

Is Compare-and-Swap Really Necessary?

Pankaj Khanchandani¹ and Roger Wattenhofer¹

¹ETH Zurich, Switzerland

kpankaj@ethz.ch, wattenhofer@ethz.ch

Abstract

The consensus number of a synchronization primitive, such as compare-and-swap or fetch-and-add, is the maximum number of processes n among which binary consensus can be solved by using read-write registers and registers supporting the synchronization primitive. As per Herlihy's seminal result, any synchronization primitive with consensus number n can be used to construct a *wait-free* and *linearizable* implementation of any non-trivial concurrent object or a data structure that is shared among n processes. As the compare-and-swap primitive has infinite consensus number and is widely supported by multi-processors, synchronization tasks have been typically solved using the compare-and-swap synchronization primitive.

In this paper, we show that having compare-and-swap as the quintessential synchronization primitive for solving wait-free synchronization is not necessary. It is not necessary as we give an $O(1)$ time wait-free and linearizable implementation of a compare-and-swap register for processes $1, 2, \dots, n$ using registers that support the two weak synchronization primitives half-max and max-write, each with consensus number one. Thus, any algorithm that uses powerful compare-and-swap registers to solve some arbitrary synchronization problem among processes $1, 2, \dots, n$ can be transformed into an algorithm that has the same asymptotic time complexity and only uses weak consensus number one primitives.

1 Introduction

Any modern multi-processor chip supports the compare-and-swap synchronization primitive which is widely used for coordination among multiple processes. The reason dates back to 1991, when Maurice Herlihy [6] introduced *consensus numbers* in his seminal work to quantify the power of a synchronization primitive. The consensus number of a synchronization primitive is defined as the maximum number of processes among which binary consensus can be solved by using read-write registers and registers supporting the synchronization primitive. Herlihy showed that if a synchronization primitive has consensus number n , then it can be used to design a *linearizable* and *wait-free* implementation of any concurrent object that is shared among n processes (such as concurrent queues, stacks, etc.). The wait-free property implies that a process completes its operation on the object in a bounded number of steps irrespective of the schedule of the other processes. Linearizability implies that each operation appears to take effect instantaneously at some point during the execution of the operation. As the compare-and-swap primitive has infinite consensus number, it can be used to design a linearizable and wait-free implementation of any concurrent object. So, compare-and-swap is often considered a most powerful synchronization primitive, and as such widely supported by hardware. As a result, concurrent algorithms and data structures have been traditionally designed using the compare-and-swap primitive, eg., [11, 7, 10].

Recently, Ellen et al. [2] showed that combining two simple primitives of consensus number one each can also achieve consensus number infinity. Thus, they showed that a set of simple consensus number one primitives are sufficient to design a linearizable and wait-free implementation of any shared object, just like compare-and-swap is. However, this is only a sufficiency argument and utilizes Herlihy’s universal wait-free construction of any shared object, which is not very efficient and can take up to $O(n)$ time per operation in a system with n processes. This brings us to the following natural question: Is there a set of simple (low consensus number) primitives that is not only sufficient like compare-and-swap but also equally efficient?

One way to approach this question is to solve a specific synchronization problem with some low consensus number primitives and compare the solution with a solution that uses compare-and-swap. For example, Gelashvili et al. [4] implement a log data structure using primitives of consensus number at most two, with the same performance as the log data structure implemented using compare-and-swap. However, this approach answers the efficiency question only partially, i.e., with respect to a specific synchronization problem. Instead, we answer the question in affirmative for *any* synchronization task by giving an $O(1)$ time linearizable and wait-free implementation of a compare-and-swap register for processes $1, 2, \dots, n$ using two consensus number one primitives that we call half-max and max-write. Thus, we show that it is possible to transform any algorithm among processes $1, 2, \dots, n$ that uses compare-and-swap registers and takes $O(T)$ time into an $O(T)$ time algorithm that only uses consensus number one primitives!

2 Related Work

Herlihy’s result [6] tells us that it is not possible to use low consensus number synchronization primitives to achieve higher consensus number. However, it relies on the assumption that the individual synchronization primitives are not applied on the same register. Ellen et al. [2] question this assumption by obtaining a higher consensus number using a collection of low consensus number primitives that are applied on the same register. One of their examples solves binary consensus among any given number of processes using the consensus number one primitives *multiply* and *decrement*.

This was followed by couple of works that highlight the use of combining low consensus number primitives for specific synchronization tasks. Gelashvili et al. [4] give an efficient implementation of a log data structure by just using low consensus number primitives. In [9], we use some low consensus number primitives along with the compare-and-swap primitive to improve the time bound of a wait-free queue from $O(n)$ to $O(\sqrt{n})$ where n is the number of processes. Earlier, Golab et al. [5] give a blocking implementation of comparison primitives by just using read-write registers and constant number of remote memory references. As the implementation is blocking, it cannot be used for wait-free synchronization tasks. So, there is no prior work that shows that a collection of low consensus number primitives is both sufficient and efficient like compare-and-swap registers for an arbitrary synchronization task.

3 An Overview of the Method

Our method is based on the observation that if several compare-and-swap operations attempt to simultaneously change the value in the register, only one of them succeeds. So, instead of updating the final value of the register for each operation, we first determine the single operation that succeeds and update the final value accordingly. This is achieved by using two consensus number one primitives: *max-write* and *half-max*.

The max-write primitive takes two arguments. If the first argument is greater than or equal to the value in the first half of the register, then the first half of the register is replaced with the first argument and the second half is replaced with the second argument. Otherwise, the register is left unchanged. This primitive helps in keeping a version number along with a value.

The half-max primitive takes a single argument and replaces the first half of the register with that argument if the argument is larger. Otherwise, the register remains unchanged. This primitive is used along with the max-write primitive to determine the single successful compare-and-swap operation out of several concurrent ones. The task of determining the successful compare-and-swap operation can be viewed as a variation of tree-based combining (as in [3, 8] for example). The difference is that we do not use a tree as it would incur $\Theta(\log n)$ time overhead. Instead, our method does the combining in constant time as we will see later.

In the following section, we formalize the model and the problem. In [Section 5](#), we give an implementation of the compare-and-swap operation using registers that support the half-max, max-write, read and write operations. In [Section 6](#), we prove its correctness and show that the compare-and-swap operation runs in $O(1)$ time. In [Section 7](#), we argue that the consensus numbers of the max-write and half-max primitives are both one. Finally, we conclude and discuss some extensions in [Section 8](#).

4 Model

A *sequential object* is defined by the tuple (S, O, R, T) . Here, S is the set of all possible *states* of the object, O is the set of *operations* that can be performed on the object, R is the set of possible *return values* of all the operations and $T : S \times O \rightarrow S \times R$ is the *transition function* that specifies the next state of the object and the return value given a state of the object and an operation applied on it.

A *register* is a sequential object and supports the operations *read*, *write*, *half-max* and *max-write*. The *read()* operation returns the current value (state) of the register. The *write(v)* operation updates the value of the register to v . The *half-max(x)* operation replaces the value in the first half of the register, say a , with $\max\{x, a\}$. The *max-write($x \mid y$)* operation replaces the first half of the register, say a , with x and second half of the register with y if and only if $x \geq a$. The register operations are *atomic*, i.e., if different processes execute them simultaneously, then they execute sequentially in some order. In general, atomicity is implied whenever we use the word operation in the rest of the text.

An *implementation* of a sequential object is a collection of *functions*, one for each operation defined by the object. A function specifies a sequence of *instructions* to be executed when the function is executed. An instruction is an operation on a register or a computation on local variables, i.e., variables exclusive to a process.

A *process* is a sequence of functions to be executed. Thus, a process defines a sequence of instructions to be executed as determined by the individual functions in its sequence. The processes have identifiers $1, 2, \dots, n$. When a process executes a function, it is said to *call* that function. A *schedule* is a sequence of process identifiers. Given a schedule S , an *execution* $E(S)$ is the sequence of instructions obtained by replacing each process identifier in the schedule with the next instruction to be executed by the corresponding process.

Given an execution and a function called by a process, the *start* of the function call is the point in the execution when the first register operation of the function call appears. Similarly, the *end* of the function call is the point in the execution when the last register operation of the function call appears. A function call A is said to occur *before* another function call B , if the call A ends before call B starts. Thus, the function calls of an implementation of an object O form a partial order $P_O(E)$ with respect to an execution E . An implementation on an object O is *linearizable* if there is a total order $T_O(E)$ that extends the partial order $P_O(E)$ for any given execution E so that the actual return value of

every functional call in the order $T_O(E)$ is same as the return value determined by applying the specification of the object to the order $T_O(E)$. The total order $T_O(E)$ is usually defined by associating a *linearization point* with each function call, which is a specific point in the execution when the call takes effect. An implementation is *wait-free* if every function call returns within a bounded number of steps of the calling process irrespective of the schedule of the other processes.

Our goal is to develop a wait-free and linearizable implementation of the *compare-and-swap* register. It supports *read* and *compare-and-swap* operations. The *read()* operation returns the current value of the register. The *compare-and-swap*(a, b) operation returns true and updates the value of the register to b if the value in the register is a . Otherwise, it returns false and does not change the value.

5 Algorithm

Figure 1 shows the (shared) registers that are used by the algorithm. There are arrays A and R of size n each. The i^{th} entry of the array A consists of two *fields*: the field c keeps a count of the number of compare-and-swap operations executed by the process i , the field val is used to store or *announce* the second argument of the compare-and-swap operation that the process i is executing. The i^{th} entry of the array R consists of the fields c and ret . The field ret is used for storing the *return* value of the c^{th} compare-and-swap operation executed by the process i . The register V stores the current *value* of the compare-and-swap object in the field val along with its version number in the field seq . The fields seq , pid and c of the register P respectively store the next version number, the *process identifier* of the process that executed the latest successful compare-and-swap operation and the count of compare-and-swap operations issued by that process. For all the registers, the individual fields are of equal sizes except for the register P . The first half of this register stores the field seq where as the second half stores the other two fields, pid and c .

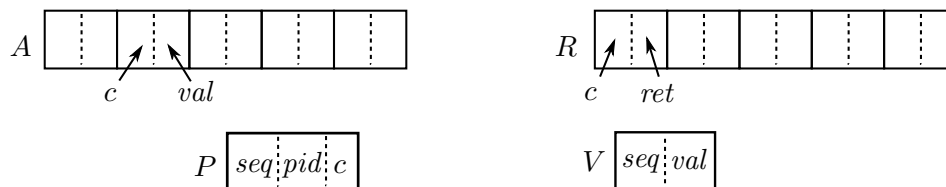


Figure 1: An overview of data structures used by Algorithm 1.

Algorithm 1 gives an implementation of the compare-and-swap register. To execute the read function, a process simply reads and returns the current value of the object as stored in the register V (Lines 20 and 21). To execute the compare-and-swap function, a process starts by reading the current value of the object (Line 2). If the first argument of the function is not equal to the current value, then it returns false (Lines 3 and 4). If both

the arguments are same as the current value, then it can simply return true as the new value is same as the initial one (Lines 5 and 6).

Otherwise, the process competes with the other processes executing the compare-and-swap function concurrently. First, the process increments its local counter (Line 7). Then, the new value to be written by the process is announced in the respective entry of the array A (Line 8) and the return value of the function is initialized to false by writing to the respective entry in the array R (Line 9). The process starts competing with the other concurrent processes by writing its identifier to the register P (Line 10). The competition is finished by writing a version number larger than used by the competing processes (Line 11).

Algorithm 1: The compare-and-swap and the read functions. The symbol $|$ is a field separator. The symbol $_$ is a variable that is not used. The variable id is the identifier of the process executing the function. At initialization, we have $c = 0$ and $V = (0 | x)$, where x is the initial value of the compare-and-swap object.

<pre> 1 compare-and-swap(a, b) 2 ($seq val$) $\leftarrow V.read()$; 3 if $a \neq val$ then 4 return false; 5 if $a = b$ then 6 return true; 7 $c \leftarrow c + 1$; 8 $A[id].write(c b)$; 9 $R[id].write(c false)$; 10 $P.max-write(seq + 1 id c)$; 11 $P.half-max(seq + 2)$; 12 ($seq pid cp$) $\leftarrow P.read()$; 13 ($ca val$) $\leftarrow A[pid].read()$; 14 if seq is even and $cp = ca$ then 15 $R[pid].max-write(ca true)$; 16 $V.max-write(seq val)$; 17 ($_ ret$) $\leftarrow R[id].read()$; 18 return ret; </pre>	<pre> 19 read() 20 ($_ val$) $\leftarrow V.read()$; 21 return val; </pre>
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Once the winner of the competing processes is determined, the winner and the value announced by it is read (Lines 12 and 13), the winner is informed that it won after appropriate checks (Lines 15, 14) and the current value is updated (Line 16). The value to be returned is then read from the designated entry of array R (Line 17).

In the following section, we analyze Algorithm 1 and show that it is a linearizable and $O(1)$ time wait-free implementation of the compare-and-swap object.

6 Analysis

Let us first define some notations. We refer to a field f of a register X by $X.f$. The term $X.f_k^i$ is the value of the field $X.f$ just after process i executes Line k during a call. We omit the call identifier from the notation as it will always be clear from the context. Similarly, v_k^i is the value of a variable v , that is local to the process i , just after it executes Line k during a call. The term $X.f_e$ is the value of a field $X.f$ at the end of an execution.

To prove that our implementation is linearizable, we first need to define the linearization points. The linearization point of the compare-and-swap function executed by a process i is given by [Definition 1](#). There are four main cases. If the process returns from [Line 4](#) or [Line 6](#), then the linearization point is the read operation in [Line 2](#) as such an operation does not change the value of the object (Cases [1](#) and [2](#)). Otherwise, we look for the execution of [Line 16](#) that wrote the sequence number $V.seq_2^i + 2$ to the field $V.seq$ for the first time. This is the linearization point of the process i if its compare-and-swap operation was successful as determined by the value of $P.pid$ ([Case 3a](#)). Otherwise, the failed compare-and-swap operations are linearized just after the successful one ([Case 3b](#)). The calls that have not yet taken effect are linearized at the end ([Case 4](#)).

Definition 1. *The compare-and-swap call by a process i is linearized as follows.*

1. If $V.val_2^i \neq a_1^i$, then the linearization point is the point when i executes [Line 2](#).
2. If $V.val_2^i = a_1^i = b_1^i$, then the linearization point is the point when i executes [Line 2](#).
3. If $V.val_2^i = a_1^i \neq b_1^i$ and $V.seq_e \geq V.seq_2^i + 2$, then let p be the point when [Line 16](#) is executed by a process j so that $V.seq_{16}^j = V.seq_2^i + 2$ for the first time.
 - (a) If $pid_{12}^j = i$, then the linearization point is p .
 - (b) If $pid_{12}^j \neq i$, then the linearization point is just after p .
4. If $V.val_2^i = a_1^i \neq b_1^i$ and $V.seq_e < V.seq_2^i + 2$, then the linearization point is at the end, after all the other linearization points.

Note that we assume in [Case 3](#) that if $V.seq_e \geq V.seq_2^i + 2$, then there is an execution of [Line 16](#) by a process j with the value $V.seq_{16}^j = V.seq_2^i + 2$. So, we first show in the following lemmas that this is indeed true.

Lemma 1. *The value of $V.seq$ is always even.*

Proof. We have $V.seq = 0$ at initialization. The modification only happens in [Line 16](#) with an even value. $\square$

Lemma 2. *Whenever $V.seq$ changes, it increases by 2.*

Proof. Say that the value of the field was changed to $V.seq_{16}^i$ when a process i executed [Line 16](#). Then, the value seq_{14}^i is even and so is the value $V.seq_{16}^i$. Thus, the value $V.seq_{16}^i$ was written to $P.seq$ by a process j and that $V.seq_2^j = V.seq_{16}^i - 2$. As the field $V.seq$ is only modified by a max-write operation so it only increases. Thus, we have $V.seq \geq V.seq_{16}^i - 2$ just before i modifies it. As $V.seq$ is even by [Lemma 1](#) and i modifies it, we have $V.seq = V.seq_{16}^i - 2$ before the modification. So, the value increases by 2. $\square$

Lemma 3. *The linearization point as given by [Definition 1](#) is well-defined.*

Proof. The linearization point as given by [Definition 1](#) clearly exists for all the cases except for [Case 3](#). Here, we only need to show that if $V.seq_e \geq V.seq_2^i + 2$, then there exists an execution of [Line 16](#) by a process j so that $V.seq_{16}^j = V.seq_2^i + 2$. As $V.seq_2^i$ is even by [Lemma 1](#) and the value of $V.seq$ only increases in steps of 2 by [Lemma 2](#), it follows from $V.seq_e \geq V.seq_2^i + 2$ that $V.seq_2^i + 2$ was written to $V.seq$ at some point. $\square$

To show that the implementation is linearizable, we need to prove two main statements. First, the linearization point is within the start and end of the corresponding function call. Second, the value returned by a finished call is same as defined by the sequence of linearization points up to the linearization point of the call. In the following two lemmas, we show the first of these statements.

Lemma 4. *If it holds that $V.val_2^i = a_1^i \neq b_1^i$ for a compare-and-swap call by a process i , then the value of $V.seq$ is at least $V.seq_2^i + 2$ at the end of the call.*

Proof. We define a set of processes $S = \{j : V.seq_2^j = V.seq_2^i\}$. Consider the process $k \in S$ that is the first one to execute [Line 14](#). As the first field of $P.seq$ is always modified by a max operation and process k writes $V.seq_2^i + 2$ to that field, we have $seq_{14}^k = seq_{12}^k \geq V.seq_2^i + 2$. If $seq_{12}^k > V.seq_2^i + 2$, then $V.seq_{12}^k \geq V.seq_2^i + 2$ and we are done.

So, we only need to check the case when $seq_{12}^k = V.seq_2^i + 2$. As $V.seq_2^i$ is even by [Lemma 1](#), so is $seq_{14}^k = seq_{12}^k$. Moreover, the process $pid_{12}^k \in S$ as some process(es) (including k) executed [Line 10](#). As $A[pid_{12}^k].c$ always increases whenever modified ([Line 8](#)), we have $ca_{13}^k \geq cp_{12}^k$. But, if $ca_{13}^k > cp_{12}^k$, then the process pid_{12}^k finished even before the process k , a contradiction. So, it holds that $ca_{13}^k = cp_{12}^k$ and the process k executes [Line 16](#).

Now, the execution of [Line 16](#) by the process k either changes the value of $V.seq$ or does not. If it does, then $V.seq_{16}^k = V.seq_2^i + 2$ and we are done. Otherwise, someone already changed the value of $V.seq$ to at least $V.seq_2^i + 2$ because of [Lemma 2](#). $\square$

Lemma 5. *The linearization point as given by [Definition 1](#) is within the corresponding call duration.*

Proof. The statement is true for [Cases 1](#) and [2](#) as the instruction corresponding to the linearization point is executed by the process i itself.

For [Case 3](#), we analyze the case of finished and unfinished call separately. Say that the call is unfinished. As $V.seq_e \geq V.seq_2^i + 2$ and $V.seq_2^i$ is the value of $V.seq$ at the start

of the call, the linearization point as given by [Definition 1](#) is after the call starts. Now, assume that the call is finished. We know from [Lemma 4](#) that the value of $V.seq$ is at least $V.seq_2^i + 2$ when the call ends. So, the point when [Line 16](#) writes $V.seq_2^i + 2$ to $V.seq$ is within the call duration.

We know from [Lemma 4](#) that if the call finishes, then we have $V.seq_e \geq V.seq_2^i + 2$. So, if $V.seq_e < V.seq_2^i + 2$, then the call is unfinished and it is fine to linearize it at the end as done for [Case 4](#). $\square$

Now, we need to show that the value returned by the calls is same as the value determined by the order of linearization points. We show this in the following lemmas.

Lemma 6. *Assume that $x = seq_{12}^i = seq_{12}^j$ for two distinct processes i and j and that x is even. Then, it holds that $pid_{12}^i = pid_{12}^j$ and $cp_{12}^i = cp_{12}^j$.*

Proof. W.l.o.g. assume that the process i executes [Line 12](#) before the process j does so. As $x = seq_{12}^i = seq_{12}^j$ by assumption, the only way in which the field $P.pid$ can change until the process j executes [Line 12](#), is by a max-write operation on P with the value x as the first field. This is not possible as x is even and the max-write on P is only executed with odd value as the first field ([Line 10](#)). So, it holds that $pid_{12}^i = pid_{12}^j$. Similarly, we have $cp_{12}^i = cp_{12}^j$. $\square$

Lemma 7. *As long as the value of $V.seq$ remains same, the value of $V.val$ does not change.*

Proof. Say that a process i is the first one to write a value x to $V.seq$. The value written to the field $V.val$ by the process i is val_{13}^i . To have a different value of $V.val$ with x as the value of $V.seq$, another process j must execute [Line 16](#) with $seq_{12}^j = x$ but $val_{13}^j \neq val_{13}^i$. As $seq_{12}^j = x = seq_{12}^i$, it follows from [Lemma 6](#) that $pid_{12}^j = pid_{12}^i$ and $cp_{12}^j = cp_{12}^i$. As the condition in [Line 14](#) is true for both the processes i and j , it then follows that $ca_{13}^i = ca_{13}^j$. As the field $A[pid_{12}^j].val$ is updated only once for a given value of $A[pid_{12}^j].c$ ([Line 8](#)), it holds that $val_{13}^i = val_{13}^j$ and the claim follows. $\square$

Lemma 8. *Say that $seq_{12}^i = x$ is even and $pid_{12}^i = j$ during a call by a process i , then it holds for a call by the process j that $V.seq_2^j = x - 2$.*

Proof. As $seq_{12}^i = x$, some process h modified P by executing [Line 10](#) or [Line 11](#) with x as the first argument. As x is even and $V.seq_2^h$ is even by [Lemma 1](#), the process h modified p by executing [Line 11](#). So, it holds that $V.seq_2^h = x - 2$. Also, process h executed [Line 10](#) with $x - 1$ as the first field. As $pid_{12}^i = j$, the process j also executed [Line 10](#) with $x - 1$ as the first field after the process h did so. So, it holds that $V.seq_2^j = x - 2$. $\square$

Lemma 9. *For every even value $x \in [2, V.seq_e]$, there is an execution of [Line 16](#) by a process i so that $seq_{12}^i = x$ and the first such execution is the linearization point of some call.*

Proof. Consider an even value $x \in [2, V.seq_e]$. Then, we know from [Lemma 2](#) that x is written to $V.seq$ by an execution of [Line 16](#). Let p be the point of first execution of [Line 16](#) by a process j so that $seq_{12}^j = x$. So, it holds for the process $pid_{12}^j = h$ that $V.seq_2^h = x - 2$ using [Lemma 8](#). As point p is the first time when x is written to the field $V.seq$, it holds that $V.seq_{16}^j = x$. Thus, p is the linearization point of the process h by [Definition 1](#). $\square$

Lemma 10. *The value $V.val$ is only modified at a [Case 3a](#) linearization point.*

Proof. Let q be a [Case 3a](#) linearization point. Say that the value of $V.seq$ is updated to x at q . Let p be the first point in the execution when the value of $V.seq$ is $x - 2$. Using [Lemma 9](#), we conclude that p is either a linearization point (for $x - 2 \geq 2$) or the initialization point (for $x - 2 = 0$). Using [Lemma 7](#), the value of $V.val$ is not modified between p and q . $\square$

We want to use the above lemma in an induction argument on the linearization points to show that the values returned by the corresponding calls are correct. First, we introduce some notations for $k \geq 1$. The term $L.val_k$ is the value of the abstract compare-and-swap object after the k^{th} linearization point. The terms $V.seq_k$ and $V.val_k$, respectively, are the values of $V.seq$ and $V.val$ after the k^{th} linearization point. All these notations refer to the respective values just after initialization for $k = 0$. For $k \geq 1$, the term $L.ret_k$ is the expected return value of the call corresponding to the k^{th} linearization point. Due to space constraints, we only give the statement of the following two lemmas without proof. The proof is an induction on the linearization points and checks the different linearization point cases separately.

Lemma 11. *After $k \geq 0$ linearization points, it holds that $L.val_k = V.val_k$ except for [Case 4](#) linearization points. For $k \geq 1$, the $L.ret_k$ values are false for [Case 1](#), true for [Case 2](#), true for [Case 3a](#) and false for [Case 3b](#).*

Lemma 12. *If the k^{th} linearization point for $k \geq 1$ corresponds to a finished call by a process i , then the value returned by the call is $L.ret_k$.*

We can now state the following main theorem about [Algorithm 1](#).

Theorem 13. *[Algorithm 1](#) is a wait-free and linearizable implementation of the compare-and-swap register where both the compare-and-swap and read functions take $O(1)$ time.*

Proof. We conclude that the compare-and-swap function as given by [Algorithm 1](#) is linearizable by using [Lemma 5](#) and [Lemma 12](#). The read operation is linearized at the point of execution of [Line 20](#). Clearly, this is within the duration of the call. To check the return value, let LP_k be the linearization point of the read operation and $LP_{k'}$ be the linearization point previous to LP_k . Then, we have $V.val_k = V.val_{k'}$ using [Lemma 10](#). So, it holds that $V.val_k = L.val_{k'}$ using [Lemma 11](#). Moreover, both the compare-and-swap and read functions end after executing $O(1)$ steps and the implementation is wait-free. $\square$

7 Consensus Numbers

In this section, we prove that each of the max-write and the half-max primitives has consensus number one. Trivially, both operations can solve binary consensus for a single process (itself) by just deciding on the input value. To show that these operations cannot solve consensus for more than one process, we use the standard valency and indistinguishability argument. We give a short proof for the reader familiar with such an argument. The detailed proof can be found in the appendix.

We know that the initial configuration is a bivalent configuration (both 0 and 1 outputs are possible) whereas the terminating configuration is univalent (either 0 or 1 only is a possible output). Thus, a wait-free algorithm to solve binary consensus reaches a critical configuration C where an operation by any process changes the configuration to a univalent one. Say that next operation s_a by process A changes the configuration to a 0-valent configuration and the next operation s_b by process B changes the configuration to a 1-valent configuration. The operations s_a and s_b cannot be operations on different registers as the configurations $Cs_a s_b$ is indistinguishable from the configuration $Cs_b s_a$. It is also not possible that one of s_a or s_b , say s_a , is a read operation as the configurations $Cs_a s_b$ and Cs_b are indistinguishable to process B . Both s_a and s_b cannot be a write operation as the configurations Cs_a and $Cs_b s_a$ are indistinguishable to A . If both the operations are max-write, then let s_a be the operation with the greater (or equal) first argument. Then, the configurations Cs_a and $Cs_b s_a$ are indistinguishable to A as s_a overwrites s_b .

8 Conclusion

The algorithm that we presented uses $O(n)$ space to implement a single compare-and-swap register due to the arrays A and R . If an algorithm uses m compare-and-swap registers, then we can run m separate instances of the presented algorithm. In that case, we use $O(mn)$ space. But, we can save space if we observe that the algorithm uses the arrays A and R to store the information about the latest pending call per process. As there is at most one pending call per process even while implementing m compare-and-swap registers, we can conceptually run m instances of the presented algorithm by using a single array A and a single array R . In that case, the space required is $O(m + n)$.

The half-max and max-write primitives have consensus number one each just like read-write registers. Moreover, these primitives do not return any value and are based on the max operation which is commutative. So, they can potentially yield a simpler and scalable hardware implementation unlike compare-and-swap, which is known to not scale well [1]. Using our result, one can transform an algorithm that uses compare-and-swap registers into an algorithm that uses the half-max and max-write primitives. The transformation takes constant time per operation and works for any algorithm among processes $1, 2, \dots, n$ in general, even if the algorithm not wait-free. Therefore, our result gives us an opportunity, to rethink from scratch, the primitives to implement in a modern multi-processor.

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A Proof of Lemma 11

Proof. We prove the claim by induction on k . For the base case of $k = 0$, the claim is true as $V.val$ is initialized with the initial of the compare-and-swap object. Let LP_k be the k^{th} linearization point for $k \geq 1$ and say that it corresponds to a call by a process i . We have the following cases.

Case 1: Let $LP_{k'}$ be the linearization point previous to LP_k . By induction hypothesis, it holds that $L.val_{k'} = V.val_{k'}$. By Lemma 10, the value of $V.val$ does not change until LP_k . As we have a read operation at LP_k , it holds that $V.val_{k'} = V.val_k$. By Definition 1, we know that $V.val_k \neq a_1^i$. So, it holds that $L.val_{k'} = V.val_{k'} = V.val_k \neq a_1^i$. Thus, it follows from the specification of the compare-and-swap object that $L.val_k = L.val_{k'} = V.val_k$. Moreover, we have $L.ret_k = false$ as $L.val_{k'} = V.val_k \neq a_1^i$.

Case 2: Again, we let $LP_{k'}$ to be the linearization point previous to LP_k . As argued in the previous case, it holds that $V.val_{k'} = V.val_k$. By Definition 1, we know that $V.val_k = a_1^i = b_1^i$. So, it holds that $L.val_{k'} = V.val_{k'} = V.val_k = a_1^i$. Thus, it follows from the object's specification that $L.val_k = b_1^i = V.val_k$. Further, we have $L.ret_k = true$ as $L.val_{k'} = a_1^i$.

Case 3a: Consider the point $LP_{k'}$ when the value $V.seq_k - 2$ was written to $V.seq$ for the first time. As $V.seq_k$ is even by Lemma 1, it follows from Lemma 9 that $LP_{k'}$ is a linearization point or the initialization point. Using definition of Case 3a, LP_k is the first point when the value $V.seq_k$ was written to the field $V.seq$. So, we have $V.seq_2^i = V.seq_{k'}^i$. Thus, it holds that $V.val_2^i = V.val_{k'}^i$ by Lemma 7. Therefore, $V.val_2^i = L.val_{k'}^i$ as $L.val_{k'}^i = V.val_{k'}^i$ by induction hypothesis. Using definition of Case 3a, it also holds that $a_1^i = V.val_2^i$. Thus, we have $a_1^i = L.val_{k'}^i$ and $L.val_k = b_1^i$.

Now, assume that the instruction at LP_k was executed by a process j . Using definition of Case 3a, we have $i = pid_{12}^j$. As LP_k is the first time when the value of $V.seq$ is $V.seq_k = V.seq_2^i + 2$, we conclude that the process i is not finished until LP_k by using Lemma 4. As $seq_{12}^j = V.seq_k = V.seq_2^i + 2$, it is true that some process i' has $V.seq_{12}^{i'} = V.seq_2^i$ and that the process executed Line 10 until LP_k . As $i = pid_{12}^j$, the process $i' = i$. Moreover, the process i did this during the call corresponding to the linearization point LP_k as it follows from Lemma 4 that there is a unique call for any process h given a fixed value of $V.seq_2^h$. Thus, the process i already executed Line 8 with b_1^i as the value of the second field. This field has not changed as the call by process i is not finished until LP_k . So, we have $val_{13}^j = b_1^i$ and that $V.val_k = b_1^i$ as well. Because $a_1^i = L.val_{k'}$ as shown before, we also have $L.ret_k = true$.

Case 3b: Let $LP_{k'}$ and $LP_{k''}$ be the first points when the value $V.seq_k$ and $V.seq_k - 2$ is written to $V.seq$ respectively ($LP_{k'}$ is just before the point LP_k as defined by Case 3b). Let i and j be the processes that execute the calls corresponding to the points LP_k and $LP_{k'}$ respectively. By definition of Case 3b, we have $V.seq_2^i = V.seq_{k'} - 2$. As process j wrote $V.seq_{k'}$ to $V.seq$, we have $V.seq_2^j = V.seq_{k'} - 2$ as well. So, we have $V.val_2^i = V.val_2^j$ using Lemma 7. Using definition of Case 3a and Case 3b, respectively, we have $a_1^j = V.val_2^j \neq b_1^j$ and $a_1^i = V.val_2^i$. So, we have $a_1^i \neq b_1^j$. We have $b_1^j = L.val_{k'}$ as argued in the previous

case, so it holds that $L.val_k = L.val_{k'}$. By induction hypothesis, we have $L.val_{k'} = V.val_{k'}$. Moreover, there no operations after $LP_{k'}$ and until LP_k by definition of [Case 3b](#). So, we have $V.val_{k'} = V.val_k$ and thus $L.val_k = V.val_k$. Also, we have $L.ret_k = false$ as $a_1^i \neq b_1^j = L.val_{k'}$. $\square$

B Proof of [Lemma 12](#)

Proof. Say the k^{th} linearization point is a [Case 1](#) point. Using its definition, the value returned by the corresponding call is *false* as the condition in [Line 3](#) holds true. Using [Lemma 11](#), we have $L.ret_k = false$ as well for [Case 1](#). Next, assume that the k^{th} linearization point is a [Case 2](#) point. Then, the value returned by the corresponding call is *true* as the condition in [Line 5](#) is true by definition. Using [Lemma 11](#), we have $L.ret_k = true$ as well for [Case 2](#).

Now, consider that the k^{th} linearization point is a [Case 3a](#) point. Say that the process j executes the operation at the linearization point. As $pid_{12}^j = i$ by definition of [Case 3a](#), the process i already executed [Line 10](#) with the first field as $V.seq_k - 1$. So, the process i also initialized $R[i]$ to $(cp_{12}^j | false)$ in [Line 9](#). Moreover, the process j wrote the value $(cp_{12}^j | true)$ to $R[i]$ afterwards using a max-write operation. Thus, the value of $R[i].ret$ after LP_k is *true*. This field is not changed by i until it returns. And, other processes only write *true* to the field. So, the call returns *true* which is same as the value of $L.ret_k$ given by [Lemma 11](#).

Next, consider that the k^{th} linearization point is a [Case 3b](#) point. Let p be the point when the process i initializes $R[i]$ to a value $(x | false)$ during the call ([Line 9](#)). Consider a process j that tries to write *true* to $R[i].ret$ after p (by executing [Line 15](#)). So, it holds that $pid_{12}^j = i$ and that seq_{12}^j is even. Now, we consider three cases depending on the relation between seq_{12}^j and $V.seq_k$. First, consider that $seq_{12}^j > V.seq_k$. As $pid_{12}^j = i$ and seq_{12}^j is even, we have $V.seq_2^i = seq_{12}^j - 2$ using [Lemma 8](#). So, we have $V.seq_2^i > V.seq_k - 2$. This cannot happen until i finishes as $V.seq_2^i = V.seq_k - 2$ for the current call by i using definition of [Case 3b](#). Second, consider that $seq_{12}^j = V.seq_k$. Using definition of [Case 3b](#), there is a process h so that $pid_{12}^h \neq i$ and $seq_{12}^h = V.seq_k$. As $seq_{12}^j = V.seq_k$ by assumption, we have $pid_{12}^j \neq i$ using [Lemma 6](#). This contradicts our assumption that $pid_{12}^j = i$. Third, consider that $seq_{12}^j < V.seq_k$. As $pid_{12}^j = i$ and seq_{12}^j is even, we have $V.seq_2^i = seq_{12}^j - 2$ using [Lemma 8](#). So, we have $V.seq_2^i < V.seq_k - 2$. This corresponds to a previous call by the process i as $V.seq_2^i = V.seq_k - 2$ for the current call by i . So, it holds that $ca_{13}^j < x$ and execution of [Line 15](#) has no effect. Thus, the process i returns *false* for [Case 3b](#) which matches the $L.ret_k$ value given by [Lemma 11](#).

If the k^{th} linearization point is a [Case 4](#) point, then we know from [Lemma 4](#) that the call is unfinished and we need not consider it. $\square$

C Proof of Consensus Numbers

First, we define some terms. A *configuration* of the system is the value of the local variables of each process and the value of the shared registers. The *initial* configuration is the input 0 or 1 for each process and the initial values of the shared registers. A configuration is called a *bivalent* configuration if there are two possible executions starting from the configuration so that in one of them all the processes terminate and decide 0 and in the other all the processes terminate and decide 1. A configuration is called *0-valent* if in all the possible executions starting from the configuration, the processes terminate and decide 0. Similarly, a configuration is called *1-valent* if in all the possible executions starting from the configuration, the processes terminate and decide 1. A configuration is called a *univalent* configuration if it is either 0-valent or 1-valent. A bivalent configuration is called *critical* if the next step by any process changes it to a univalent configuration. Consider an initial configuration in which there is a process X with the input 0 and a process Y with the input 1. This configuration is bivalent as X outputs 0 if it is made to run until it terminates and Y outputs 1 if it is made to run until it terminates. As the terminating configuration is univalent, a critical configuration is reached assuming that the processes solve wait-free binary consensus.

Assume that the max-write operation can solve consensus between two processes A and B . Then, a critical configuration C is reached. W.l.o.g., say that the next step s_a by the process A leads to a 0-valent configuration C_0 and that the next step s_b by the process B leads to a 1-valent configuration C_1 . In a simple notation, $C_0 = Cs_a$ and $C_1 = Cs_b$. We have the following cases.

1. s_a and s_b are operations on different registers: The configuration C_0s_b is indistinguishable from the configuration C_1s_a . Thus, the process B decides the same value if it runs until termination from the configurations C_0s_b and C_1s_a , a contradiction.
2. s_a and s_b are operations on the same register and at least one of them is a read operation: W.l.o.g., assume that s_a is a read operation. Then, the configuration C_0s_b is indistinguishable to C_1 with respect to B as the read operation by A only changes its local state. Thus, the process B decides the same value if it runs until termination from the configurations C_0s_b and C_1 , a contradiction.
3. s_a and s_b are write operations on the same register: Then, the configuration C_0s_b is indistinguishable from the configuration C_1 as s_b overwrites the value written by s_a . Thus, the process B will decide the same value from these configurations, a contradiction.
4. s_a and s_b are max-write operations on the same register: Say that the arguments of these operations are $a|x$ and $b|y$ for A and B respectively. W.l.o.g., assume that $b \geq a$. Then, the configurations C_0s_b is indistinguishable from C_1 as the operation

s_b will overwrite both the fields of the register. Thus, the process B will decide the same value from these configurations, a contradiction.

So, the critical configuration cannot be reached and the processes A and B cannot solve consensus using the max-write primitive. For the half-max primitive, observe that it also cannot solve consensus between the processes A and B as otherwise max-write primitive would as well. Thus, the consensus number is one for the max-write as well as the half-max primitive.