

LETTER

General Closed-Form Transfer Function Expressions for Fast Filter Bank

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SUMMARY The existing literature focuses on the applications of fast filter bank due to its excellent frequency responses with low complexity. However, the topic is not addressed related to the general transfer function expressions of the corresponding subfilters for a specific channel. To do this, in this paper, general closed-form transfer function expressions for fast filter bank are derived. Firstly, the cascaded structure of fast filter bank is modelled by a binary tree, with which the index of the subfilter at each stage within the channel can be determined. Then the transfer functions for the two outputs of a subfilter are expressed in a unified form. Finally, the general closed-form transfer functions for the channel and its corresponding subfilters are obtained by variables replacement if the prototype lowpass filters for the stages are given. Analytical results and simulations verify the general expressions. With such closed-form expressions lend themselves easily to analysis and direct computation of the transfer functions and the frequency responses without the structure graph.

key words: fast filter bank, bit reversed order, binary tree, prototype lowpass filter, transfer function, closed-form expression

1. Introduction

Fast filter bank (FFB) was first introduced in [1]. FFB was treated as a general form of fast fourier transform (FFT) or the extension of frequency response masking (FRM) technology with much better frequency response at cost of slightly higher complexity. It was implemented with a cascaded structure and attracted more attentions due to its excellent performance.

References [2]–[5] developed the prototype filters or the structures for fast filter bank. A mixed-radix fast filter bank structure was proposed to design the filters with variable fractional delay [2]. In [3], an iterative method was proposed to design the prototype filters as an optimization problem with minimizing the number of multiplications subject to the constrained ripples in the passband and stopband. Compared to other methods, the proposed method required less number of multiplications [4], [5].

Many applications were related to the signals processing in the field of wireless applications. Fast filter bank was used for spectrum sensing in military radio receivers as a multi-resolution filter bank [6]. In [7] and [8], new low-complexity reconfigurable fast filter banks were presented for wireless communication applications such as spectrum sens-

ing and channelization by replacing the subfilter at first stage with an improved modified frequency transformation-based variable digital filter. Fast filter bank pairs were applied in the filter bank based multicarrier (FBMC) systems combining with offset quadrature amplitude modulation (OQAM) in [9].

All above references discussed various applications about fast filter bank depending on the structure graph, because there is not adequate research on calculating the general closed-form transfer functions of some specific channels and the corresponding subfilters at the stages. With the increasing the channel number, it brings difficulty in calculating the transfer functions of a specific channel and its corresponding subfilters at each stage without the structure graph. To do this, in this paper, fast filter bank is modelled by a binary tree, with which the general closed-form transfer function expressions for fast filter bank are derived under the assumption that the prototype lowpass filters are determined by one of methods mentioned in the references. With the closed-form transfer function expressions, the characteristics of each channel and its subfilters are easy to compute and analysis the responses in time or frequency domain.

The rest of this paper is organized as follows. Section 2 reviews fast filter bank briefly. Section 3 derives the general closed-form transfer functions of any channel and its corresponding subfilters at the stages for FFB. Section 4 verifies the closed-form transfer function expressions by spectrum and the path within the specific channels.

2. Review of Fast Filter Bank

FFB was regarded as a generalized sliding FFT to improve the frequency response performance with a cascaded structure [1]. It can be divided into two different types of filter banks, that is, analysis FFB (AFFB) and synthesis FFB (SFFB). The structure of AFFB is shown in Fig. 1. SFFB can be implemented with a symmetrical structure by changing the direction of the flow of data in the structure of AFFB. In general, for a L -stage AFFB, it can realize $N = 2^L$ channels. At the k -th stage, there are 2^k subfilters in this stage, where k ranges from 0 to $L - 1$. To be specific, the number of stages in Fig. 1, L , is 3 and the channels number N equals to 8. For the sake of expository simplicity and explanation, some symbols are defined and listed in Table 1.

The structures of all the subfilters for the AFFB are similar. A (k, m) subfilter can be treated as a system with one input and two outputs, where m is an integer between

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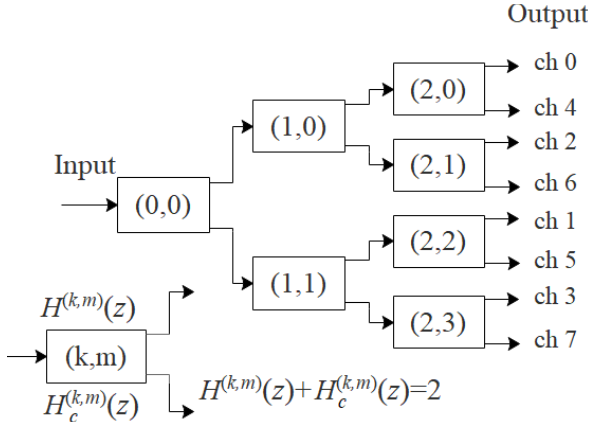


Fig. 1 Structure of analysis FFB.

Table 1 Symbols and descriptions.

Symbols	Descriptions
(k, m)	abbreviation for the m -th subfilter at the k -th stage
$H^{(k)}(z)$	transfer function of the prototype filter at the k -th stage
$h^{(k)}(n)$	impulse response coefficient of $H^{(k)}(z)$ at time n
$H^{(k,m)}(z)$	transfer function of the primary output of (k, m)
$H_c^{(k,m)}(z)$	transfer function of the complementary output of (k, m)
$H_{n_0}(z)$	transfer function of the n_0 -th channel
$H_{n_0}^{(k)}(z)$	transfer function of the n_0 -th channel at the k -th stage

0 and $2^k - 1$. The upper path is termed as the primary output with the transfer function $H^{(k,m)}(z)$, while the lower path is defined as the complementary output with the transfer function $H_c^{(k,m)}(z)$. The summation of two outputs transfer functions is a constant shown in Fig. 1.

At the k -th stage, it is further assumed that the prototype lowpass filter $H^{(k)}(z)$ is an odd length symmetrical and noncausal finite impulse response (FIR) filter with the length of $2M_k + 1$ in Eq. (1). Causality can be reinstated by delaying the noncausal impulse response by an appropriate number of samples.

$$H^{(k)}(z) = \sum_{n=-M_k}^{M_k} h^{(k)}(n)z^{-n} \quad h^{(k)}(n) = h^{(k)}(-n) \quad (1)$$

With transfer function of the prototype lowpass filter at the k -th stage, the primary output transfer function of the m -th subfilter at this stage, $H^{(k,m)}(z)$, can be obtained by replacing each z in the prototype lowpass filter $H^{(k)}(z)$ with $W_N^{\tilde{m}} z^{2^{L-k-1}}$, where $\tilde{m}$ is the bit reversed version of m in $L-1$ bits. Hence, the transfer functions of two outputs in the m -th subfilter at the k -th stage can be computed in Eq. (2), where $W_N = e^{-j2\pi/N}$.

$$\begin{aligned} H^{(k,m)}(z) &= H^{(k)}(W_N^{\tilde{m}} z^{2^{L-k-1}}) \\ H_c^{(k,m)}(z) &= 2 - H^{(k)}(W_N^{\tilde{m}} z^{2^{L-k-1}}) \end{aligned} \quad (2)$$

3. Closed-Form Transfer Functions

Considering that the structure of AFFB is symmetrical to

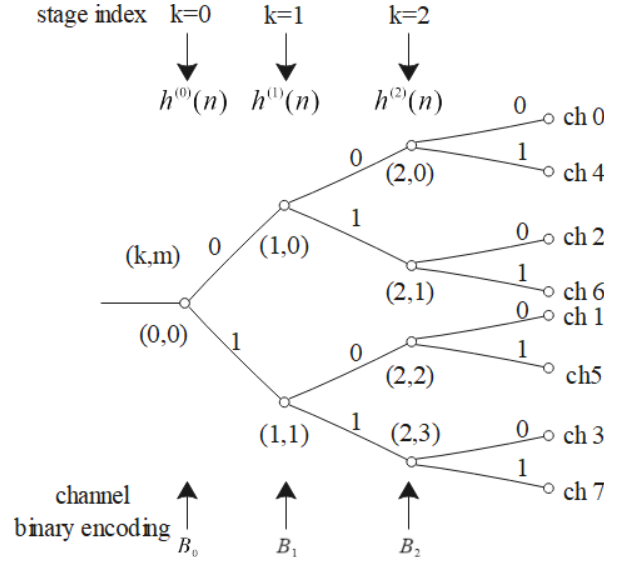


Fig. 2 Equivalent model of FFB.

that of SFFB with the same transfer functions of the corresponding subfilters, AFFB is chosen as the model to derive the closed-form transfer function expressions.

It is assumed that the prototype lowpass filters for the stages, $H^{(k)}(z)$, are given. In order to derive the general transfer functions, the following problems should be solved step by step.

- 1) For a specific channel, the position or index of the corresponding subfilter at each stage should be determined.
- 2) For the subfilter in 1) of the specific channel, unified transfer functions of two outputs should be expressed.
- 3) According to the given prototype lowpass filters for the stages, general closed-form transfer function expressions can be obtained by determining the proper outputs of the subfilter and generalizing the corresponding unified transfer functions in 2).

A binary tree is used to model the structure of AFFB in Fig. 1 and the equivalent structure is illustrated in Fig. 2. The subfilters in AFFB are modelled as the nodes in the binary tree. The two paths within a subfilter are described as the branches. For a node, the upper branch is assigned to the value '0' representing the primary transfer function $H^{(k,m)}(z)$ and the lower branch is set to '1' denoting the complementary transfer function $H_c^{(k,m)}(z)$.

The channel index, n_0 , can be expressed in L bits with binary encoding with $n_0 = [B_{L-1}B_{L-2} \dots B_0]_2$, where B_{L-1} is the most significant bit (MSB) on the left and B_0 is the least significant bit (LSB) on the right. The binary encoding is illustrated with reversed order in the binary tree graph in Fig. 2.

3.1 Index Determination of the Subfilter

For channel n_0 , it is assumed that the index of its corresponding subfilter at the k -th stage is m_k , where k ranges from 0 to $L-1$ and m_k is an integer between 0 and $2^k - 1$. In the Fig. 2,

Table 2 Channel number, binary code and indicies.

n_0	code	m_0	m_1	m_2	n_0	code	m_0	m_1	m_2
0	[000] ₂	0	0	0	4	[100] ₂	0	0	0
1	[001] ₂	0	1	2	5	[101] ₂	0	1	2
2	[010] ₂	0	0	1	6	[110] ₂	0	0	1
3	[011] ₂	0	1	3	7	[111] ₂	0	1	3

the channels number, its binary code, and the corresponding index m_k are listed in Table 2.

For channels number $N = 8$, stage number L equals to 3. Therefore, the channel n_0 is binary encoded in L bits with $n_0 = [B_2 B_1 B_0]_2$, where the range of n_0 is from 0 to 7. From Table 2, it is deduced that m_0 equal to 0 for all n_0 . The m_1 equals to the value of B_0 . The m_2 is accordance with the value of the reversed order of lower two bits of n_0 in decimal, that is, the corresponding decimal value of $[B_0 B_1]_2$.

From the above analysis, it can be concluded that m_k is the value of lower k bits of n_0 in reversed order. Let p_k be the value of lower k bits of n_0 , the expressions for m_k and p_k are formulated in Eq. (3). It is obvious that m_k is the bit reversed version of p_k in k bits.

$$\begin{aligned} p_k &= \begin{cases} 0, & \text{if } k = 0 \\ [B_{k-1} B_{k-2} \dots B_0]_2, & \text{if } k > 0 \end{cases} \\ m_k &= \begin{cases} 0, & \text{if } k = 0 \\ [B_0 B_1 \dots B_{k-1}]_2, & \text{if } k > 0 \end{cases} \end{aligned} \quad (3)$$

3.2 Unified Transfer Functions of the Subfilter Outputs

For channel n_0 , the exact output path should be asserted when the subfilter index is determined in Sect. 3.1, because there are two outputs for a subfilter. From Fig. 2, it can be seen that at the k -th stage, if the k -th bit of binary code, B_k , is 0, the primary output is in the channel n_0 with the transfer function $H^{(k,m_k)}(z)$. Otherwise, the complementary output is in the channel n_0 with the transfer function $H_c^{(k,m_k)}(z)$. Based on the relationship of two outputs, the transfer function of channel n_0 , $H_{n_0}^{(k)}(z)$ can be calculated with the unified transfer function in Eq. (4).

$$H_{n_0}^{(k)}(z) = 2 \cdot B_k + (1 - 2 \cdot B_k) \cdot H^{(k,m_k)}(z) \quad (4)$$

For example, for channel $n_0 = 3$ in Fig. 2, the binary code is expressed with $n_0 = 3 = [B_2 B_1 B_0]_2 = [011]_2$, that is, $B_2 = 0$, $B_1 = 1$ and $B_0 = 1$. According to Eq. (3), the values of m_0 , m_1 and m_2 are 0, 1 and 3. Substituting m_k into Eq. (4), the transfer function expressions of the subfilters at the stages for channel 3 can be computed in Eq. (5). The results imply that the total path of channel 3 includes the lower path of the (0,0) subfilter, the lower path of the (1,1) subfilter and the upper path of the (2,3) subfilter, which is consistent with the path in Fig. 2.

$$\begin{aligned} H_3^{(0)}(z) &= 2 - H^{(0,0)}(z) \\ H_3^{(1)}(z) &= 2 - H^{(1,1)}(z) \\ H_3^{(2)}(z) &= H^{(2,3)}(z) \end{aligned} \quad (5)$$

3.3 General Transfer Function Expressions

In Eq. (4), the transfer function of the channel n_0 at the k -th stage, $H_{n_0}^{(k)}(z)$, is expressed with the primary output transfer function $H^{(k,m_k)}(z)$. In this subsection, the general transfer function of $H^{(k,m_k)}(z)$ will be deduced with the given prototype lowpass filter $H^{(k)}(z)$.

From the literature [1], it is known that $H^{(k,m_k)}(z)$ can be obtained by replacing z in $H^{(k)}(z)$ with $W_N^{\tilde{m}_k} z^{2^{L-1-k}}$, where W_N^* represents $e^{-i2\pi \bullet / N}$ and $\tilde{m}_k$ is the bit reversed version of m_k in $L - 1$ bits. Therefore, $H^{(k,m_k)}(z)$ and $H^{(k)}(z)$ satisfy the relationship in Eq. (6).

$$H^{(k,m_k)}(z) = H^{(k)}(W_N^{\tilde{m}_k} z^{2^{L-1-k}}) \quad (6)$$

Considering that m_k is the bit reversed version of p_k in k bits in Eq. (3) and $\tilde{m}_k$ is the bit reversed version of m_k in $L - 1$ bits, $\tilde{m}_k$ can be computed by p_k with the form in Eq. (7) directly.

$$\tilde{m}_k = p_k \cdot 2^{L-1-k} \quad (7)$$

Combing Eqs. (3), (4) and (6), the general transfer function expression of the subfilter for channel n_0 at the k -th stage, $H_{n_0}^{(k)}(z)$, can be determined with the form in Eq. (8).

$$H_{n_0}^{(k)}(z) = 2B_k + (1 - 2B_k)H^{(k)}(W_N^{p_k} z^{2^{L-1-k}}) \quad (8)$$

Multiplying the transfer functions of the subfilters at all stages, the general transfer function expression for channel n_0 , H_{n_0} can form in Eq. (9), where ‘ $\prod$ ’ represents the multiplications of the elements.

$$H_{n_0}(z) = \prod_{k=0}^{L-1} H_{n_0}^{(k)}(z) \quad (9)$$

4. Simulations and Analysis

In this section, simulations are carried out to verify the general transfer function expressions. The parameters for simulations are from [1]. For the 64-channel AFFB, the nonzero coefficients for the prototype lowpass filters are taken from TABLE I in the Ref. [1]. Same as the reference, channel 8 is investigated, that is, in this simulation, the following parameters are used: $L = 6$, $N = 2^L = 64$, $n_0 = 8$.

Encoding channel n_0 in binary format in L bits, n_0 can be expressed with

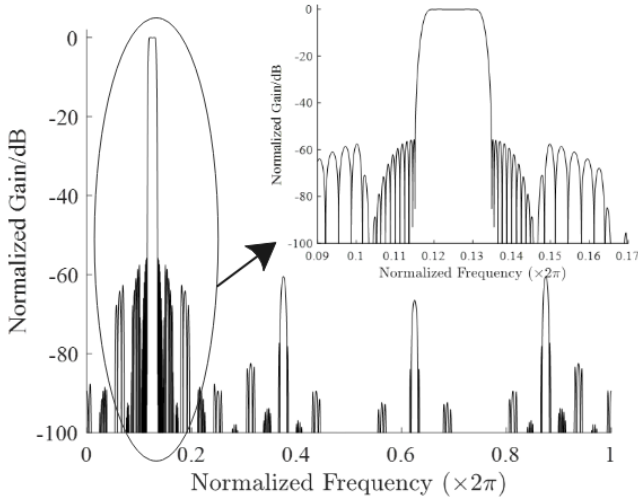
$$n_0 = 8 = [B_5 B_4 B_3 B_2 B_1 B_0]_2 = [001000]_2$$

Using Eq. (8), one can calculate the general transfer function expressions at each stage for channel n_0 shown in Table 3.

From Table 3, the transfer functions of the corresponding subfilters from the stage 0 to the stage $L - 1$ can be determined for the channel 8 of 64-channel AFFB without the cascaded structure graph in Eq. (10).

Table 3 Subfilter transfer functions for channel 8.

stage index(k)	B_k	m_k	general transfer function
0	0	0	$H_{n_0}^{(0)} = H^{(0,0)}(z) = H^{(0)}(z^{32})$
1	0	0	$H_{n_0}^{(1)} = H^{(1,0)}(z) = H^{(1)}(z^{16})$
2	0	0	$H_{n_0}^{(2)} = H^{(2,0)}(z) = H^{(2)}(z^8)$
3	1	0	$H_{n_0}^{(3)} = H_c^{(3,0)}(z) = 2 - H^{(3)}(z^4)$
4	0	1	$H_{n_0}^{(4)} = H^{(4,1)}(z) = H^{(4)}(W_N^{16} z^2)$
5	0	2	$H_{n_0}^{(5)} = H^{(5,2)}(z) = H^{(5)}(W_N^8 z)$

**Fig. 3** Frequency response of channel 8.

$$\begin{aligned}
 &H^{(0,0)}(z) \rightarrow H^{(1,0)}(z) \rightarrow H^{(2,0)}(z) \rightarrow \\
 &H_c^{(3,0)}(z) \rightarrow H^{(4,1)}(z) \rightarrow H^{(5,2)}(z)
 \end{aligned} \quad (10)$$

As a sequence, the path of channel 8 can be determined in the subfilters at each stage shown in Eq. (11), which can be verified with cascaded structure graph.

$$\begin{aligned}
 &\text{upper path of } (1, 0) \rightarrow \text{upper path of } (2, 0) \rightarrow \\
 &\text{lower path of } (3, 0) \rightarrow \text{upper path of } (4, 1) \rightarrow \\
 &\text{upper path of } (5, 2)
 \end{aligned} \quad (11)$$

Substituting the general transfer functions of the subfilters into Eq. (9), the transfer function of channel n_0 in z domain, $H_{n_0}(z)$, can be obtained. The normalized frequency response of $H_{n_0}(z)$ is depicted in Fig. 3. Referring to the frequency response of channel 8 in [1], the two frequency responses are same. Therefore, the general closed-form transfer function expressions are verified.

5. Conclusion

In this paper, a method is proposed to express the general

closed-form transfer functions for fast filter bank, which is not addressed in the existing literature. For the cascaded structure of fast filter bank, it is modelled by a binary tree and the branches are marked with binary value representing the transfer function. For a specific channel, the position or index of the subfilter at each stage can be determined with the model and then followed by unified expressions for the two outputs of the subfilter. When the prototype low-pass filters for the stages are given, the general closed-form transfer function expressions can be obtained by variable replacement in the prototype lowpass filter. Analytical results verify the general closed-form transfer functions.

Acknowledgments

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