Static and Dynamic Coherent Robust Control for a Class of Uncertain Quantum Systems

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***Abstract*—This paper concerns a class of uncertain linear quantum systems subject to quadratic perturbations in the system Hamiltonian. In order to obtain improved control performance, we propose two methods to design coherent robust controllers for the system. One is to formulate a static quantum controller by adding a controller Hamiltonian to the given system, and the other is to build a dynamic quantum controller which is directly coupled to the given system.**

I. Introduction

In recent years, there has been considerable interests focusing on quantum control due to its applications in metrology, quantum optics, quantum computation, quantum communication, chemical physics and other quantum technologies [1]-[17]. In quantum control, feedback control plays a vital role in manipulating a quantum mechanical system to achieve some pre-required closed loop properties such as stability [18], [19], robustness [20], [21], entanglement [22] or other performance requirements [23], [24], [25]. In particular, linear quantum optics is widely studied in control areas and a quantum optical system can often be described by a set of linear quantum stochastic differential equations (QSDEs) [19].

In the conventional picture of quantum feedback control, digital or analog electronic devices are often implemented as controllers [1]. However, this tends to destroy quantum coherence and involves the destruction of quantum information in the process of making measurements. Hence, recent research has been focused on using a fully quantum system as a controller, which is referred to as a coherent controller, e.g., [2], [23]. Compared to measurement feedback control, the advantages of coherent control are to preserve quantum coherence, to achieve improved performance and to obtain a high speed processing bandwidth [3], [9]. However, when we consider coherent feedback control using the QSDE description, the issue of physical realizability of controllers arises [2], [19]. That is, in QSDEs framework, the state space matrices defining the coherent controller are required to satisfy certain conditions in order that the controller represents a physically meaningful quantum system. Instead of considering linear QSDEs with arbitrary coefficients and then imposing physical realizability constraints on them, we use the (S, L, H) parameters which specify the state space matrices of the linear QSDEs, so that the physical realizability constraints are satisfied automatically and become irrelevant. In this (S, L, H) representation, H is a Hamiltonian operator, L is a vector of coupling operators and S is a scattering matrix [3]. The quantity L describes the interface between the system and the field, and the operator H defines the self-energy of the system. Since the parameters (S, L, H) already describe a physically realizable system, we do not need to be concerned about physical realizability conditions.

Quantum feedback controllers have been designed with a number of different techniques. For example, an H^∞ synthesis approach [19], [25] and an LQG method [2] have been used to design a quantum controller for a class of linear quantum stochastic systems. [7] used a transfer function method to analyze the robustness of feedback quantum systems. Coherent feedback control methods were developed to achieve specific control goals, such as backaction evasion, generation of a quantum nondemolished variable, and generation of a decoherence-free subsystem, in terms of controllability and observability properties or transfer functions of linear systems [16]. A general framework for passive coherent feedback in the Gaussian regime was proposed in [17] and proved that no protocol under this framework can be better than 3dB squeezing limit at steady state. Nevertheless, few papers have considered quantum controller design based on an (S, L, H) description. Based on the parameters (S, L, H), we are going to design a guaranteed cost coherent controller not only to robustly stabilize the uncertain quantum system, but also to guarantee a specific level of performance for any admissible value of the uncertainties.

The objective of this paper is to propose two different control methods and also compare their performance. One is to build a static guaranteed cost controller for uncertain quantum systems by adding controller Hamiltonian. Though static controller method has been considered in [23], there we restricted the matrix in Lyapunov operator V to be scalar multiplied by identity matrix. However, in this paper, we eliminate this constraint to allow the matrix in Lyapunov operator V to be any symmetric positive definite matrix by adding a numerical procedure and represent the system model in terms of the quadrature form instead of annihilation-creation form. Therefore, we extend the results in [23] to allow for more general cases.

More importantly, one main contribution of this paper is to design the other kind of controllers, that is, dynamic coherent controllers for uncertain quantum systems. The static coherent controller in this paper or in [23] has only been focused on adding a controller Hamiltonian with plant variables. That

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is, the static controller uses the same mode as the nominal system. Different from the static one, the dynamic controller design method introduces another quantum system as our desired controller. Thus, the controller variables are different from plant variables. Quantum controller design methods have been developed using various techniques [2], [18], [19], and the coupling between the plant and the controller in these papers are mainly via a field coupling which we call indirect coupling and the system models are only based on QSDEs description. However, little attention has been paid to quantum control design to deal with uncertain quantum systems in terms of dynamic guaranteed cost controller using direct coupling. Therefore, in this paper, we construct a directly coupled quantum controller for the given system as our desired dynamic controller. Here, direct coupling refers to that two independent quantum systems may interact by exchanging energy [8], [9] and this energy exchange is often described by an interaction Hamiltonian. The introduction of dynamic controllers adds new complexity compared with the static controller design in [23].

In this paper, the guaranteed cost coherent controller design is given in terms of LMI and nonlinear equality conditions. We use a nonlinear change of variables to convert the problem into a rank constrained LMI problem which can be solved using an alternating projections algorithm [26]. Although some of the results in this paper were presented in the conference paper [24], the static controller design method was not provided there. Also, the conference paper did not include any proofs of the stated results and did not present illustrative numerical results.

This paper is organized as follows. In Section II, we define the nominal quantum system under consideration as a linear system using parameters (S, L, H). Then an uncertain perturbation of the Hamiltonian is introduced in terms of a commutator decomposition and sector bound conditions in Section III. In Section IV, the cost function for uncertain linear quantum systems subject to quadratic perturbation of the Hamiltonian is defined and a small gain type performance analysis result is presented. In Section V, we introduce a controller Hamiltonian and present a theorem to show the construction of this static guaranteed cost quantum controller. In Section VI, a dynamic controller system is directly coupled to the uncertain quantum system. The corresponding controller design and numerical procedures are presented. An illustrative example is presented to demonstrate the coherent controller methods in Section VII. We also make a performance comparison between static coherent controller design and dynamic coherent controller design. Conclusions are presented in Section VIII.

II. System Description

The open quantum system under consideration is an uncertain linear quantum system defined by parameters (S, L, H), where H refers to the system Hamiltonian and can be decomposed as $H = H_P + H_{un}$. Here, H_P denotes a known nominal Hamiltonian and H_{un} denotes a perturbation Hamiltonian contained in a specified set of Hamiltonians $\mathcal{W}$ [20]. We assume that H_P is in the form of

$$H_P = \frac{1}{2}\begin{bmatrix} q^T & p^T \end{bmatrix} M_P \begin{bmatrix} q \\ p \end{bmatrix} \tag{1}$$

where M_P is a real symmetric matrix with dimension $2n_P \times 2n_P$. Here, q is a vector of position operators and p is a vector of momentum operators. The commutation relations between position and momentum operators are described as follows

$$\begin{aligned} &\left[\begin{bmatrix} q \\ p \end{bmatrix}, \begin{bmatrix} q \\ p \end{bmatrix}^T \right] = \begin{bmatrix} q \\ p \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}^T \\ &- \left(\begin{bmatrix} q \\ p \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}^T \right)^T = 2i\theta = \Sigma, \end{aligned} \tag{2}$$

where $\theta = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. The coupling operator L is of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}, \tag{3}$$

where $N_1 \in \mathbb{C}^{m_P \times n_P}$ and $N_2 \in \mathbb{C}^{m_P \times n_P}$. We also have

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = N_P \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_1^\# & N_2^\# \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}. \tag{4}$$

Here, $^\#$ refers to the complex conjugate matrix. The scattering matrix S is associated with the photon exchange between the field channels. Since the effect of interaction between different fields is not considered in most cases, it is assumed that $S = I$.

To proceed, we define the corresponding generator operator as follows $\mathcal{G}(\mathbf{X}) = -i[\mathbf{X}, H] + \mathcal{L}(\mathbf{X})$, where $\mathcal{L}(\mathbf{X}) = \frac{1}{2}L^\dagger[\mathbf{X}, L] + \frac{1}{2}[L^\dagger, \mathbf{X}]L$. Here, the notation † stands for the adjoint transpose of a vector of operators and $[\mathbf{X}, H] = \mathbf{X}H - H\mathbf{X}$ describes the commutator between two operators. The following lemma will be used in the main results presented in this paper.

Lemma 1: [21] Consider an open quantum system defined by (S, L, H) and driven by vacuum fields. Suppose there exist non-negative self-adjoint operator V and self-adjoint operator W on the underlying Hilbert space such that

$$\mathcal{G}(V) + W \leq \lambda \tag{5}$$

where λ is a real number. Then for any plant state, we have

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle W(t)\rangle dt \leq \lambda. \tag{6}$$

Here $W(t)$ denotes the Heisenberg evolution of the operator W and $\langle\cdot\rangle$ denotes quantum expectation; e.g., see [21] and [3].

Lemma 1 is applied to this class $\mathcal{P}$ of system operator V in what follows. A self-adjoint "Lyapunov" operators V can be described in the following form

$$V = \begin{bmatrix} q^T & p^T \end{bmatrix} X \begin{bmatrix} q \\ p \end{bmatrix} \tag{7}$$

where $X \in \mathbb{R}^{2n_P \times 2n_P}$ is a symmetric positive definite matrix. We define a set of non-negative self-adjoint operators $\mathcal{P}$ as follows

$$\mathcal{P} = \left\{ \begin{array}{c} V \text{ of the form (7) where } X > 0 \text{ is a} \\ \text{symmetric positive definite matrix} \end{array} \right\}. \tag{8}$$

III. Perturbation of the Hamiltonian

In this section, we introduce a perturbation for the quantum system under consideration. First, we define the perturbation of the Hamiltonian in terms of a commutator decomposition. Then, we introduce the formulation of the quadratic perturbation in the system Hamiltonian.

A. Commutator Decomposition

For the set of non-negative self-adjoint operators $\mathcal{P}$ and given real parameters $\gamma > 0$, $\delta \geq 0$, a particular set of perturbation Hamiltonians $\mathcal{W}_1$ is defined in terms of the commutator decomposition

$$[V, H_{un}] = [V, z^T]w - w^T[z, V] \tag{9}$$

for $V \in \mathcal{P}$, where w and z are given vectors of self-adjoint operators. $\mathcal{W}_1$ is then defined in terms of sector bound condition:

$$w^T w \leq \frac{1}{\gamma^2} z^T z + \delta. \tag{10}$$

We define

$$\mathcal{W}_1 = \left\{ \begin{array}{c} H_{un} : \exists\ w, z \text{ such that (9) and (10)} \\ \text{are satisfied } \forall\ V \in \mathcal{P} \end{array} \right\}. \tag{11}$$

Lemma 2: Consider an open quantum system (S, L, H) where $H = H_P + H_{un}$ and $H_{un} \in \mathcal{W}_1$, and the set of non-negative self-adjoint operators $\mathcal{P}$. If there exists a $V \in \mathcal{P}$ and a real constant $\lambda \geq 0$ such that

$$-i[V, H_P] + \mathcal{L}(V) + [V, z^T][z, V] + \frac{1}{\gamma^2} z^T z + W \leq \lambda, \tag{12}$$

then $\quad \limsup_{T\to\infty} \frac{1}{T} \int_0^T \langle W(t)\rangle dt \leq \lambda + \delta.$

Proof: As we know $V \in \mathcal{P}$ and $H_{un} \in \mathcal{W}_1$,

$$\mathcal{G}(V) = -i[V, H_P] + \mathcal{L}(V) - i[V, z^T]w + iw^T[z, V]. \tag{13}$$

Since V is self-adjoint, we have $[V, z^T]^\dagger = [z, V]$. Therefore,

$$\begin{aligned} 0 &\leq ([V, z^T] - iw^T)([V, z^T] - iw^T)^\dagger \\ &= [V, z^T][z, V] + i[V, z^T]w - iw^T[z, V] + w^T w. \end{aligned} \tag{14}$$

Substituting (14) into (13) and using the sector bound condition (10), the following inequality is obtained:

$$\mathcal{G}(V) \leq -i[V, H_P] + \mathcal{L}(V) + [V, z^T][z, V] + \frac{1}{\gamma^2} z^T z + \delta. \tag{15}$$

It follows from (12) that $\mathcal{G}(V) + W \leq \lambda + \delta$. Consequently, the result follows from Lemma 1. □

B. Quadratic Hamiltonian Perturbation

A set of quadratic perturbation uncertainties is defined in the following form

$$H_{un} = \frac{1}{2} \left[\begin{array}{cc} \zeta_q^T & \zeta_p^T \end{array} \right] \Delta \left[\begin{array}{c} \zeta_q \\ \zeta_p \end{array} \right] \tag{16}$$

where $\Delta^T = \Delta$. We assume the relationship

$$z = \left[\begin{array}{c} \zeta_q \\ \zeta_p \end{array} \right] = E \left[\begin{array}{c} q \\ p \end{array} \right]. \tag{17}$$

Hence, we write

$$H_{un} = \frac{1}{2} \left[\begin{array}{cc} q^T & p^T \end{array} \right] E^T \Delta E \left[\begin{array}{c} q \\ p \end{array} \right]. \tag{18}$$

The matrix Δ is subject to the norm bound

$$\|\Delta\| \leq \frac{2}{\gamma} \tag{19}$$

where $\|.\|$ refers to the matrix induced norm and we have the following definition

$$\mathcal{W}_2 = \left\{ \begin{array}{c} H_{un} \text{ of the form (16) such that} \\ \text{condition (19) is satisfied} \end{array} \right\}. \tag{20}$$

Lemma 3: For any set of self-adjoint operators $\mathcal{P}$,

$$\mathcal{W}_2 \subset \mathcal{W}_1 \tag{21}$$

Proof: Given any $H_{un} \in \mathcal{W}_2$, let

$$w = \frac{1}{2} \left[\begin{array}{cc} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{array} \right] \left[\begin{array}{c} \zeta_q \\ \zeta_p \end{array} \right] = \frac{1}{2} \Delta E \left[\begin{array}{c} q \\ p \end{array} \right] \tag{22}$$

and we have the relationship (17). Hence,

$$H_{un} = w^T z = \frac{1}{2} \left[\begin{array}{cc} q^T & p^T \end{array} \right] E^T \Delta E \left[\begin{array}{c} q \\ p \end{array} \right]. \tag{23}$$

Then, for any $V \in \mathcal{P}$,

$$\begin{aligned} [V, z^T]w =& \frac{1}{2} \left(\begin{array}{l} V\zeta_q^T \Delta_{11}\zeta_q + V\zeta_q^T \Delta_{12}\zeta_p \\ +V\zeta_p^T \Delta_{12}^T \zeta_q + V\zeta_p^T \Delta_{22}\zeta_p \end{array} \right) \\ & - \frac{1}{2} \left(\begin{array}{l} \zeta_q^T V\Delta_{11}\zeta_q + \zeta_q^T V\Delta_{12}\zeta_p \\ +\zeta_p^T V\Delta_{12}^T \zeta_q + \zeta_p^T V\Delta_{22}\zeta_p \end{array} \right) \end{aligned} \tag{24}$$

Also,

$$\begin{aligned} w^T[z, V] =& \frac{1}{2} \left(\begin{array}{l} \zeta_q^T \Delta_{11}\zeta_q V + \zeta_q^T \Delta_{12}\zeta_p V \\ +\zeta_p^T \Delta_{12}^T \zeta_q V + \zeta_p^T \Delta_{22}\zeta_p V \end{array} \right) \\ & - \frac{1}{2} \left(\begin{array}{l} \zeta_q^T V\Delta_{11}\zeta_q + \zeta_q^T V\Delta_{12}\zeta_p \\ +\zeta_p^T V\Delta_{12}^T \zeta_q + \zeta_p^T V\Delta_{22}\zeta_p \end{array} \right) \end{aligned} \tag{25}$$

Hence,

$$\begin{aligned} & [V, z^T]w - w^T[z, V] \\ &= \frac{1}{2} \left(\begin{array}{l} V\zeta_q^T \Delta_{11}\zeta_q + V\zeta_q^T \Delta_{12}\zeta_p \\ +V\zeta_p^T \Delta_{12}^T \zeta_q + V\zeta_p^T \Delta_{22}\zeta_p \end{array} \right) \\ &- \frac{1}{2} \left(\begin{array}{l} \zeta_q^T \Delta_{11}\zeta_q V + \zeta_q^T \Delta_{12}\zeta_p V \\ +\zeta_p^T \Delta_{12}^T \zeta_q V + \zeta_p^T \Delta_{22}\zeta_p V \end{array} \right) \\ &= VH_{un} - H_{un}V = [V, H_{un}] \end{aligned} \tag{26}$$

Also,

$$\frac{1}{4} \left[\begin{array}{cc} \zeta_q^T & \zeta_p^T \end{array} \right] \Delta\Delta \left[\begin{array}{c} \zeta_q \\ \zeta_p \end{array} \right] \leq \frac{1}{\gamma^2} \left[\begin{array}{cc} \zeta_q^T & \zeta_p^T \end{array} \right] \left[\begin{array}{c} \zeta_q \\ \zeta_p \end{array} \right] \tag{27}$$

Equation (27) shows that the w, z being considered satisfy the condition (10) with any $\delta \geq 0$.

Therefore, we have $\mathcal{W}_2 \subset \mathcal{W}_1$. □

IV. Performance Analysis

In this section, we evaluate the performance of the given uncertain linear quantum system. First, we define the associated cost function for a quantum system as

$$\mathcal{J} = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle \begin{bmatrix} q^T & p^T \end{bmatrix} R \begin{bmatrix} q \\ p \end{bmatrix} \rangle dt \tag{28}$$

where $R > 0$. We denote that

$$W = \begin{bmatrix} q^T & p^T \end{bmatrix} R \begin{bmatrix} q \\ p \end{bmatrix}. \tag{29}$$

In order to introduce our results on performance analysis, we require the following algebraic identities.

Lemma 4: Consider $V \in \mathcal{P}, H_P$ is of the form (1) and L is of the form (3). Then we have

$$[V, H_P] = \begin{bmatrix} q \\ p \end{bmatrix}^T (X\Sigma M_P - M_P\Sigma X) \begin{bmatrix} q \\ p \end{bmatrix}, \tag{30}$$

and

$$\mathcal{L}(V) = -\frac{1}{2}\begin{bmatrix} q \\ p \end{bmatrix}^T (N_P^\dagger J N_P \Sigma X + X\Sigma N_P^\dagger J N_P) \begin{bmatrix} q \\ p \end{bmatrix} + \mathrm{Tr}(X\Sigma N_P^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N_P \Sigma), \tag{31}$$

where $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Moreover,

$$[\begin{bmatrix} q \\ p \end{bmatrix}, \begin{bmatrix} q \\ p \end{bmatrix}^T X \begin{bmatrix} q \\ p \end{bmatrix}] = 2\Sigma X \begin{bmatrix} q \\ p \end{bmatrix}. \tag{32}$$

Proof: The proof of these identities follows via straightforward but tedious calculations using (2). □

Lemma 5: For $V \in \mathcal{P}$ and z defined in (17),

$$[z, V] = 2E\Sigma X \begin{bmatrix} q \\ p \end{bmatrix}, \tag{33}$$

$$[V, z^T][z, V] = 4\begin{bmatrix} q \\ p \end{bmatrix}^T X\Sigma^\dagger E^T E \Sigma X \begin{bmatrix} q \\ p \end{bmatrix}, \tag{34}$$

$$z^T z = \begin{bmatrix} q \\ p \end{bmatrix}^T E^T E \begin{bmatrix} q \\ p \end{bmatrix}. \tag{35}$$

Proof: The result follows from Lemma 4. □

We now present our main result in this section.

Theorem 1: Consider an uncertain quantum system (S, L, H), where $H = H_P + H_{un}$, H_P is in the form of (1), L is of the form (3) and $H_{un} \in \mathcal{W}_2$. If

$$\begin{bmatrix} A^T X + XA + \frac{E^T E}{\gamma^2} + R & 2X\Sigma^\dagger E^T \\ 2E\Sigma X & -I \end{bmatrix} < 0 \tag{36}$$

has a symmetric solution $X > 0$, then

$$\mathcal{J} = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle W(t)\rangle dt = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle \begin{bmatrix} q^T & p^T \end{bmatrix} R \begin{bmatrix} q \\ p \end{bmatrix}\rangle dt \le \tilde{\lambda} + \delta \tag{37}$$

where $\tilde{\lambda} = \mathrm{Tr}(X\Sigma N_P^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N_P \Sigma)$.

Proof: The proof is similar to that in Theorem 1 of [23], and the detailed proof is omitted. □

V. Static Coherent Controller

In this section, we aim to design a coherent guaranteed cost controller for the given uncertain quantum system by adding a controller Hamiltonian H_{K1}. As there are no additional dynamic variables defined by this Hamiltonian, we refer this kind of quantum controller as a static coherent controller.

The controller Hamiltonian H_{K1} is assumed to be in the form

$$H_{K1} = \frac{1}{2}\begin{bmatrix} q^T & p^T \end{bmatrix} F^T K F \begin{bmatrix} q \\ p \end{bmatrix} \tag{38}$$

where $F \in \mathbb{R}^{2n_P \times 2n_P}$ and $K \in \mathbb{R}^{2n_P \times 2n_P}$ is a symmetric matrix. An associated cost function $\mathcal{J}$ is defined in the following form

$$\mathcal{J} = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle \begin{bmatrix} q \\ p \end{bmatrix}^T (R + \rho F^T K F F^T K F) \begin{bmatrix} q \\ p \end{bmatrix}\rangle dt, \tag{39}$$

where $\rho \in (0, \infty)$ is a weighting factor. We have

$$W = \begin{bmatrix} q \\ p \end{bmatrix}^T (R + \rho F^T K F F^T K F) \begin{bmatrix} q \\ p \end{bmatrix}. \tag{40}$$

A. Controller Design

Theorem 2: Consider an uncertain quantum system (S, L, H), where $H = H_P + H_{un} + H_{K1}$, H_P is in the form of (1), L is of the form (3) and $H_{un} \in \mathcal{W}_2$, and the controller Hamiltonian H_{K1} is in the form of (38). If there exist symmetric matrices K, $X > 0$ and $Y = KF\theta^T X$ such that

$$\begin{bmatrix} B & 4X\theta^T E^T & F^T K F \\ 4E\theta X & -I & 0 \\ F^T K F & 0 & -I/\rho \end{bmatrix} < 0 \tag{41}$$

where $B = A^T X + XA + 2F^T Y + 2Y^T F + E^T E/\gamma^2 + R$ and $A = -i\Sigma M_P - \frac{1}{2}\Sigma N_P^\dagger J N_P$, then the associated cost function satisfies the bound

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle W(t)\rangle dt \le \tilde{\lambda} + \delta, \tag{42}$$

where $W(t)$ is in the form of (40) and $\tilde{\lambda} = \mathrm{Tr}(X\Sigma N_P^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N_P \Sigma)$.

Proof: Suppose the inequality (41) is satisfied. Using the Schur complement [27], we have

$$\begin{bmatrix} B + 16X\theta^T E^T E\theta X & F^T K F \\ F^T K F & -I/\rho \end{bmatrix} < 0. \tag{43}$$

Also, it follows from the Schur complement that (43) is equivalent to the following inequality

$$A^T X + XA + 2F^T Y + 2Y^T F + E^T E/\gamma^2 + 16X\theta^T E^T E\theta X + R + \rho F^T K F F^T K F < 0. \tag{44}$$

Substituting $Y = KF\theta^T X$ into the inequality (44), we obtain

$$(A - i\Sigma F^T K F)^T X + X(A - i\Sigma F^T K F) + 4X\Sigma^\dagger E^T E\Sigma X + E^T E/\gamma^2 + R + \rho F^T K F F^T K F < 0. \tag{45}$$

It follows in a straightforward way from (45) that $A - i\Sigma F^T K F$ is Hurwitz. Based on Lemma 4 and Lemma 5, we have

$$\begin{aligned}
&-i[V, H_P + H_{K1}] + \mathcal{L}(V) + [V, z^T][z, V] + \frac{1}{\gamma^2} z^T z + W \\
&= \begin{bmatrix} q \\ p \end{bmatrix}^T \begin{pmatrix} (A - i\Sigma F^T K F)^T X \\ +X(A - i\Sigma F^T K F) \\ +4X\Sigma^\dagger E^T E \Sigma X + E^T E/\gamma^2 \\ +R + \rho F^T K F F^T K F \end{pmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \\
&\quad + \mathrm{Tr}(X\Sigma N_P^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N_P \Sigma).
\end{aligned}$$

It follows from Lemma 2 and Lemma 3 that the following conditions are satisfied;

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \langle W(t) \rangle dt \leq \tilde{\lambda} + \delta, \tag{46}$$

where $\tilde{\lambda} = \mathrm{Tr}(X\Sigma N_P^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N_P \Sigma)$. □

B. Numerical Procedure

A controller design using Theorem 2 can be formulated as the minimization of cost bound $\tilde{\lambda}$ subject to LMI (41) and the equality constraint $Y = KF\theta^T X$. Due to the fact that the equality constraint is not linear, we cannot use the traditional LMI technique. Therefore, this problem is transformed into a rank constrained LMI problem [26]. The way to deal with this problem is to linearize the equality $Y = KF\theta^T X$ by introducing appropriate matrix lifting variables and the associated equality constraints, and transforming them into an LMI with a rank constraint.

We set $Z_{x1} = X, Z_{x2} = K$ and introduce appropriate matrix lifting variables $Z_{v1} = KF\theta^T, Z_{v2} = KF\theta^T X$. Then we define Z as a symmetric matrix with dimension $10n_P \times 10n_P$ and $Z = VV^T$, where $V = [I \quad Z_{x1}^T \quad Z_{x2}^T \quad Z_{v1}^T \quad Z_{v2}^T]^T$.

We require the symmetric matrix Z to satisfy the following conditions:

$$\begin{array}{ll} Z \geq 0; & Z_{0,0} - I = 0; \\ Z_{x1} - Z_{x1}^T = 0; & Z_{x2} - Z_{x2}^T = 0; \\ Z_{v1} - Z_{x2} F\theta^T = 0; & Z_{v2} - Z_{v1} Z_{x1} = 0 \end{array} \tag{47}$$

and a rank constraint

$$\mathrm{rank}(Z) \leq 2n_P. \tag{48}$$

An upper bound condition is given as

$$\mathrm{Tr}(XJN^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} NJ) < \beta. \tag{49}$$

Therefore, a way to apply Theorem 2 is to solve (41) subject to conditions (47), (48) and (49) using a rank constrained LMI. To solve the rank constrained LMI problem above, we can work in Matlab using the algorithm given in [28] and also use the Yalmip optimization prototyping environment [29] and SeDuMi solver [30].

VI. Dynamic Coherent Controller

In this section, we design a guaranteed cost dynamic coherent controller that is realized by directly coupling to the original uncertain quantum system. The directly coupled controller and the quantum plant may interact with each other by exchanging energy via an interaction Hamiltonian. We should also note that the associated operators of the nominal system commute with all the associated operators of the dynamic controller. In the following part, we introduce the design of dynamic quantum controllers and then formulate the augmented quantum system consisting of the quantum plant and the directly coupled quantum controller using an (S, L, H) parameterization.

We consider a linear dynamic quantum controller Hamiltonian in the form of

$$H_{K2} = \frac{1}{2} \begin{bmatrix} q_K \\ p_K \end{bmatrix}^T K_{22} \begin{bmatrix} q_K \\ p_K \end{bmatrix}, \tag{50}$$

where $K_{22} \in \mathbb{R}^{2n_K \times 2n_K}$ is a symmetric matrix. In what follows, we take the interaction Hamiltonian to be

$$\begin{aligned}
H_{int} =& \frac{1}{2} \begin{bmatrix} q_K \\ p_K \end{bmatrix}^T K_{12}^T F \begin{bmatrix} q \\ p \end{bmatrix} \\
&+ \frac{1}{2} \begin{bmatrix} q \\ p \end{bmatrix}^T F^T K_{12} \begin{bmatrix} q_K \\ p_K \end{bmatrix},
\end{aligned} \tag{51}$$

where $K_{12} \in \mathbb{R}^{2n_P \times 2n_K}$. Also, we consider a static coherent controller in the form of (38).

It follows from (38), (50) and (51) that the total controller Hamiltonian is in the form of

$$\begin{aligned}
H_K &= H_{int} + H_{K1} + H_{K2} \\
&= \frac{1}{2} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T \begin{bmatrix} F^T K_{11} F & F^T K_{12} \\ K_{12}^T F & K_{22} \end{bmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}.
\end{aligned} \tag{52}$$

Here, we denote $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix}$. Consequently, the total Hamiltonian for the augmented quantum linear system (excluding the perturbation Hamiltonian) that comprises the nominal quantum plant and the quantum controller is

$$H_{P+K} = H_P + H_K. \tag{53}$$

We know that the coupling operator L_P and $\begin{bmatrix} L_P \\ L_P^\# \end{bmatrix}$ for the plant are defined in (3) and (4), respectively. The coupling operator for the controller is assumed to be in the form

$$L_K = \begin{bmatrix} N_{K1} & N_{K2} \end{bmatrix} \begin{bmatrix} q_K \\ p_K \end{bmatrix}, \tag{54}$$

where $N_{K1} \in \mathbb{C}^{m_K \times n_K}$ and $N_{K2} \in \mathbb{C}^{m_K \times n_K}$.
Also, we have

$$\begin{bmatrix} L_K \\ L_K^\# \end{bmatrix} = N_K \begin{bmatrix} q_K \\ p_K \end{bmatrix} = \begin{bmatrix} N_{K1} & N_{K2} \\ N_{K1}^\# & N_{K2}^\# \end{bmatrix} \begin{bmatrix} q_K \\ p_K \end{bmatrix}. \tag{55}$$

Hence, the overall coupling operator for the directly coupled system comprising the plant and the controller is of the form

$$L = \begin{bmatrix} L_P \\ L_K \end{bmatrix} = \begin{bmatrix} N_{P1} & N_{P2} & 0 & 0 \\ 0 & 0 & N_{K1} & N_{K2} \end{bmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}. \tag{56}$$

Also, we write

$$\begin{aligned} \begin{bmatrix} L \\ L^{\#} \end{bmatrix} &= \begin{bmatrix} L_P \\ L_K \\ L_P^{\#} \\ L_K^{\#} \end{bmatrix} = N \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix} \\ &= \begin{bmatrix} N_{P1} & N_{P2} & 0 & 0 \\ 0 & 0 & N_{K1} & N_{K2} \\ N_{P1}^{\#} & N_{P2}^{\#} & 0 & 0 \\ 0 & 0 & N_{K1}^{\#} & N_{K2}^{\#} \end{bmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}. \end{aligned} \tag{57}$$

The "Lyapunov" operator V for the augmented system is considered to be a non-negative self-adjoint operator in the form of

$$V = \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix} \tag{58}$$

where $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \in \mathbb{R}^{2(n_P+n_K)\times 2(n_P+n_K)}$ is a symmetric positive definite matrix.

Hence, a set of non-negative self-adjoint operators $\tilde{\mathcal{P}}$ can be defined as

$$\tilde{\mathcal{P}} = \left\{ \begin{array}{c} V \text{ of the form (58) where } X > 0 \text{ is a} \\ \text{symmetric positive definite matrix} \end{array} \right\}. \tag{59}$$

Since all variables of the plant are assumed to commute with all variables of the controller, we have the following commutation relation

$$\left[\begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}, \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T \right] = 2i\Theta = \Xi \tag{60}$$

where $\Theta = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}$. We introduce the permutation matrix P_{n+m}, where the symbol P_{n+m} refers to a $2(n+m)\times 2(n+m)$ matrix. The permutation matrix P_{n+m} is defined in such way that if we consider a column vector $a = [a_1 \quad a_2 \quad \dots \quad a_{n+m} \quad \dots \quad a_{2(n+m)}]^T$, then $P_{n+m}a = [a_1 \quad a_2 \quad \dots \quad a_n \quad a_{n+m+1} \quad a_{n+m+2} \quad \dots \quad a_{2n+m} \quad a_{n+1} \quad a_{n+2} \quad \dots \quad a_{n+m} \quad a_{2n+m+1} \quad \dots \quad a_{2(n+m)}]^T$. Recall the property of an $m \times m$ permutation matrix that it is a full-rank real matrix whose columns comprise standard basis vector for $\mathbb{R}^m$, that is, vectors in $\mathbb{R}^m$ contains precisely a single element with value 1 and all the remaining elements are 0. A permutation matrix P also has the unitary property $PP^T = P^TP = I$ [31]. Another piece of notation we need to introduce is $J_n = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$. Hence, we have

$$\begin{aligned} &P_{m_P+m_K} N = \begin{bmatrix} N_P & 0 \\ 0 & N_K \end{bmatrix}, \\ &N = P_{m_P+m_K}^T \begin{bmatrix} N_P & 0 \\ 0 & N_K \end{bmatrix}, \\ &P_{m_P+m_K}^T J_{m_P+m_K} P_{m_P+m_K} = \begin{bmatrix} J_{m_P} & 0 \\ 0 & J_{m_K} \end{bmatrix}. \end{aligned} \tag{61}$$

In order to present the main result, we require some algebraic identities.

Lemma 6: Suppose $\mathcal{V} \in \tilde{\mathcal{P}}$, H_{P+K} is in the form of (53) and L is in the form of (56). Then, we have that

$$\begin{aligned} &-i[V, H_{P+K}] + \mathcal{L}(V) \\ &= \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T \begin{pmatrix} (\overline{A} - i\Xi\overline{F}^T K\overline{F})^T X \\ +X(\overline{A} - i\Xi\overline{F}^T K\overline{F}) \end{pmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix} \\ &\quad + \text{Tr}(X\Xi N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N\Xi), \end{aligned} \tag{62}$$

where $\overline{A} = \begin{bmatrix} -i\Sigma M_P - \frac{1}{2}\Sigma N_P^\dagger J N_P & 0 \\ 0 & -\frac{1}{2}\Sigma N_K^\dagger J N_K \end{bmatrix}$, $\overline{F} = \begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix}$.

Proof: According to Lemma 4, we know that

$$\begin{aligned} &-i[V, H_{P+K}] + \mathcal{L}(V) \\ &= \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T \begin{pmatrix} (-i\Xi M - \frac{1}{2}\Xi N^\dagger J N)^T X \\ +X(-i\Xi M - \frac{1}{2}\Xi N^\dagger J N) \end{pmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix} \\ &\quad + \text{Tr}(X\Xi N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N\Xi), \end{aligned} \tag{63}$$

where $M = \begin{bmatrix} M_P + F^T K_{11} F & F^T K_{12} \\ K_{12}^T F & K_{22} \end{bmatrix}$. To shorten the equation, we denote $\mathbf{A} = -i\Xi M - \frac{1}{2}\Xi N^\dagger J N$ and we have

$$\begin{aligned} \mathbf{A} = &-i\Xi \begin{bmatrix} M_P + F^T K_{11} F & F^T K_{12} \\ K_{12}^T F & K_{22} \end{bmatrix} \\ &- \frac{1}{2}\Xi N^\dagger J_{m_P+m_K} N \end{aligned} \tag{64}$$

Following (61), we get

$$\begin{aligned} \mathbf{A} = &-i\Xi \begin{bmatrix} M_P + F^T K_{11} F & F^T K_{12} \\ K_{12}^T F & K_{22} \end{bmatrix} \\ &- \frac{1}{2}\Xi \begin{bmatrix} N_P^\dagger & 0 \\ 0 & N_K^\dagger \end{bmatrix} \begin{bmatrix} J_{m_P} & 0 \\ 0 & J_{m_K} \end{bmatrix} \begin{bmatrix} N_P & 0 \\ 0 & N_K \end{bmatrix} \\ = &\begin{bmatrix} -i\Sigma M_P - \frac{1}{2}\Sigma N_P^\dagger J N_P & 0 \\ 0 & -\frac{1}{2}\Sigma N_K^\dagger J N_K \end{bmatrix} \\ &- i\Xi\overline{F}^T K\overline{F} \\ = &\overline{A} - i\Xi\overline{F}^T K\overline{F}. \end{aligned} \tag{65}$$

Therefore, the result follows from (63) and (65). □

We know that only the plant is subject to uncertain quantum perturbation. Hence, we rewrite the quadratic perturbation uncertainty (18) and (17) in the following form

$$H_{un} = \frac{1}{2}\begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T \begin{bmatrix} E & 0 \end{bmatrix}^T \Delta \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}, \tag{66}$$

$$z = \begin{bmatrix} \zeta_q \\ \zeta_p \end{bmatrix} = \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} q^T & p^T & q_K^T & p_K^T \end{bmatrix}^T. \tag{67}$$

Now we are in the position to define an associated cost function as

$$\mathcal{J} = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle W\rangle dt, \tag{68}$$

$$W = \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T (\overline{R} + \rho \overline{F}^T K \overline{F}\overline{F}^T K \overline{F}) \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}, \tag{69}$$

where $\overline{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$, $R > 0$ and $\rho \in (0, \infty)$ is a weighting factor.

A. Controller Design

In this subsection, we present the main result on dynamic coherent controller design.

Theorem 3: Consider an uncertain quantum system (S_P, L_P, H_{P+un}), where $H_{P+un} = H_P + H_{un}$, H_P is in the form of (1), L_P is of the form (3) and $H_{un} \in \mathcal{W}_2$. It is directly coupled to a dynamic quantum controller (S_K, L_K, H_K). Then the closed loop quantum system is defined by (S, L, H) where $H = H_P + H_{un} + H_K$, H_K is in the form of (52) and L is in the form of (56). If there exist symmetric matrices K, $X > 0$ and $Y = K\overline{F}\Theta^T X$ such that

$$\begin{bmatrix} B & 4X\Theta^T\overline{E}^T & \overline{F}^T K \overline{F} \\ 4\overline{E}\Theta X & -I & 0 \\ \overline{F}^T K \overline{F} & 0 & -I/\rho \end{bmatrix} < 0 \tag{70}$$

where $\overline{E} = \begin{bmatrix} E & 0 \end{bmatrix}$, $B = \overline{A}^T X + X\overline{A} + 2\overline{F}^T Y + 2Y^T\overline{F} + \overline{E}^T\overline{E}/\gamma^2 + \overline{R}$ and $\overline{A}$ is in the form of $\overline{A} = \begin{bmatrix} -i\Sigma M_P - \frac{1}{2}\Sigma N_P^\dagger J N_P & 0 \\ 0 & -\frac{1}{2}\Sigma N_K^\dagger J N_K \end{bmatrix}$, then the associated cost function satisfies the bound

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle W(t)\rangle dt \le \tilde{\lambda} + \delta, \tag{71}$$

where $W(t)$ is in the form of (69) and

$$\tilde{\lambda} = \mathrm{Tr}(X\Xi N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N\Xi). \tag{72}$$

Proof: Suppose the conditions of the theorem are satisfied. Using the Schur complement, (70) is equivalent to

$$\begin{bmatrix} B + 16X\Theta^T\overline{E}^T\overline{E}\Theta X & \overline{F}^T K\overline{F} \\ \overline{F}^T K\overline{F} & -I/\rho \end{bmatrix} < 0. \tag{73}$$

Applying the Schur complement to (73), we obtain

$$\begin{aligned} &\overline{A}^T X + X\overline{A} + 2\overline{F}^T Y + 2Y^T\overline{F} + \overline{E}^T\overline{E}/\gamma^2 \\ &+ 16X\Theta^T\overline{E}^T\overline{E}\Theta X + \overline{R} + \rho\overline{F}^T K\overline{F}\overline{F}^T K\overline{F} < 0. \end{aligned} \tag{74}$$

Substituting $Y = K\overline{F}\Theta^T X$ into (74), we have

$$\begin{aligned} &(\overline{A} - i\Xi\overline{F}^T K\overline{F})^T X + X(\overline{A} - i\Xi\overline{F}^T K\overline{F}) \\ &+ 4X\Xi^\dagger\overline{E}^T\overline{E}\Xi X + \overline{E}^T\overline{E}/\gamma^2 + \overline{R} + \rho\overline{F}^T K\overline{F}\overline{F}^T K\overline{F} < 0 \end{aligned} \tag{75}$$

It follows straightforwardly from (75) that $\overline{A} - i\Xi\overline{F}^T K\overline{F}$ is Hurwitz. Following Lemma 5, (66) and (67), we have

$$\begin{aligned} &[V, z^T][z, V] + \frac{1}{\gamma^2} z^T z \\ &= \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T \begin{pmatrix} 4X\Xi^\dagger\overline{E}^T\overline{E}\Xi X \\ +\overline{E}^T\overline{E}/\gamma^2 \end{pmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}. \end{aligned} \tag{76}$$

Using (76) and Lemma 6, the following equation is obtained.

$$\begin{aligned} &-i[V, H_{P+K}] + \mathcal{L}(V) + [V, z^T][z, V] + \frac{1}{\gamma^2} z^T z + W \\ &= \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix}^T \begin{pmatrix} (\overline{A} - i\Xi\overline{F}^T K\overline{F})^T X \\ +X(\overline{A} - i\Xi\overline{F}^T K\overline{F}) \\ +4X\Xi^\dagger\overline{E}^T\overline{E}\Xi X \\ +\overline{E}^T\overline{E}/\gamma^2 + \overline{R} \\ +\rho\overline{F}^T K\overline{F}\overline{F}^T K\overline{F} \end{pmatrix} \begin{bmatrix} q \\ p \\ q_K \\ p_K \end{bmatrix} \\ &+ \mathrm{Tr}(X\Xi N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N\Xi). \end{aligned} \tag{77}$$

According to Lemma 3 and Lemma 2, we obtain the desired results in (71), (72). □

B. Numerical Procedure

In order to construct a guaranteed cost dynamic controller in Theorem 3, we should not only satisfy the linear matrix inequality (70), but also guarantee the equality $Y = K\overline{F}\Theta^T X$. Since the above equality is a non-convex constraint, we need to introduce some additional variables and an auxiliary equality to convert the equality $Y = K\overline{F}\Theta^T X$ into a linear equality and a rank constraint. In this subsection, we use a similar approach to that in Section V-B.

We define Z as a symmetric matrix with dimension $5(2n_P + 2n_K) \times 5(2n_P + 2n_K)$ and $Z = VV^T$, where $V = [I \quad Z_{x1}^T \quad Z_{x2}^T \quad Z_{v1}^T \quad Z_{v2}^T]^T$. Let $Z_{x1} = X, Z_{x2} = K$ and matrix lifting variables $Z_{v1} = K\overline{F}\Theta^T, Z_{v2} = K\overline{F}\Theta^T X$. Moreover, we require the symmetric matrix Z to satisfy the following conditions:

$$\begin{aligned} &Z \ge 0; && Z_{0,0} - I = 0; \\ &Z_{x1} - Z_{x1}^T = 0; && Z_{x2} - Z_{x2}^T = 0; \\ &Z_{v1} - Z_{x2}\overline{F}\Theta^T = 0; && Z_{v2} - Z_{v1}Z_{x1} = 0 \end{aligned} \tag{78}$$

and a rank constraint

$$\mathrm{rank}(Z) \le 2n_P + 2n_K. \tag{79}$$

An upper bound condition is as follows

$$\mathrm{Tr}(X\Xi N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N\Xi) < \beta. \tag{80}$$

Therefore, controller design in Theorem 3 is converted to solve (70) subject to (78), (79) and (80). This rank constraint LMI problem can be solved using Matlab toolbox LMIrank in Yalmip optimization environment with SeDuMi solver.

VII. Illustrative Example

In order to illustrate the controller design methods in this paper, we consider an example corresponding to a degenerate parametric amplifier (DPA) [32] and its (S, L, H) description has the following form

$$S_P = I, \quad L_P = \frac{1}{2}\begin{bmatrix} \sqrt{\kappa} & \sqrt{\kappa}i \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix},$$
$$H_{P+un} = \frac{1}{2}\begin{bmatrix} q & p \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix},$$

where $H_{P+un} = H_P + H_{un}$. The construction of a DPA is illustrated in Fig. 1. (a). The Hamiltonian of the nominal system is

$$H_P = \frac{1}{2}\begin{bmatrix} q & p \end{bmatrix}\begin{bmatrix} -1.25 & 1 \\ 1 & -1.25 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix},$$

and the Hamiltonian of the perturbation system is

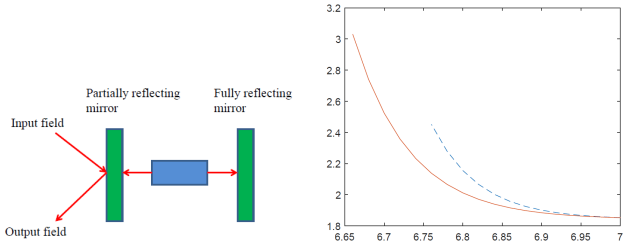
$$H_{un} = \frac{1}{2}\begin{bmatrix} q & p \end{bmatrix}\begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}.$$

This leads to the corresponding parameters in Theorem 2 and Theorem 3 as follows:

$$M_P = \begin{bmatrix} -1.25 & 1 \\ 1 & -1.25 \end{bmatrix}, N_P = \frac{1}{2}\begin{bmatrix} \sqrt{\kappa} & \sqrt{\kappa}i \\ \sqrt{\kappa} & -\sqrt{\kappa}i \end{bmatrix},$$
$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Delta = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix},$$
$$N_K = \frac{1}{2}\begin{bmatrix} \sqrt{0.5} & \sqrt{0.5}i \\ \sqrt{0.5} & -\sqrt{0.5}i \end{bmatrix}, A = \begin{bmatrix} -\frac{\kappa}{2}+1 & -1.25 \\ 1.25 & -\frac{\kappa}{2}-1 \end{bmatrix}.$$

By a numerical method, we find that the nominal plant is mean square stable for the range $\kappa \in (7.9, \infty)$ without any controller. Nevertheless, a static coherent controller can stabilize the system for the range of $\kappa \in (6.75, \infty)$. Meanwhile, a dynamic coherent controller can guarantee that the system is stable for the range of $\kappa \in (6.65, \infty)$. Therefore, quantum controllers can stabilize uncertain quantum systems for a larger range of κ than the one without a controller.

With regard to the stability and the performance of the quantum systems, the importance of the static controller to the given system has already been addressed in [23]. Here, we focus our attention on a performance comparison between the given system with a static quantum controller and that with a dynamic quantum controller, which is shown in Fig. 1. (b). As displayed in Fig. 1. (b), for the same value of κ, the upper bound of cost performance for the system with a static quantum controller is larger than the one with a dynamic quantum controller. For example, when $\kappa = 6.8$, the upper bound of cost function for the uncertain quantum system with a static quantum controller is 2.15, while the upper bound of cost function for the given system with a dynamic quantum controller is 2.01. Therefore, compared to the static controller, the dynamic quantum controller can guarantee the uncertain quantum system stable for a larger range of κ values and at the meantime can achieve a lower performance cost upper bound. Hence, this leads to a closed loop system having better performance.



(a) (b)

Fig. 1. (a) Schematic diagram of degenerate parametric amplifier. (b) Guaranteed cost bounds for uncertain quantum systems with a dynamic controller (solid line) and with a static controller (dash line).

VIII. Conclusions

This paper has evaluated the performance of uncertain linear quantum systems subject to quadratic perturbations in the system Hamiltonian. We proposed methods for static coherent quantum controller design and dynamic coherent quantum controller design. We converted the controller design problem into a rank constrained LMI problem and solved the problem based on an alternating projections algorithm. An example was presented to illustrate these two controller design methods. By comparison, we have shown that a dynamic controller can have an improved control performance over a static one.

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