

Robust Time-Varying Control Barrier Functions with Sector-Bounded Nonlinearities*

Felix Biertümpfel, Jungbae Chun and Peter Seiler

Abstract—This paper presents a novel approach for ensuring safe operation of systems subject to input nonlinearities and time-varying safety constraints. We extend the time-varying barrier function framework to address time-varying safety constraints and explicitly account for control-dependent nonlinearities at the plant input. Guaranteed bounds on the input-output behavior of these nonlinearities are provided through pointwise-in-time quadratic constraints. The result is a class of robust time-varying control barrier functions that define a safety filter. This filter ensures robust safety for all admissible nonlinearities while minimally modifying the command generated by a baseline controller. We derive a second-order cone program (SOCP) to compute this safety filter online and provide feasibility conditions for ball-constrained inputs. The proposed approach is demonstrated on a spacecraft docking maneuver.

I. INTRODUCTION

Engineering problems must meet stringent safety requirements in uncertain and time-varying environments. For example, a driver assistance system must avoid unexpected obstacles and spacecraft need to avoid late-detected space debris during maneuvering. However, these applications usually have well-established control design and verification pipelines resulting in reluctance to perform radical design changes. Thus, novel safety features should minimally alter the original design while showing robustness to comply with certification standards.

Control barrier functions (CBFs) [1], [2] are one popular approach to address these issues. These are online safety filters that guarantee safety by slightly adjust baseline control commands through quadratic programs. CBFs are naturally defined for systems with relative degree one. Higher degrees are handled by exponential control barrier functions (ECBFs) [3] and higher-order control barrier functions (HOCBFs) [4]–[6]. Time-varying (exponential) barrier functions (TVCBFs) [7], [8] address time-varying safe sets.

Existing CBF methods rely on either system models [2] or data-driven approaches [9]–[11]. To handle uncertainties and disturbances, robust control barrier functions have been proposed to ensure robust safety with respect to time-invariant safe sets [12]. The approach in [13] leverages quadratic constraints to handle unmodeled input nonlinearities. A generalization to integral quadratic constraints in [14]

allows for uncertainties with memory. Other formulations address vector field perturbations with L_∞ disturbances [15], or apply mixed-monotone decompositions and robust optimization [16]. Additional robustness schemes include [17]–[19], with recent links to predictive control [20]. Robust CBF approaches have been extended to time-varying safe sets to address norm-bounded disturbances [21] or estimated parametric uncertainties [22]. However, robust methods that simultaneously account for time-varying safety constraints and input nonlinearities have not yet been explored.

The key contribution of our paper is to address this gap using a novel robust time-varying control barrier function (RTVCBF). The proposed framework, introduced in Section III, extends time-varying barrier function methods [7], [8] (Section II-B) to systems with unknown nonlinearities at the plant input while ensuring safety with respect to time-varying safe sets. In particular, we employ pointwise-in-time quadratic constraints to bound the input-output behavior of the nonlinearities by leveraging derivations in [13]. This is summarized in Section II-C. We use this to derive robust safety conditions for time-varying safe sets that hold for all admissible nonlinearities. We construct these conditions explicitly for systems with relative degree two. This is an important class in practice and the results can be easily generalized, with only notational changes, to systems with relative degree different from two. The robustly safe control actions are computed from a second-order cone program (SOCP). In addition, we establish conditions for the feasibility of the SOCP for ball-constrained inputs in Section III-C. This setting is highly relevant for systems subject to limits on the maximum simultaneous torque or force magnitude as it is commonly the case in aerospace applications [23]. Our results also hold for parametric uncertainties at the plant input. Finally, the proposed framework is illustrated in Section IV through a CubeSat docking example.

II. PRELIMINARIES

A. Problem Formulation

We consider the feedback interconnection of a controller k and an uncertain system G_ϕ shown in Fig. 1. The uncertain plant consists of the interconnection of a known nonlinear, input-affine system G and an unknown nonlinearity ϕ :

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t)) (u(t) + w(t)), \\ w(t) &= \phi(u(t), t),\end{aligned}\tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ and $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ denote the state and input at time t , respectively. The functions $f : \mathcal{D} \rightarrow \mathbb{R}^n$

*This work was supported by the EU under Grant No. 101153910. The responsibility for the content of this paper is with its authors.

F. Biertümpfel is with the Chair of Flight Mechanics and Control at TU Dresden, Dresden, and the Department of Electrical and Computer Engineering at the University of Michigan, Ann Arbor felix.biertuempfel@tu-dresden.de, felixb@umich.edu.

J. Chun and P. Seiler are with the Department of Electrical and Computer Engineering at the University of Michigan, Ann Arbor, jungbaec@umich.edu, pseiler@umich.edu

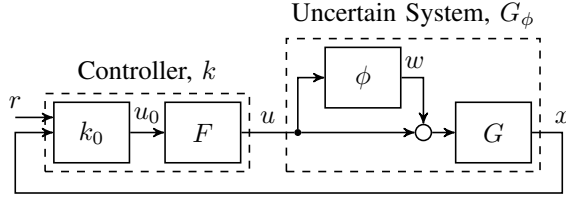


Fig. 1: Closed loop interconnection with input nonlinearity ϕ , baseline controller k_0 and safety filter F .

and $g : \mathcal{D} \rightarrow \mathbb{R}^{n \times m}$ are locally Lipschitz continuous. The admissible sets of states and inputs are denoted by $\mathcal{D}$ and $\mathcal{U}$.

The nonlinearity $\phi : \mathcal{U} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ is assumed to be memoryless but possibly time-varying and can model, for example, unknown nonlinear effects. We also assume that the nonlinearity is bounded in the sector $[-\theta, \theta]$ with $0 < \theta < 1$. This implies that the input $u(t)$ and output $w(t)$ of ϕ satisfy the following inequality at each point in time:

$$\|w(t)\|_2 \leq \theta \|u(t)\|_2 \quad \forall t \geq 0. \quad (2)$$

Thus $\theta \geq 0$ measures the amount of nonlinear perturbation at the plant input. No perturbation corresponds to $\theta = 0$, i.e. $w = 0$, and is referred to as nominal system G in the remainder of the paper. Systems with more general sector-bounded nonlinearities can be represented in this form via a standard loop-shifting process [24, Sec. 6.5]. Based on this constraint, we define the set of feasible outputs of the nonlinearity ϕ for a given input $u(t) \in \mathcal{U}$ as:

$$\mathcal{W}(u(t)) := \{w(t) : \|w(t)\|_2 \leq \theta \|u(t)\|_2\}. \quad (3)$$

The feedback interconnection includes a baseline controller k_0 which is designed for reference tracking. It has access to state-feedback measurements and a reference input vector $r(t) \in \mathcal{R} \subset \mathbb{R}^r$. The latter is confined to the admissible set $\mathcal{R}$. The baseline controller is defined by a Lipschitz continuous function $k_0 : \mathcal{D} \times \mathcal{R} \rightarrow \mathcal{U}$.

We assume this baseline controller provides adequate performance but fails to provide “safe” operation. The notion of safety refers to a safe set $\mathcal{C} \subset \mathcal{D}$. This set defines the, possibly time-varying, region in the state space in which the system must remain for safe operation [7]. This safe set is defined as the zero-superlevel set of a twice continuously differentiable, time-varying C^2 function $h : \mathcal{D} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$:

$$\mathcal{C}(t) := \{x \in \mathcal{D} : h(x, t) \geq 0\}. \quad (4)$$

The boundary and interior of the safe set are denoted by $\partial\mathcal{C}$ and $\text{Int}(\mathcal{C})$, respectively. We assume $\mathcal{C}$ is non-empty with no isolated points and zero is a regular value of h .

The goal is to design a safety filter F to ensure that the uncertain system G_ϕ , defined in (1), operates safely for all nonlinearities in the sector $[-\theta, \theta]$, i.e., $x(0) \in \mathcal{C}(0)$ implies $x(t) \in \mathcal{C}(t)$ for all $t \geq 0$. This is equivalent to the forward invariance of $\mathcal{C}(t)$ for the uncertain closed-loop with all possible nonlinearities. Moreover, the safety filter should maintain safety while minimally altering the command from the baseline controller k_0 . The overall controller, k , consists of both the baseline controller k_0 and the safety filter F as shown in Fig. 1. The closed-loop dynamics with the uncertain system and this combined controller are given by:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t)) (u(t) + w(t)), \\ u(t) &= k(x(t), r(t)), \\ w(t) &= \phi(u(t), t). \end{aligned} \quad (5)$$

We will refer to (5) as nominal closed loop for perturbation levels $\theta = 0$, i.e. $w = 0$. The remainder of the section reviews the ingredients required to design a “safe” controller.

B. Time-Varying Exponential Control Barrier Functions

Time-varying exponential control barrier functions (TVCBFs) were introduced in [7]. The nominal closed-loop system is given by (5) with $\theta = 0$ and hence $w = 0$. TVCBFs ensure safety of this nominal closed loop with respect to time-variant safe sets $\mathcal{C}(t)$ defined by (4). We review the formulation for relative degree two. This is a common case due to its relevance for position and rotational dynamics. However, all subsequent derivations generalize to relative degrees other than two, requiring only notational changes. The following Definition 1 formalizes time-varying control invariance of a time-varying safe set $\mathcal{C}(t)$ with respect to the nominal closed loop system.

Definition 1: (Time-Varying Control Invariance [7]) A set $\mathcal{C}(t)$ is time-varying control invariant with respect to the nominal closed loop, (5) with $w = 0$, if there exists a control law $k : \mathcal{D} \times \mathcal{R} \rightarrow \mathcal{U}$ such that the following holds:

$$x(0) \in \mathcal{C}(t) \Rightarrow x(t) \in \mathcal{C}(t) \quad \forall t \geq 0. \quad (6)$$

The nominal closed loop is safe with respect to $\mathcal{C}(t)$ if $\mathcal{C}(t)$ is time-varying control invariant.

A modified Lie derivative of $h(x, t)$ along f is required for time-varying safe sets. It is defined as $\bar{L}_f^i h := (\frac{\partial}{\partial t} + L_f)^i h$ for non-negative integers i [7]. The next definition is a specific case of [7, Definition 1] for time-varying exponential barrier functions with relative degree two.

Definition 2: (Time-Varying Exponential Control Barrier Function) Consider the nominal closed-loop system, (5), with $w = 0$. A C^2 function $h : \mathcal{D} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ with a relative degree two is a time-varying control barrier function if there exists an $\alpha > 0$ such that for all $(x, t) \in \mathcal{D} \times \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \sup_{u \in \mathcal{U}} [\bar{L}_f^2 h(x, t) + L_g \bar{L}_f h(x, t)u] \\ \geq -2\alpha \bar{L}_f h(x, t) - \alpha^2 h(x, t). \end{aligned} \quad (7)$$

The existence of a TVCBF $h(x)$ implies that there exists a control input u that can render the safe set $\mathcal{C}(t)$ forward invariant and, in particular, prevent trajectories from crossing its boundary $\partial\mathcal{C}(t)$. Define the following set of control inputs at a state x and time t :

$$\begin{aligned} \mathcal{U}_{\text{TVCBF}}(x, t) := \{u \in \mathcal{U} : \bar{L}_f^2 h(x, t) + 2\alpha \bar{L}_f h(x, t) \\ + \alpha^2 h(x, t) + L_g \bar{L}_f h(x, t)u \geq 0\}. \end{aligned}$$

Also, define the set $\mathcal{C}_1(t) := \{x \in \mathcal{D} : \bar{L}_f h(x, t) + \alpha h(x, t) \geq 0\}$. The following corollary is an extension of [7, Proposition 1] for multi-input systems and control systems with reference signals r .

Corollary 1: Assume a C^2 function $h : \mathcal{D} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ has relative degree two with respect to the nominal closed loop, (5) with $w = 0$. Moreover, assume h satisfies (7) for

some $\alpha > 0$. Then, any controller $k : \mathcal{D} \times \mathcal{R} \rightarrow \mathcal{U}$ that is Lipschitz continuous in x and r , and satisfies $k(x, r) \in \mathcal{U}_{\text{TVCBF}}(x, t) \forall (x, r, t) \in \mathcal{D} \times \mathcal{R} \times \mathbb{R}_{\geq 0}$ renders the set $\mathcal{C}(t)$ forward invariant provided that $x_0 \in \mathcal{C}(0) \cap \mathcal{C}_1(0)$.

Proof: By assumption, h has relative degree two with respect to the nominal system. Hence $L_g h(x, t) = 0$ and $L_g \bar{L}_f h(x, t) \neq 0$ for all $(x, t) \in \mathcal{D} \times \mathbb{R}_{\geq 0}$. This implies $\bar{L}_f h(x, t) = \dot{h}(x, t)$ and for any $u(x, t) \in \mathcal{U}_{\text{TVCBF}}(x, t)$ the inequality $\ddot{h}(x, t) + 2\alpha\dot{h}(x, t) + \alpha^2 h(x, t) \geq 0$ is satisfied. Furthermore, the polynomial $s^2 + 2\alpha s + \alpha^2$ has a repeated root $-\alpha$ with $\alpha > 0$. Since $x_0 \in \mathcal{C}(0) \cap \mathcal{C}_1(0)$, it follows that $h(x_0, 0) \geq 0$ and $\dot{h}(x_0, 0) + \alpha h(x_0, 0) \geq 0$. Thus, Lemma 1 in [7] implies that $x(t) \in \mathcal{C}(t)$ for all $t \geq 0$. ■

C. Robust Exponential Control Barrier Functions

This section provides a short overview of time-invariant robust exponential control barrier functions [13]. Define a time-invariant C^2 function $h : \mathcal{D} \rightarrow \mathbb{R}$ and the set

$$\mathcal{C} := \{x \in \mathcal{D} : h(x) \geq 0\}. \quad (8)$$

The safety of the uncertain closed loop (5) is assured if $x(0) \in \mathcal{C}$ implies that $x(t) \in \mathcal{C}$ for all $t \geq 0$. The robust control invariance concept proposed in [12, Definition 4.4] provides a way to ensure safety for uncertain systems. A specific formulation for sector-bounded nonlinearities ϕ was given in [13, Definition 4]. This definition is repeated next.

Definition 3: (Robust Control Invariance [13]) A set $\mathcal{C} \subset \mathcal{D}$ is robustly control invariant with respect to the uncertain closed loop (5), if there exists a control law $k : \mathcal{D} \times \mathcal{R} \rightarrow \mathcal{U}$ such that the following holds for all admissible nonlinearities ϕ in the sector $[-\theta, \theta]$:

$$x(0) \in \mathcal{C} \Rightarrow x(t) \in \mathcal{C} \quad \forall t \geq 0. \quad (9)$$

The uncertain closed loop (5) is robustly safe with respect to $\mathcal{C}$ if $\mathcal{C}$ is robustly control invariant.

Exponential control barrier functions can be used to guarantee safety for the nominal closed loop system, i.e., $\theta = 0$ [2], [3]. Robust exponential control barrier functions guarantee robust control invariance of the uncertain closed-loop system (5). This robust version was proposed in [14]. The following definition presents an explicit formulation for an uncertain system of relative degree two.

Definition 4: (Robust Exponential Control Barrier Functions) Given a safe set $\mathcal{C} \subset \mathcal{D}$ defined by (8), a C^2 function $h : \mathcal{D} \rightarrow \mathbb{R}$ with a relative degree two is a robust exponential control barrier function for the uncertain system (5) for all nonlinearities ϕ in the sector $[-\theta, \theta]$, if there exists $\alpha > 0$ such that for all $x \in \mathcal{D}$,

$$\sup_{u \in \mathcal{U}} \inf_{w \in \mathcal{W}(u)} [L_f^2 h(x) + L_g L_f h(x)(u + w)] \geq -2\alpha L_f h(x) - \alpha^2 h(x). \quad (10)$$

III. ROBUST TIME-VARYING CBFs

A. Formulation

In this section, we propose robust time-varying control barrier functions (RTVCBFs), which are designed to provide safety guarantees for the uncertain closed-loop dynamics (5).

We first formalize the notion of robust time-varying control invariance. This generalizes Definitions 3 and 1.

Definition 5: (Robust Time-Varying Control Invariance) A set $\mathcal{C}(t)$ is robustly time-varying control invariant with respect to the uncertain dynamics (5) if there exists a control law $k : \mathcal{D} \times \mathcal{R} \rightarrow \mathcal{U}$ such that the following holds for all admissible nonlinearities ϕ in the sector $[-\theta, \theta]$:

$$x(0) \in \mathcal{C}(t) \Rightarrow x(t) \in \mathcal{C}(t) \quad \forall t \geq 0. \quad (11)$$

The system (5) is robustly safe with respect to $\mathcal{C}(t)$ if $\mathcal{C}(t)$ is robust time-varying control invariant.

A robust time-varying exponential control barrier function can be used to formulate safety filters F that guarantee robust safety of uncertain closed loop systems (5) with respect to the time-varying safe sets $\mathcal{C}(t)$ (4). The following definition provides an explicit formulation of such control barrier functions for relative degree two.

Definition 6: (Robust Time-Varying Exponential Control Barrier Functions) Given a safe set $\mathcal{C}(t) \subset \mathcal{D}$ defined by (4), a C^2 function $h : \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ with a relative degree two is a robust time-varying control barrier function for the uncertain closed loop (5) for all admissible nonlinearities ϕ in the sector $[-\theta, \theta]$, if there exists $\alpha > 0$ such that for all $(x, t) \in \mathcal{D} \times \mathbb{R}_{\geq 0}$,

$$\sup_{u \in \mathcal{U}} \inf_{w \in \mathcal{W}(u)} [\bar{L}_f^2 h(x, t) + L_g \bar{L}_f h(x, t)(u + w)] \geq -2\alpha \bar{L}_f h(x, t) - \alpha^2 h(x, t). \quad (12)$$

The worst-case behavior of the nonlinearity is represented by a perturbation w that minimizes the left-hand side of the inequality (12). Our goal is to determine a safe control input $u \in \mathcal{U}$ under the corresponding worst-case perturbation $w^* \in \mathcal{W}$. The worst-case uncertain input $w^*(u)$, obtained as the solution of the inner optimization problem in (12), is

$$w^*(u) = -\theta \|u\|_2 \frac{L_g \bar{L}_f h(x, t)^\top}{\|L_g \bar{L}_f h(x, t)\|_2}. \quad (13)$$

Note, that for $\theta \rightarrow 0$ the worst-case disturbance $w^*(u)$ approaches zero and the nominal TVCBF formulation in Definition 2 is recovered. Substituting $w^*(u)$ for w in condition (12) provides

$$\sup_{u \in \mathcal{U}} [\bar{L}_f^2 h(x, t) + L_g \bar{L}_f h(x, t)(u + w^*(u))] \geq -2\alpha \bar{L}_f h(x, t) - \alpha^2 h(x, t). \quad (14)$$

The constraint (14) is equivalent to (12). Next, define the set of control inputs

$$\mathcal{U}_{\text{RTVCBF}}(x, t) := \{u \in \mathcal{U} : \bar{L}_f^2 h(x, t) + 2\alpha \bar{L}_f h(x, t) + \alpha^2 h(x, t) + L_g \bar{L}_f h(x, t)(u + w^*(u)) \geq 0\}$$

If the supremum in (14) is attained, then the above set is non-empty. The following Theorem 1 states an existence condition for a robust time-varying control barrier function guaranteeing that the uncertain closed loop (5) is robustly safe with respect to $\mathcal{C}(t)$.

Theorem 1: Assume a C^2 function $h : \mathcal{D} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ has relative degree two with respect to the uncertain closed loop

(5) and satisfies (14) for some $\alpha > 0$. Then, any controller $k : \mathcal{D} \times \mathcal{R} \rightarrow \mathcal{U}$ that is Lipschitz continuous in x and r , and satisfies $k(x, r) \in \mathcal{U}_{\text{TVCBF}}(x, t) \forall (x, r, t) \in \mathcal{D} \times \mathcal{R} \times \mathbb{R}_{\geq 0}$ renders the set $\mathcal{C}(t)$ robust time-varying forward invariant provided that $x_0 \in \mathcal{C}(0) \cap \mathcal{C}_1(0)$.

Proof: By assumption h has relative degree two with respect to the uncertain closed loop dynamics (5). Thus $L_g h(x, t) = 0$ and $L_g \bar{L}_f h(x, t) \neq 0$ for all $(x, t) \in \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. This implies $\bar{L}_f h(x, t) = \dot{h}(x, t)$ and for any $u(x, t) \in \mathcal{U}_{\text{RTVCBF}}(x, t)$ the inequality $\dot{h}(x, t) + 2\alpha \dot{h}(x, t) + \alpha^2 h(x, t) \geq 0$ holds for all inputs $w \in \mathcal{W}$ satisfying (2). Furthermore, the polynomial $s^2 + 2\alpha s + \alpha^2$ has a repeated root $-\alpha$ with $\alpha > 0$. Since $x_0 \in \mathcal{C}(0) \cap \mathcal{C}_1(0)$, it follows that $h(x_0, 0) \geq 0$ and $\dot{h}(x_0, 0) + \alpha h(x_0, 0) \geq 0$. Thus, Lemma 1 in [7] implies that $x(t) \in \mathcal{C}(t)$ for all $t \geq 0$. ■

B. Control Algorithm

Based on the approach presented in [13], we propose an efficient online method for computing a safe control input u that minimally interferes with the baseline controller k_0 . The problem can be posed as the optimization program:

$$\begin{aligned} u^*(x) &= \arg \min_{u \in \mathcal{U}} \frac{1}{2} \|u - k_0(x, r)\|^2 \\ \text{s.t. } &\bar{L}_f^2 h(x, t) + 2\alpha \bar{L}_f h(x, t) + \alpha^2 h(x, t) \\ &+ L_g \bar{L}_f h(x, t)(u + w^*(u)) \geq 0. \end{aligned} \quad (15)$$

Substituting for the the worst-case perturbation $w^*(u)$ as given in Eq. (13)) and expanding the cost function provides

$$\begin{aligned} u^*(x) &= \arg \min_{u \in \mathcal{U}} \left[\frac{1}{2} u^\top u - k_0(x, r)^\top u \right] \\ \text{s.t. } &\bar{L}_f^2 h(x, t) + 2\alpha \bar{L}_f h(x, t) + \alpha^2 h(x, t) \\ &+ L_g \bar{L}_f h(x, t)u - \theta \|u\|_2 \|L_g \bar{L}_f h(x, t)\|_2 \geq 0. \end{aligned} \quad (16)$$

For an uncertainty level $\theta = 0$, i.e., $w = 0$, the nominal system behavior is recovered and (16) reduces to the standard time-varying exponential control barrier function quadratic program. For $\theta > 0$, the decision variable u enters nonlinearly in the inequality constraint through $\|u\|_2$. To address this, the optimization can be reformulated into a convex problem in two steps. First, following [13] introduce a slack variable q to rewrite (16) as

$$\begin{aligned} \begin{bmatrix} u^*(x) \\ q^*(x) \end{bmatrix} &= \arg \min_{u \in \mathcal{U}, q \in \mathbb{R}_{>0}} [q - k_0(x, r)^\top u] \\ \text{s.t. } &\bar{L}_f^2 h(x, t) + 2\alpha \bar{L}_f h(x, t) + \alpha^2 h(x, t) \\ &+ L_g \bar{L}_f h(x, t)u \geq \theta \|L_g \bar{L}_f h(x, t)\|_2 \|u\|_2 \\ &2q \geq \|u\|_2^2. \end{aligned} \quad (17)$$

The reformulated problem is a minimization over $u \in \mathcal{U}$ and $q \in \mathbb{R}_{>0}$ with $2q^* = u^{*\top} u^*$. Finally, the second constraint is recast as a rotated second-order cone (SOC) condition following [25, Sec. 10.1]. This gives the Robust TVCBF-

second-order-cone-program (Robust TVCBF-SOCP):

$$\begin{aligned} \begin{bmatrix} u^*(x) \\ q^*(x) \end{bmatrix} &= \arg \min_{u \in \mathcal{U}, q \in \mathbb{R}_{>0}} [q - k_0(x, r)^\top u] \\ \text{s.t. } &\theta \|L_g \bar{L}_f h(x, t)\|_2 \|u\|_2 \leq \bar{L}_f^2 h(x, t) \\ &+ 2\alpha \bar{L}_f h(x, t) + \alpha^2 h(x, t) + L_g \bar{L}_f h(x, t)u \\ &\left\| \begin{bmatrix} \sqrt{2}u \\ q-1 \end{bmatrix} \right\|_2 \leq q+1. \end{aligned} \quad (18)$$

This can be efficiently solved online.

C. Feasibility

The feasibility of the optimization (18) is essential to guarantee robust safety of the control system. We consider two cases: 1) unconstrained control inputs and 2) Euclidean norm bounded control inputs. The latter appear in problems assuming maximum torque and force magnitude bounds, e.g., in thrust-vectoring control of space launchers or spacecraft.

First, we consider the case of unconstrained control inputs. To simplify the notation, we define

$$\begin{aligned} c_1(x, t) &:= \bar{L}_f^2 h(x, t) + 2\alpha \bar{L}_f h(x, t) + \alpha^2 h(x, t) \\ c_2(x, t) &:= L_g \bar{L}_f h(x, t). \end{aligned}$$

The main constraint in (18) can be rewritten in the following simpler form based on this notation:

$$\theta \|c_2(x, t)\|_2 \|u\|_2 \leq c_1(x, t) + c_2(x, t)u$$

The next theorem formalizes the feasibility analysis of the RTVCBF-SOCP for the case of unconstrained control inputs.

Theorem 2: Assume a C^2 function $h : \mathcal{D} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ has relative degree two with respect to the uncertain system dynamics (5). If $\theta \in [0, 1]$ and the control input is unconstrained then the RTVCBF-SOCP problem is feasible for all $(x, t) \in \mathcal{D} \times \mathbb{R}_{\geq 0}$.

Proof: The optimization (17) and the RTVCBF-SOCP (18) are equivalent. Hence we consider (17) in the remainder of the proof. Since h has relative degree two with respect to the uncertain dynamics (5), $c_2(x, t) \neq 0 \forall (x, t) \in \mathcal{D} \times \mathbb{R}_{\geq 0}$.

If $c_1(x, t) \geq 0$ then select $u(t) := 0 \in \mathbb{R}^m$ and $q(t) := 0$. The first constraint in (17) simplifies to $c_1(x, t) \geq 0$ which is satisfied by assumption. The second constraint in (17) holds trivially. Thus, $[u(t)^\top, q(t)]^\top$ is a feasible solution for (17).

If $c_1(x, t) < 0$ then select

$$\begin{aligned} u(t) &:= \left(-\frac{c_1(x, t)}{(1 - \theta)\|c_2(x, t)\|_2^2} \right) c_2(x, t)^\top \in \mathbb{R}^m \\ q(t) &:= \frac{1}{2} \|u\|_2^2 \end{aligned} \quad (19)$$

This choice of $u(t)$ (combined with $0 < \theta < 1$) yields

$$\begin{aligned} c_1(x, t) + c_2(x, t)u &= -\frac{\theta}{1 - \theta} c_1(x, t) \\ \theta \|c_2(x, t)\|_2 \|u\|_2 &= \frac{\theta}{1 - \theta} |c_1(x, t)| \end{aligned}$$

Since $c_1(x, t) < 0$ for this case, it follows that

$$c_1(x, t) + c_2(x, t)u = \theta \|c_2(x, t)\|_2 \|u\|_2$$

Thus, $[u(t)^\top, q(t)^\top]^\top$ is a feasible solution for (17). ■

Next, we consider the case of control inputs with a Euclidean norm bounded by $u_{\max} > 0$, i.e., $\|u(t)\|_2 \leq u_{\max}$. The feasibility of the RTVCBF-SOCP under this constraint is established in the next theorem.

Theorem 3: Assume a C^2 function $h : \mathcal{D} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ has relative degree two with respect to the uncertain dynamics (5). If $\theta \in [0, 1)$ and the Euclidean norm of the control input $u(t)$ is bounded by $u_{\max} > 0$ then the RTVCBF-SOCP problem is feasible for $(x, t) \in \mathcal{D} \times \mathbb{R}_{\geq 0}$ if and only if

$$u_{\max} \geq -\frac{c_1(x, t)}{(1 - \theta)\|c_2(x, t)\|_2} \quad (20)$$

Proof: The proof is similar to that given for Theorem 2. Note that $u(t) = 0$ is feasible when $c_1(x, t) \geq 0$. Hence it suffices to only consider the case when $c_1(x, t) < 0$.

($\Leftarrow$) Assume Equation (20) holds. Then select $u(t)$ and $q(t)$ as in (19). It was shown in the proof for Theorem 2 that this choice is feasible. Moreover, it follows directly from (20) that this choice for the control input satisfies $\|u(t)\|_2 \leq u_{\max}$.

($\Rightarrow$) Assume Equation 20 is violated. Then for any $u(t)$ with $\|u(t)\|_2 \leq u_{\max}$, we have

$$\begin{aligned} c_2(x, t)u(t) - \theta\|c_2(x, t)\|_2\|u(t)\|_2 \\ \leq (1 - \theta)\|c_2(x, t)\|_2\|u(t)\|_2 < -c_1(x, t) \end{aligned}$$

The last inequality uses

$$\|u(t)\|_2 \leq u_{\max} < -\frac{c_1(x, t)}{(1 - \theta)\|c_2(x, t)\|_2}.$$

Hence, for any $\|u(t)\| \leq u_{\max}$,

$$c_1(x, t) + c_2(x, t)u < \theta\|c_2(x, t)\|_2\|u\|_2.$$

Thus, the RTVCBF-SOCP constraint is violated. ■

The feasibility condition in Theorem 3 can be checked before the execution of the SOCP. Solving the SOCP can be avoided if the condition is violated. This information can also be used to command a maximum control effort or trigger another form of emergency measure. Alternatively, it can motivate a re-design of the system hardware to provide larger actuator limits if needed for safety.

IV. NUMERICAL EXAMPLE

The example considers a planar docking maneuver of a CubeSat with a decommissioned spacecraft in low earth orbit (LEO). This is an essential maneuver in debris removal missions using CubeSats and requires safe and robust autonomy.

A. Satellite Dynamics

The CubeSat's docking dynamics are formulated with respect to a local vertical local horizontal (LVLH) coordinate frame fixed to the target and defined following [26]. These dynamics have the form of (1) with

$$\begin{aligned} f(x(t)) &= \begin{bmatrix} \frac{v_x}{v_y} \\ 2\omega(t)v_y + \dot{\omega}(t)p_y + \omega(t)^2 p_x + \frac{\mu}{r_{\text{CS}}(t)^2} - \mu \frac{r_T(t) + p_x}{r_{\text{CS}}(t, x)^3} \\ -2\omega(t)v_x - \dot{\omega}(t)p_x + \omega(t)^2 p_y - \mu \frac{p_y}{r_{\text{CS}}(t, x)^3} \end{bmatrix} \\ g(x(t)) &= \begin{bmatrix} 0 & 0 & \frac{1}{m_{\text{CS}}} & 0 \\ 0 & 0 & 0 & \frac{1}{m_{\text{CS}}} \end{bmatrix}^\top. \end{aligned} \quad (21)$$

The state vector $x = [p_x, p_y, v_x, v_y]^\top$ is comprised of the relative positions (p) and velocities (v) of the CubeSat radial to (x) and in direction (y) of the target velocity, respectively. The control input $u = [T_x, T_y]^\top$ is the thrust in x and y direction. The CubeSat is modeled after the OAAN CubeSat [27] with mass $m_{\text{CS}} = 4$ kg. Four cold gas thrusters provide ± 300 mN of thrust T_x and T_y . The maximum simultaneous thrust T is limited, due to power constraints, by $T_x^2 + T_y^2 \leq (300\text{mN})^2$. The nonlinearity ϕ is bounded to the sector $\pm\theta = 0.75$, equivalent to 75% maximum plant input uncertainty, to account for large thrust uncertainties for small satellites.

In (21), $r_{\text{CS}} = \sqrt{(r_T + p_x)^2 + p_y^2}$ is the CubeSat's and r_T the target's distance to the center of the central body, respectively. The LVLH rotation rate and angular acceleration are denoted by ω and $\dot{\omega}$, respectively. The orbit-coefficient time histories are calculated for ICESat-2 (NORAD 43613) using the publicly available two-line element sets [28], propagated using a two-body Kepler propagation. The gravitational parameter μ has a value of $3.986 \cdot 10^{14} \text{ m}^3/\text{s}^2$.

B. Docking Scenario

The maneuver starts at $p_x = -10$ m and $p_y = 0$ m and $v_x = v_y = 0$, i.e., with zero initial velocity. The initial target docking position is located at $p_{f,x} = 0$ m and $p_{f,y} = 0$. During the approach, the CubeSat identifies a cluster of space debris with a radius of $r_0 = 0.35$ m at $p = [-5 \text{ m}, 0 \text{ m}] = [c_{x0}, c_{y0}]$ that drifts with a constant velocity $v_{c,x} = 1$ mm/s and $v_{c,y} = 2$ mm/s. More objects are identified while the CubeSat gets closer to the target which increases the cluster size by $s_r = 1$ mm/s. Note that r_0 includes a 0.05 m safety margin to account for the CubeSat's cross-section. While avoiding the obstacle, the CubeSat must also satisfy a line-of sight (LoS) constraint to maintain visual contact with the docking location. This constrain is modeled as a cone centered at p_f , with half-cone angle $\gamma = 5$ deg and width of 0.1 m at p_f . The cone moves with a constant y velocity $v_{\text{LoS},y} = -1$ mm/s as the CubeSat updates the docking position p_f . As the derivations in Section III generalize directly to multiple barrier functions h_i , $i \in \{1, 2, \dots, n\}$, we can, according to the two safety requirements, define the safe set as in (4) via

$$h_1 = (p_x - \bar{c}_x(t))^2 + (p_y - \bar{c}_y(t))^2 - r_c(t)^2 \quad (22)$$

$$h_2 = \tan(\gamma)p_x + n_1(t) - p_y \quad (23)$$

$$h_3 = p_y - \tan(-\gamma)p_x - n_2(t), \quad (24)$$

where $\bar{c}_x(t) = c_{x0} + v_{c,x}t$, $\bar{c}_y(t) = c_{y0} + v_{c,y}t$, $r_c = r_0 + s_r t$, $n_i = n_{i,0} + v_{\text{LoS},y}$ with $i = 2, 3$. The relevant terms to compute $\bar{L}_f h_i(x, t)$, $\bar{L}_f^2 h_i(x, t)$ and $L_g \bar{L}_f h_i(x, t)$ follow the descriptions in Section III. The TVCBF and RTVCBF place the repeated design poles at $s = -\alpha = 0.5$. The TVCBF control signal is calculated using Matlab's `quadprog`. The algorithm `coneprog` is used in case of the RTVCB-SOCP.

The CubeSat shall approach the target with a constant velocity of 0.03 m/s and follow the y component of the docking location p_f . Thus, a linear quadratic regulator (LQR), K , is designed for the linearized CubeSat dynamics. The cost matrices are chosen as $Q = \text{diag}(10, 10^3, 10^8, 10^5)$ and

$R = \text{diag}(10^7, 10^7)$ yielding $K = \begin{bmatrix} 0.001 & 0 & 3.164 & 0 \\ 0 & 0.010 & 0 & 0.300 \end{bmatrix}$. The LQR controller follows the reference command $r = \begin{bmatrix} 0 & \frac{n_1(t)+n_2(t)}{2} & 0.03 & 0 \end{bmatrix}$ using the control law $u_0(r, x) = K \cdot (r - x)$. We implement three control architectures following Fig. 1: 1) baseline LQR controller, 2) baseline LQR with nominal TVECBF safety filter, and 3) baseline LQR with the novel RTVECBF safety filter. Architecture 3) also checks the feasibility condition in Theorem 3. For guaranteed infeasibility, it skips the SOCP and commands the maximum control effort.

C. Results

The docking trajectories using the worst-case perturbation w^* corresponding to h_1 are shown in the top plot of Fig. 2. Pure LQR control (—) cannot evade the obstacle. The safety filter using the nominal TVECBF (—) performs an evasive maneuver, but violates the safety criterion at $t = 152$ s. The CubeSat crosses the debris field at this time instance (—) which moved from its initial location (—) at $t = 0$ s. This is visualized by the negative distance to the safety boundary shown in the bottom plot of Fig. 2. The novel safety filter based on the RTVCBF (—) safely avoids the obstacle. This is achieved by choosing a more cautious and slower trajectory by applying more thrust that account for the system uncertainty as shown in the middle plot of Fig. 2.

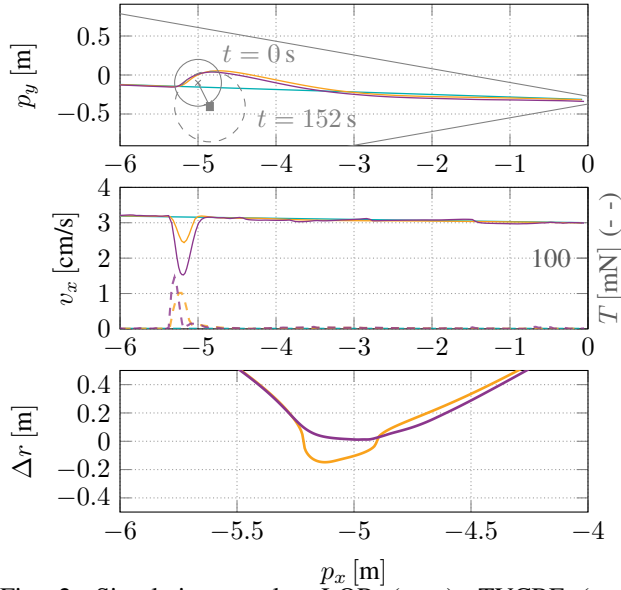


Fig. 2: Simulation results: LQR (—), TVECBF (—), RTVCBF (—)

V. CONCLUSION

The paper introduced an approach for robust time-varying control barrier functions that accounts for time-varying safe sets and plant input nonlinearities. The safety filter is formulated as the solution of a second-order cone program. A feasibility criterion is derived for this optimization. The approach was demonstrated on a CubeSat docking scenario. Future work will further investigate feasibility conditions under different assumptions on the input constraints.

REFERENCES

- [1] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 3861–3876, Aug. 2017.
- [2] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *European Control Conf.*, 2019, pp. 3420–3431.
- [3] Q. Nguyen and K. Sreenath, "Exponential control barrier functions for enforcing high relative-degree safety-critical constraints," in *American Control Conf.*, 2016, pp. 322–328.
- [4] W. Xiao and C. Belta, "Control barrier functions for systems with high relative degree," in *IEEE Conf. on Decis. Control*, 2019, pp. 474–479.
- [5] X. Tan, W. S. Cortez, and D. V. Dimarogonas, "High-order barrier functions: Robustness, safety, and performance-critical control," *IEEE Trans. Autom. Control*, vol. 67, no. 6, pp. 3021–3028, 2021.
- [6] H. Wang, J. Peng, J. Xu, F. Zhang, and Y. Wang, "High-order control barrier functions-based optimization control for time-varying nonlinear systems with full-state constraints: A dynamic sub-safe set approach," *Int. J. Robust Nonlinear Control*, vol. 33, no. 8, pp. 4490–4503, 2023.
- [7] X. Xu, "Constrained control of input-output linearizable systems using control sharing barrier functions," *Autom.*, vol. 87, pp. 195–201, 2018.
- [8] L. Lindemann and D. Dimarogonas, "Control barrier functions for signal temporal logic tasks," *IEEE Control Syst. Lett.*, vol. 3, no. 1, pp. 96–101, 2018.
- [9] A. Taylor, A. Singletary, Y. Yue, and A. Ames, "Learning for safety-critical control with control barrier functions," in *Proc. 2nd Conf. Learn. Dyn. Cont.*, ser. Proc. MLR, 2020, pp. 708–717.
- [10] J. Choi, F. Castaneda, C. J. Tomlin, and K. Sreenath, "Reinforcement learning for safety-critical control under model uncertainty, using control lyapunov functions and control barrier functions," *arXiv*, 2020.
- [11] M. Bajelani and K. van Heusden, "Data-driven input-output control barrier functions," *IEEE Control Syst. Lett.*, vol. 9, pp. 216–221, 2025.
- [12] F. Blanchini and S. Miani, *Set-theoretic methods in control*. Springer, 2015.
- [13] J. Buch, S.-C. Liao, and P. Seiler, "Robust control barrier functions with sector-bounded uncertainties," *IEEE Control Syst. Lett.*, vol. 6, pp. 1994–1999, 2021.
- [14] P. Seiler, M. Jankovic, and E. Hellstrom, "Control barrier functions with unmodeled input dynamics using integral quadratic constraints," *IEEE Control Syst. Lett.*, vol. 6, pp. 1664–1669, 2021.
- [15] Q. Nguyen and K. Sreenath, "Robust safety-critical control for dynamic robotics," *IEEE Trans. Autom. Control*, vol. 67, no. 3, pp. 1073–1088, 2022.
- [16] T. Pati and S. Z. Yong, "Robust control barrier functions for control affine systems with time-varying parametric uncertainties," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 10588–10594, 2023.
- [17] J. J. Choi, D. Lee, K. Sreenath, C. J. Tomlin, and S. L. Herbert, "Robust control barrier-value functions for safety-critical control," in *Proc. IEEE Conf. Decis. Control*, 2021, pp. 6814–6821.
- [18] A. Safari and J. B. Hoagg, "Time-varying soft-maximum control barrier functions for safety in an a priori unknown environment," in *American Control Conf.*, 2024, pp. 3698–3703.
- [19] V. Hamdipoor, N. Meskin, and C. G. Cassandras, "Safe control synthesis using environmentally robust control barrier functions," *Eur. J. Control*, vol. 74, p. 100840, 2023.
- [20] J. Huang, H. Wang, K. Margellos, and P. Goulart, "Predictive control barrier functions: bridging model predictive control and control barrier functions," *arXiv*, 2025.
- [21] I. Tezuka, T. Kuramoto, and H. Nakamura, "Input-to-state constrained safety zeroing control barrier function and its application to time-varying obstacle avoidance for electric wheelchair," *IFAC-PapersOnLine*, vol. 55, no. 41, pp. 44–51, 2022.
- [22] Y. Kim and C. Sloth, "Robust adaptive time-varying control barrier function with application to robotic surface treatment," *arXiv*, 2025.
- [23] Z. Wang, "A survey on convex optimization for guidance and control of vehicular systems," *Annu. Rev. Control*, vol. 57, p. 100957, 2024.
- [24] H. K. Khalil, *Nonlinear systems*. Prentice Hall, 2002, vol. 3.
- [25] G. C. Calafiore and L. El Ghaoui, *Optimization models*. Cambridge University Press, 2014.
- [26] K. T. Alfriend, S. R. Vadali, P. Gurfil, J. P. How, and L. S. Breger, "Nonlinear models of relative dynamics," in *Spacecraft Formation Flying: Dynamics, Control and Navigation*. Butterworth-Heinemann, 2010, ch. 4, pp. 59–82.
- [27] J. Pei, "Ground demonstration on the autonomous docking of two 3U cubesats using a novel permanent magnet docking mechanism," in *55th AIAA Aerospace Sciences Meeting*. AIAA, Jan. 2017.
- [28] Celestrak. Gp element set (tle) for ICESat-2 (NORAD 43613). [Online]. Available: <https://celestrak.org/NORAD/elements/gp.php?CATNR=43613&FORMAT=TLF>