac{b-a}{4} \cdot \big(1+\frac{1}{n}\big)\big) \geq \sum_{n=1}^{\infty} c = \infty$$

In contradiction to $\mu\big(\big[\frac{b-a}{4}, \frac{b-a}{2}\big]\big) \leq \mu\big([0,1]\big) = l$

□

For section 5:
Proof of Lemma 5.4:

PROOF. Denote:

$$y \in \underset{x \in [q^c_{1/4}, q^c_{3/4}]}{\arg\max} \Big[\int_{\frac{q^c_{1/4}+x}{2}}^{\frac{q^c_{3/4}+x}{2}} f_c(t)dt \Big]$$

The integral inside the $\arg\max$ is continuous in the closed and bounded interval $[q^c_{1/4}, q^c_{3/4}]$, hence by the Extreme Value Theorem it must attain a maximum at least once, so the $\arg\max$ is well defined. Assume by contradiction that $\int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{3/4}+y}{2}} f_c(t)dt > \frac{1}{4}$. From the first property of LM distribution and definition of quantiles we know that $y \notin \{q^c_{1/4}, q^c_{3/4}\}$. There are 3 possible cases:

(1) $f(\frac{q^c_{3/4}+y}{2}) > f(\frac{q^c_{1/4}+y}{2})$: since $y < q^c_{3/4}$, there exists $\epsilon > 0$ such that $y + 2\epsilon < q^c_{3/4}$. Therefore:

$$\int_{\frac{q^c_{1/4}+y+2\epsilon}{2}}^{\frac{q^c_{3/4}+y+2\epsilon}{2}} f_c(t)dt = \int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{3/4}+y}{2}} f_c(t)dt + \underbrace{\int_{\frac{q^c_{3/4}+y}{2}}^{\frac{q^c_{3/4}+y+2\epsilon}{2}} f_c(t)dt}_{none-decreasing} - \underbrace{\int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{1/4}+y+2\epsilon}{2}} f_c(t)dt}_{none-increasing}$$

$$\geq \int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{3/4}+y}{2}} f_c(t)dt + \epsilon\big[\underbrace{f_c(\frac{q^c_{3/4}+y}{2}) - f_c(\frac{q^c_{1/4}+y}{2})}_{>0}\big] > \int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{3/4}+y}{2}} f_c(t)dt$$

In contradiction to $y \in \underset{x \in [q^c_{1/4}, q^c_{3/4}]}{\arg\max} \Big[\int_{\frac{q^c_{1/4}+x}{2}}^{\frac{q^c_{3/4}+x}{2}} f_c(t)dt \Big]$.

(2) $f_c(\frac{q^c_{1/4}+y}{2}) > f_c(\frac{q^c_{3/4}+y}{2})$: since $q^c_{1/4} < y$, there exists $\epsilon > 0$ such that $q^c_{1/4} < y - 2\epsilon$. From here, similar to the previous case.

(3) $f_c(\frac{q^c_{1/4}+y}{2}) = f_c(\frac{q^c_{3/4}+y}{2})$: From the monotonicity we get that $\forall t \in [\frac{q^c_{1/4}+y}{2}, \frac{q^c_{3/4}+y}{2}]$ it holds that:

$$f_c(\frac{q^c_{1/4}+y}{2}) \geq f_c(t) \overset{integrate}{\Rightarrow} \int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{3/4}+y}{2}} f_c(\frac{q^c_{1/4}+y}{2})dt \geq \int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{3/4}+y}{2}} f_c(t)dt$$

$$\Rightarrow (\frac{q^c_{3/4}-q^c_{1/4}}{2}) f_c(\frac{q^c_{1/4}+y}{2}) \geq \int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{3/4}+y}{2}} f_c(t)dt > \frac{1}{4}$$

$$\Rightarrow (\frac{q^c_{3/4}-q^c_{1/4}}{2}) f_c(\frac{q^c_{1/4}+y}{2}) > \frac{1}{4}$$

Therefore -

$$\int_{q^c_{1/4}}^{q^c_{3/4}} f_c(t)dt = \int_{q^c_{1/4}}^{\frac{q^c_{1/4}+y}{2}} f(t)dt + \int_{\frac{q^c_{1/4}+y}{2}}^{\frac{q^c_{3/4}+y}{2}} f_c(t)dt + \int_{\frac{q^c_{3/4}+y}{2}}^{q^c_{3/4}} f_c(t)dt$$

$$> \underbrace{\int_{q^c_{1/4}}^{\frac{q^c_{1/4}+y}{2}} f_c(t)dt}_{none-increasing} + \underbrace{\int_{\frac{q^c_{3/4}+y}{2}}^{q^c_{3/4}} f_c(t)dt + \frac{1}{4}}_{none-decreasing}$$

$$\geq (\frac{q^c_{1/4}+y}{2} - q^c_{1/4}) f_c(\frac{q^c_{1/4}+y}{2}) + (q^c_{3/4} - \frac{q^c_{3/4}+y}{2}) f_c(\frac{q^c_{3/4}+y}{2}) + \frac{1}{4}$$

$$= (\frac{q^c_{3/4}-q^c_{1/4}}{2}) f_c(\frac{q^c_{1/4}+y}{2}) + \frac{1}{4} > \frac{1}{2}$$

But $(q^c_{1/4}, q^c_{3/4})$ are the quantiles, therefore we got a contradiction to $\int_{q^c_{1/4}}^{q^c_{3/4}} f_c(t)dt = \frac{1}{2}$.

□