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ALMOST EVERYWHERE MATRIX RECOVERY

YI RONG, YANG WANG, AND ZHIQIANG XU

Abstract. Matrix recovery is raised in many areas. In this paper, we build up a framework for almost everywhere matrix recovery which means to recover almost all the $P \in \mathcal{M} \subset \mathbb{F}^{p\times q}$ from $\mathrm{Tr}(A_jP), j = 1, \ldots, N$ where $A_j \in V_j \subset \mathbb{F}^{p\times q}$. We mainly focus on the following question: how many measurements are needed to recover almost all the matrices in $\mathcal{M}$? For the case where both $\mathcal{M}$ and V_j are algebraic varieties, we use the tools from algebraic geometry to study the question and present some results to address it under many different settings.

1. Introduction

1.1. **Matrix Recovery Problems.** The matrix recovery problem has gained much attention in recent years. The general formulation of the problem is that there is an unknown $m \times n$ real or complex matrix P and we would like to recover the matrix P from those measurements or samples. A typical such problem is the so-called *Netflix Problem*, where we know the value of some but not all entries, and the matrix in question has low rank. The aim is to fully recover the matrix from the partial set of entries. The Netflix Problem has seen extensive study because of its broad applications in many other areas (see [31, 10, 32]).

The Netflix Problem is a special case of matrix recovery from linear measurements, which can be phrased generally as follows: For $1 \leq j \leq N$ let $L_j : \mathbb{F}^{p\times q} \longrightarrow \mathbb{F}$ be linear maps, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Given $L_1(P), \ldots, L_N(P)$, can we recover $P \in \mathbb{F}^{p\times q}$? The ability to recover P depends on the properties of P and L_j, and we also need to have *enough* measurements.

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It is well known that a linear map $L : \mathbb{F}^{p\times q} \longrightarrow \mathbb{F}$ can always be represented in the form of $L(P) = \mathrm{Tr}(A^T P)$ for $A \in \mathbb{F}^{p\times q}$, and such a representation is one-to-one. Now let $\mathcal{A} = (A_j)_{j=1}^N$ be a sequence of matrices with each $A_j \in \mathbb{F}^{p\times q}$. We denote by $\mathbf{L}_\mathcal{A} : \mathbb{F}^{p\times q} \longrightarrow \mathbb{F}^N$ the map given by

$$\mathbf{L}_\mathcal{A}(P) = (\mathrm{Tr}(A_1^T P), \mathrm{Tr}(A_2^T P), \ldots, \mathrm{Tr}(A_N^T P))^T, \qquad P \in \mathbb{F}^{p\times q}. \tag{1.1}$$

Matrix recovery problems aim to recover a matrix in a subset of $\mathbb{F}^{p\times q}$ from linear measurements. Let $\mathcal{M} \subseteq \mathbb{F}^{p\times q}$ be the subset of interest, say the set of all rank r matrices in $\mathbb{F}^{p\times q}$. We say $\mathcal{A} = (A_j)_{j=1}^N$ where $A_j \in \mathbb{F}^{p\times q}$ has the *$\mathcal{M}$-recovery property* if every $P \in \mathcal{M}$ is uniquely determined by $\mathbf{L}_\mathcal{A}(P)$. In other words, the map $\mathbf{L}_\mathcal{A}$ is injective on $\mathcal{M}$.

One particular class of matrices of interest is the set of all rank r or less matrices, which we denote by

$$\mathcal{M}_{p\times q,r}(\mathbb{F}) := \Big\{Q \in \mathbb{F}^{p\times q} : \mathrm{rank}(Q) \le r\Big\}, \qquad \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}. \tag{1.2}$$

For example, it is known that at least $N \ge 4rn - 4r^2$ linear measurements are needed to completely recover any $P \in \mathcal{M}_{n\times n,r}(\mathbb{C})$ where $0 < r \le n/2$, and furthermore $N \ge 4rn - 4r^2$ linear measurements will suffice (see [30, 12]).

1.2. **Almost Everywhere Matrix Recovery.** As said before, by providing enough measurements, we can recover all matrices in $\mathcal{M}_{n\times n,r}(\mathbb{C})$, e.g. $N \ge 4rn - 4r^2$ random measurements. Sometimes we may have fewer measurements. Numerical experiments show that it is possible to recover most of matrices in $\mathcal{M}_{n\times n,r}(\mathbb{C})$ from $N < 4rn - 4r^2$ random measurements. So, sometimes, even though we can not be able to recover all matrices in a subset $\mathcal{M}$, we may be able to recover most of them nevertheless. This leads to the notion of *almost everywhere matrix recovery*.

Definition 1.1. Let $\mathcal{M} \subset \mathbb{F}^{p\times q}$ and $\mathcal{A} = (A_j)_{j=1}^N \in (\mathbb{F}^{p\times q})^N$. We say $\mathbf{L}_\mathcal{A}$ or simply $\mathcal{A}$ has the *$\mathcal{M}$-recovery property* if $\mathbf{L}_\mathcal{A}$ is injective on $\mathcal{M}$. It has the *almost everywhere $\mathcal{M}$-recovery property* if for almost every $P \in \mathcal{M}$, we have $\mathbf{L}_\mathcal{A}^{-1}\big(\mathbf{L}_\mathcal{A}(P)\big) \cap \mathcal{M} = \{P\}$ where $\mathbf{L}_\mathcal{A}^{-1}\big(\mathbf{L}_\mathcal{A}(P)\big) := \{Q \in \mathbb{F}^{p\times q} : \mathbf{L}_\mathcal{A}(Q) = \mathbf{L}_\mathcal{A}(P)\}$.

Here the easiest way to define "almost everywhere" is through the Hausdorff measure on $\mathcal{M}$. But since our study only focuses on $\mathcal{M}$ that are "nice" such as algebraic varieties there should be no ambiguity. For the case where $\mathcal{M}$ is an algebraic variety, to show $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property, it is enough to prove that there exists a subvariety $Y \subset \mathcal{M}$ with $\dim(Y) < \dim(\mathcal{M})$ so that $\mathbf{L}_{\mathcal{A}}^{-1}\big(\mathbf{L}_{\mathcal{A}}(P)\big) \cap \mathcal{M} = \{P\}$ for any $P \in \mathcal{M} \setminus Y$. We say $\mathcal{A} = (A_j)_{j=1}^N \in (\mathbb{F}^{p\times q})^N$ or $\mathbf{L}_{\mathcal{A}}$ has the *almost everywhere rank r matrix recovery property* if it has the almost everywhere $\mathcal{M}_{p\times q,r}(\mathbb{F})$-recover property. Note that in this case $\mathcal{M}_{p\times q,r}(\mathbb{F})$ is an algebraic variety of dimension $r(p+q)-r^2$ (see [17]).

This paper studies the following questions: Let $\mathcal{A} = (A_j)_{j=1}^N \in (\mathbb{F}^{p\times q})^N$. *What is the minimal measurement number N needed for $\mathbf{L}_{\mathcal{A}}$ to have the almost everywhere rank r matrix recovery property?* Or more generally, for a given subset $\mathcal{M} \subset \mathbb{F}^{p\times q}$, what is the minimal measurement number N needed for $\mathbf{L}_{\mathcal{A}}$ to have the almost everywhere $\mathcal{M}$-recovery property? Note that in general we also have additional constraints on measurement matrices $\mathcal{A}$. The aim of this paper is to present a series of results addressing these questions.

1.3. **Related Work.** In the context of matrix recovery, one already presents many conditions under which $\mathcal{A} = (A_j)_{j=1}^N$ has $\mathcal{M}_{n\times n,r}(\mathbb{F})$-recovery property [12, 23, 24, 26]. In [12], it is proved that if $N \geq 4nr - 4r^2$ and $A_1, \ldots, A_N$ are Gaussian random matrices, then $\mathcal{A}$ has $\mathcal{M}_{n\times n,r}(\mathbb{F})$-recovery property with probability 1. In [12], Eldar, Needell and Plan conjecture the measurement number $4nr-4r^2$ is tight. In [30], Xu confirm the conjecture for the case $\mathbb{F} = \mathbb{C}$ and also disprove it for $\mathbb{F} = \mathbb{R}$.

Under the setting of $\mathcal{M} = \{\mathbf{x}\mathbf{x}^* : \mathbf{x} \in \mathbb{F}^n\} \subset \mathbb{F}^{n\times n}$ and $A_j = \mathbf{f}_j\mathbf{f}_j^*$ with $\mathbf{f}_j \in \mathbb{F}^n, j = 1, \ldots, N$, $\mathcal{A} = (A_j)_{j=1}^N$ has the almost everywhere $\mathcal{M}$-recovery property if and only if $(\mathbf{f}_j)_{j=1}^N$ has the almost phase retrieval property. It is an active topic to present the smallest N for which $(\mathbf{f}_j)_{j=1}^N$ having the almost phase retrieval property [2, 13, 15, 22]. For the case where $\mathbb{F} = \mathbb{R}$, it is known that $N \geq d+1$ is sufficient and necessary. For $\mathbb{F} = \mathbb{C}$, it is known that $N = 2d$ generic measurements are sufficient for almost phase retrieval (see [2]). However, one still does not know whether $N = 2d$ is tight or not.

1.4. **Our Contribution.** In this paper we establish a general framework for the almost everywhere matrix recovery problem. Under our framework they are all unified under matrix recovery. One representative result in the paper is the following theorem on almost everywhere rank r matrix recovery:

Theorem 1.1. *Assume that* $1 \le r \le \frac{1}{2}\min(p,q)$ *and* $N > (p+q)r - r^2$. *Let* $\mathcal{A} = (A_j) \in (\mathbb{F}^{p\times q})^N$ *be randomly chosen under an absolutely continuous probability distribution in* $(\mathbb{F}^{p\times q})^N$, *where* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{R}$. *Then with probability one* $\mathcal{A}$ *has the almost everywhere rank* r *matrix recovery property in* $\mathbb{F}^{p\times q}$.

In Theorem 1.1 there is no constraint on the measurement matrices. However, often restrictions are put on these measurements. This turns out not to be an obstacle in general. Theorem 1.1 is actually a special case of the following general theorem:

Theorem 1.2. *Assume that* $1 \le r \le \frac{1}{2}\min(p,q)$. *Let* $V_j \subseteq \mathbb{F}^{p\times q}$ *be algebraic varieties and* $A_j \in V_j$ *for* $1 \le j \le N$, *where* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{R}$. *Set* $\mathcal{A} = (A_j)_{j=1}^N$. *If* $N < (p+q)r - r^2$ *then* $\mathcal{A}$ *does not have the almost everywhere rank* r *matrix recovery property in* $\mathbb{F}^{p\times q}$. *If* $N > (p+q)r - r^2$ *then a generic* $\mathcal{A}$ *in* $V_1 \times \cdots \times V_N$ *has the almost everywhere rank* r *matrix recovery property in* $\mathbb{F}^{p\times q}$ *if:*

(A) $V_j = \mathcal{M}_{p\times q, r_j}(\mathbb{F})$ *and* $1 \le r_j \le \min(p,q)$ *for* $1 \le j \le N$.
(B) *For* $\mathbb{F} = \mathbb{R}$ *and* $q = p = d$, V_j *is the set of* $d \times d$ *orthogonal matrices.*
(C) *For* $\mathbb{F} = \mathbb{R}$ *and* $q = p = d$, V_j *is the set of* $d \times d$ *orthogonal projection matrix of rank* $r_j \in [1, d-1]$.

The theorem holds for far more broad classes of sets V_j's. Later we shall show ways to establish this type of results in general, including some of the basic algebraic geometry tools needed for the study.

The paper is organized as follows. In Section 2, after introducing some results and notations from elementary algebraic geometry, we present Theorem 2.1 which is often used in this study. In Section 3, we prove that $\mathcal{A} = (A_j)_{j=1}^N$ with $A_j \in V_j \subset \mathbb{F}^{p\times q}$ has the almost everywhere $\mathcal{M}$-recovery property if $N > \dim(\mathcal{M})$ and V_j satisfies the admissible condition (see Definition 2.1). In Section 4, we prove the algebraic varieties introduced in Theorem 1.2 satisfy the admissible condition. This implies

Theorem 1.1 and Theorem 1.2. We furthermore use the results in Section 3 to study the minimal measurement number for the recovery of Hermitian low rank matrices from rank one measurements.

2. Background from algebraic geometry

There is a strong connection between rank r matrix recovery and the classical dimension theory in algebraic geometry. Such connection has been employed to study both matrix recovery and phase retrieval (see [30, 28, 9]). Not surprisingly, this connection also plays a key role for almost everywhere matrix recovery. Before proceeding to the main results, we first introduce some basic notations related to projective spaces and varieties.

For any complex vector space X we shall use $\mathbb{P}(X)$ to denote the induced projective space, i.e. the set of all one dimensional subspaces in X. As usual for each $\mathbf{x} \in X$ we use $[\mathbf{x}]$ to denote the induced elements in $\mathbb{P}(X)$. Similarly, for any subset $S \subset X$ we use $[S]$ or $\mathbb{P}(S)$ to denotes its projectivization in $\mathbb{P}(X)$. Throughout this paper, we say $V \subset \mathbb{C}^d$ is a projective variety if V is the locus of a collection of homogeneous polynomials in $\mathbb{C}[\mathbf{x}]$. Strictly speaking a projective variety lies in $\mathbb{P}(\mathbb{C}^d)$ and is the projectivization of the zero locus of a collection of homogeneous polynomials. But like in [28], when there is no confusion the phrase *projective variety in* $\mathbb{C}^d$ means an algebraic variety in $\mathbb{C}^d$ defined by homogeneous polynomials. We shall use *projective variety in* $\mathbb{P}(\mathbb{C}^d)$ to describe a true projective variety. Note that sometimes it is useful to consider the more general quasi-projective varieties. A set $U \subset \mathbb{C}^d$ is a *quasi-projective variety* if there exist two projective varieties V and Y with $Y \subset V$ such that $U = V \setminus Y$. The concept of dimension for a quasi-projective variety in $\mathbb{C}^d$ is very well defined, and can be found in any standard algebraic geometry text such as [17].

In studying almost everywhere $\mathcal{M}$-recovery, we shall focus entirely on those $\mathcal{M}$ that are algebraic varieties in $\mathbb{F}^{p\times q}$. Note that the set $\mathcal{M}_{p\times q,r}(\mathbb{F})$ is a projective variety as $\operatorname{rank}(Q) \leq r$ is equivalent to the vanishing of all $(r+1)\times(r+1)$ minors of Q. It is called a *determinantal variety* and has $\dim_{\mathbb{F}}(\mathcal{M}_{p\times q,r}(\mathbb{F})) = (p+q)r - r^2$ [17, Prop. 12.2].

In [28] the notion of an admissible algebraic variety with respect to a family of linear functions was introduced. The concept is equally useful in this paper.

Definition 2.1 ([28])**.** Let V be the zero locus of a finite collection of homogeneous polynomials in $\mathbb{C}^d$ with $\dim(V) > 0$ and let $\{\ell_\alpha(\mathbf{x}) : \alpha \in I\}$ be a family of (homogeneous) linear functions. We say V is *admissible* with respect to $\{\ell_\alpha(\mathbf{x}) : \alpha \in I\}$ if $\dim(V \cap \{\mathbf{x} \in \mathbb{C}^d : \ell_\alpha(\mathbf{x}) = 0\}) < \dim(V)$ for all $\alpha \in I$.

It is well known in algebraic geometry that if V is irreducible in $\mathbb{C}^d$ then $\dim(V \cap Y) = \dim(V) - 1$ for any hyperplane Y that does not contain V. Thus the above admissible condition is equivalent to the property that no irreducible component of V of dimension $\dim(V)$ is contained in any hyperplane $\ell_\alpha(\mathbf{x}) = 0$. In general without the irreducibility condition, admissibility is equivalent to that for a generic point $\mathbf{x} \in V$, any small neighborhood U of $\mathbf{x}$ has the property that $U \cap V$ is not completely contained in any hyperplane $\ell_\alpha(\mathbf{x}) = 0$. The following theorem extends a result in [28], and will play a key role in our paper.

Theorem 2.1. *For $j = 1, \dots, N$ let $L_j : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ be bilinear functions and V_j be projective varieties in $\mathbb{C}^n$. Set $V := V_1 \times \cdots \times V_N \subseteq (\mathbb{C}^n)^N$. Let $W, Y \subset \mathbb{C}^m$ be a projective variety in $\mathbb{C}^m$, $W \setminus Y$ be a quasi-projective variety. For each fixed j, assume that V_j is admissible with respect to the linear functions $\{f^{\mathbf{w}}(\cdot) = L_j(\cdot, \mathbf{w}) : \ \mathbf{w} \in W \setminus Y\}$.*

(A) *Assume that $N \geq \dim(W)$. There exists an algebraic subvariety $Z \subseteq V$ with $\dim(Z) < \dim(V)$ such that for any $\mathbf{x} = (\mathbf{v}_j)_{j=1}^N \in V \setminus Z$, the subvariety $W_{\mathbf{x}}$ given by*

$$W_{\mathbf{x}} := \Big\{\mathbf{w} \in W \setminus Y : \ L_j(\mathbf{v}_j, \mathbf{w}) = 0 \text{ for all } 1 \leq j \leq N\Big\}$$

is the empty set.

(B) *Assume that $N < \dim(W)$. There exists an algebraic subvariety $Z \subset V$ with $\dim(Z) < \dim(V)$ such that for any $\mathbf{x} = (\mathbf{v}_j)_{j=1}^N \in V \setminus Z$, the subvariety $W_{\mathbf{x}}$ given by*

$$W_{\mathbf{x}} := \Big\{\mathbf{w} \in W \setminus Y : \ L_j(\mathbf{v}_j, \mathbf{w}) = 0 \text{ for all } 1 \leq j \leq N\Big\}$$

has $\dim(W_{\mathbf{x}}) = \dim(W) - N$.

Proof. We first prove (A). For $\mathbf{x} = (\mathbf{v}_j)_{j=1}^N \in V$, define $\Phi_{\mathbf{x}} : W \setminus Y \to \mathbb{C}^N$ by $\Phi_{\mathbf{x}}(\mathbf{w}) = (L_j(\mathbf{v}_j, \mathbf{w}))_{j=1}^N$. Let $\mathcal{G}$ be the subset of $[V] \times [W \setminus Y] \subset \mathbb{P}((\mathbb{C}^n)^N) \times \mathbb{P}(\mathbb{C}^m)$ such that $([\mathbf{x}], [\mathbf{w}]) \in \mathcal{G}$ if and only if $\Phi_{\mathbf{x}}(\mathbf{w}) = 0$, i.e. $L_j(\mathbf{v}_j, \mathbf{w}) = 0$ for all j. We can view $\mathcal{G}$ as a quasi-projective variety via Segre embedding [17, Page 27]. Note that $\mathcal{G}$ is a projective variety of $\mathbb{P}((\mathbb{C}^n)^N) \times \mathbb{P}(\mathbb{C}^m)$. We consider its dimension. Let π_1 and π_2 be projections from $\mathbb{P}((\mathbb{C}^n)^N) \times \mathbb{P}(\mathbb{C}^m)$ onto the first and the second coordinates, respectively, namely

$$\pi_1([\mathbf{x}], [\mathbf{w}]) = [\mathbf{v}_1, \ldots, \mathbf{v}_N], \qquad \pi_2([\mathbf{x}], [\mathbf{w}]) = [\mathbf{w}].$$

It is easy to check that $\pi_2(\mathcal{G}) = [W \setminus Y]$, the projection of $W \setminus Y$. Thus $\dim(\pi_2(\mathcal{G})) = \dim(W \setminus Y) - 1$.

We next consider the dimension of the preimage of the $\pi_2^{-1}([\mathbf{w}_0]) \subset \mathbb{P}((\mathbb{C}^n)^N)$ for a fixed $[\mathbf{w}_0] \in \mathbb{P}(\mathbb{C}^m)$. Let $V_j' := V_j \cap H_j$ where $H_j := \{\mathbf{y} \in \mathbb{C}^n : L_j(\mathbf{y}, \mathbf{w}_0) = 0\}$ is a hyperplane. The admissibility property of V_j implies that $\dim(V_j') = \dim(V_j) - 1$. Hence after projectivization the preimage $\pi_2^{-1}([\mathbf{w}_0])$ has dimension

$$\dim(\pi_2^{-1}([\mathbf{w}_0])) = \sum_{j=1}^{N} (\dim(V_j) - 1) - 1 = \dim(V) - N - 1.$$

According to Cor.11.13 in [17], we have

$$\begin{aligned} \dim(\mathcal{G}) &= \dim(\pi_2(\mathcal{G})) + \dim(\pi^{-1}([\mathbf{w}_0])) \\ &= (\dim(W \setminus Y) - 1) + (\dim(V) - N - 1) \\ &= \dim(V) + \dim(W \setminus Y) - N - 2 \\ &\leq \dim(V) + \dim(W) - N - 2. \end{aligned}$$

If $N \geq \dim(W)$ then

$$\dim(\pi_1(\mathcal{G})) \leq \dim(\mathcal{G}) = \dim(V) + \dim(W) - N - 2 \leq \dim(V) - 2.$$

Note that $\pi_1(\mathcal{G})$ is itself a projective variety. Let Z be the lift of $\pi_1(\mathcal{G})$ into the vector space $(\mathbb{C}^n)^N$. Then

$$\dim(Z) \leq \dim(V) - 1.$$

The definition of Z implies that $W_{\mathbf{x}}$ is an empty set provided $\mathbf{x} \in V \setminus Z$.

Next we prove (B). Let $K = \dim(W \setminus Y)$. Noting $K > N$, we augment $\{V_j\}_{j=1}^N$ and $\{L_j(\mathbf{v}, \mathbf{w})\}_{j=1}^N$ to $\{V_j\}_{j=1}^K$ and $\{L_j(\mathbf{v}, \mathbf{w})\}_{j=1}^K$ via $V_j = V_1$ and $L_j(\mathbf{v}, \mathbf{w}) = L_1(\mathbf{v}, \mathbf{w})$ for

all $j > N$. Set $\hat{V} = V_1 \times \cdots \times V_K \subseteq (\mathbb{C}^n)^K$. By (1) there exists a subvariety $\hat{Z}$ of $\hat{V}$ with $\dim(\hat{Z}) < \dim(\hat{V})$ such that for any $\hat{\mathbf{x}} = (\mathbf{v}_j)_{j=1}^K \in \hat{V} \setminus \hat{Z}$ and $\mathbf{w} \in W \setminus Y$, we have $L_j(\mathbf{v}, \mathbf{w}) \neq 0$ for some $j \in [1, K]$. Now consider the sequence of nested varieties with $W_{\hat{\mathbf{x}},0} = W$ and

$$W_{\hat{\mathbf{x}},k} := \Big\{ \mathbf{w} \in W \setminus Y : \ L_j(\mathbf{v}_j, \mathbf{w}) = 0 \text{ for all } 1 \leq j \leq k \Big\}, \quad k = 1, \ldots, K.$$

Thus the above is equivalent to $W_{\hat{\mathbf{x}},K} = \emptyset$ provided $\hat{\mathbf{x}} \in \hat{V} \setminus \hat{Z}$.

Since for each fixed $\mathbf{v}_j$ the equation $L_j(\mathbf{v}_j, \mathbf{w}) = 0$ defines a hyperplane H in $\mathbb{C}^m$, it is well known that $\dim(U \cap H) \geq \dim(U) - 1$ for any variety U in $\mathbb{C}^m$. Then we have a decreasing sequence of subvarieties of $\mathbb{C}^m$

$$W \setminus Y = W_{\hat{\mathbf{x}},0} \supseteq W_{\hat{\mathbf{x}},1} \supseteq W_{\hat{\mathbf{x}},2} \supseteq \cdots \supseteq W_{\hat{\mathbf{x}},K} = \emptyset.$$

Now $\dim(W_{\hat{\mathbf{x}},0}) = \dim(W \setminus Y) = K$. By Krull's Principal Ideal Theorem, at each step the dimension can only be reduced by at most 1, we must thus have $\dim(W_{\hat{\mathbf{x}},k-1}) - 1 = \dim(W_{\hat{\mathcal{A}},k})$ for $1 \leq k \leq K$. It follows that $\dim(W_{\hat{\mathbf{x}},N}) = \dim(W) - N = K - N$.

Thus for any $\mathbf{x} = (\mathbf{v}_j)_{j=1}^N \in V$, if there exists $\mathbf{v}_j \in V_j$ for $N < j \leq K$ such that $\hat{\mathbf{x}} = (\mathbf{v}_j)_{j=1}^K \in \hat{V} \setminus \hat{Z}$ we must have $\dim(W_{\hat{\mathbf{x}},N}) = K - N$. Since $W_{\hat{\mathbf{x}},N} = W_{\mathbf{x}}$ we then have $\dim(W_{\mathbf{x}}) = K - N$. Finally, let $Z = \{\mathbf{x} = (\mathbf{v}_j)_{j=1}^N\} \subset V$ be those such that there exists no such extensions $\hat{\mathbf{x}} \in \hat{V} \setminus \hat{Z}$. We have

$$Z = \Big\{ \mathbf{x} = (\mathbf{v}_j)_{j=1}^N \in V : \ \hat{\mathbf{x}} = (\mathbf{v}_j)_{j=1}^K \in \hat{Z} \text{ for any } \mathbf{v}_j \in V_j, j > N \Big\}.$$

Since $\hat{Z}$ is variety in $(\mathbb{C}^n)^K$, Z is a variety. Clearly it has $\dim(Z) < \dim(V)$, for otherwise we would have $\dim(\hat{Z}) = \dim(\hat{V})$, which is a contradiction. ∎

For real matrix recovery we need to consider real projective varieties. Here we introduce some notations. Let V be a variety in $\mathbb{C}^d$. We shall use $V \cap \mathbb{R}^d$ to denote the real points of V. As a real variety we can define the *real dimension* of $V \cap \mathbb{R}^d$, see [17] and [3]. A key fact is that for a variety V we have $\dim_{\mathbb{R}}(V \cap \mathbb{R}^d) \leq \dim(V)$ (see Section 2.1.3 in [11] and [28]). This also holds for a quasi-projective variety since the proof uses only local properties of V (see [28]).

A particularly important class of projective varieties for our study are those $V \subseteq \mathbb{C}^d$ such that $\dim_{\mathbb{R}}(V \cap \mathbb{R}^d) = \dim(V)$. For example, $V = \mathcal{M}_{p\times q,r}(\mathbb{C})$ in $\mathbb{C}^{p\times q}$ has this property. This class is especially useful for real matrix recovery.

3. Almost Everywhere Matrix Recovery: General Results

In this section we consider the problem of almost everywhere matrix recovery. At the same we also prove results on the classical matrix recovery (i.e. everywhere matrix recovery). Let $\mathcal{M}$ be a projective variety in $\mathbb{F}^{p\times q}$ such as $\mathcal{M} = \mathcal{M}_{p\times q,r}(\mathbb{F})$, and let $P \in \mathcal{M}$. We ask how many linear measurements are needed to recover P for all $P \in \mathcal{M}$, and how many linear measurements are needed to recover P for *almost* all $P \in \mathcal{M}$.

Theorem 3.1. *Assume that $\mathcal{A} = (A_j)_{j=1}^N \in (\mathbb{F}^{p\times q})^N$ where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Let $\mathcal{M}$ be a projective variety in $\mathbb{C}^{p\times q}$ with $\dim(\mathcal{M}) = K$ and $Y_{\mathcal{A}}$ be*

$$Y_{\mathcal{A}} := \Big\{(P,Q):\ P,Q\in\mathcal{M},\ P\neq Q,\ \mathrm{Tr}(A_j^T(P-Q))=0 \text{ for } 1\le j\le N\Big\} \subset \mathbb{F}^{p\times q}\times\mathbb{F}^{p\times q}.$$

(A) *For $\mathbb{F} = \mathbb{C}$, if the (complex) quasi-projective variety $Y_{\mathcal{A}}$ has $\dim(Y_{\mathcal{A}}) < K$ then $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property.*

(B) *For $\mathbb{F} = \mathbb{R}$ let $\mathcal{M}_{\mathbb{R}} = \mathcal{M} \cap \mathbb{R}^{p\times q}$. If $\dim(\mathcal{M}) = \dim_{\mathbb{R}}(\mathcal{M}_{\mathbb{R}}) = K$ and $\dim_{\mathbb{R}}(Y_{\mathcal{A}}) < K$ then $\mathcal{A}$ has the almost everywhere $\mathcal{M}_{\mathbb{R}}$-recovery property.*

Proof. First we consider the case $\mathbb{F} = \mathbb{C}$. Let Z denote the set of matrices $P \in \mathcal{M}$ such that there exists a $Q \neq P$ in $\mathcal{M}$ such that $\mathrm{Tr}(A_j^T P) = \mathrm{Tr}(A_j^T Q)$ for all $1 \le j \le N$. The goal is to show that Z is a null set in $\mathcal{M}$. Observe that the set Z is the projection of $Y_{\mathcal{A}}$ onto the first coordinate. Since projections cannot increase dimension (see [17][Cor.11.13]), it follows that $\dim(Z) < K = \dim(\mathcal{M})$. Hence Z is a null set in $\mathcal{M}$.

Now for $\mathbb{F} = \mathbb{R}$, we already stated that the real dimension of $Y_{\mathcal{A}} \cap \mathbb{R}^{p\times q} \times \mathbb{R}^{p\times q}$ is no larger than the (complex) dimension of $Y_{\mathcal{A}}$. Thus $\dim_{\mathbb{R}}(Y_{\mathcal{A}}) < K = \dim_{\mathbb{R}}(\mathcal{M}_{\mathbb{R}})$. Let $Z_{\mathbb{R}}$ denote the set of matrices $P \in \mathcal{M}_{\mathbb{R}}$ such that there exists a $Q \neq P$ in $\mathcal{M}_{\mathbb{R}}$ with $\mathrm{Tr}(A_j^T P) = \mathrm{Tr}(A_j^T Q)$ for all $1 \le j \le N$. Again $Z_{\mathbb{R}}$ is the projection of $Y_{\mathcal{A}} \cap \mathbb{R}^{p\times q}$ onto the first coordinate. Since projections cannot increase dimension, it follows that $\dim_{\mathbb{R}}(Z_{\mathbb{R}}) < K = \dim_{\mathbb{R}}(\mathcal{M}_{\mathbb{R}})$. Hence $Z_{\mathbb{R}}$ is a null set in $\mathcal{M}_{\mathbb{R}}$. The theorem is proved. ∎

Intuitively speaking, the more the number of measurements is the smaller $\dim(Y_{\mathcal{A}})$ will be. So the question is how many measurements do we need to reach $\dim(Y_{\mathcal{A}}) <$

K. Our next theorem provides the answer for generic measurements restricted to projective varieties. First we establish a very intuitive lemma.

Lemma 3.2. *Let $\Phi : U \longrightarrow \mathbb{R}^n$ be a C^1 map, where $U \subseteq \mathbb{R}^m$ is an open set and $m > n$. Then Φ cannot be almost everywhere injective.*

Proof. Let $J := (\partial\phi_i/\partial x_j)$ be the Jacobian matrix of $\Phi = (\phi_1, \ldots, \phi_n)^T$. Let r be the maximal rank of J on U. The set of points in U at which the rank of J is r is an open set in U, and we shall show that Φ is not almost everywhere injective on this set. So without loss of generality we may assume that $\operatorname{rank}(J) = r$ everywhere on U.

We first consider the case where $r = n$. For any $\mathbf{x}_0 \in U$ let $\mathbf{y}_0 = \Phi(\mathbf{x}_0)$. Without loss of generality again we may assume that the first n columns of $J(\mathbf{x}_0)$ are linearly independent. Set $F(\mathbf{x}) = \Phi(\mathbf{x}) - \mathbf{y}_0$. By the Implict Function Theorem there exist functions $\psi_{n+1}(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)$ in a small neighborhood of $\mathbf{x}_0$ such that

$$F(x_1, \ldots, x_n, \psi_{n+1}(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)) = 0,$$

namely

$$\Phi(x_1, \ldots, x_n, \psi_{n+1}(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)) = \mathbf{y}_0.$$

Thus $\Phi^{-1}(\Phi(\mathbf{x}_0))$ contain more than just $\mathbf{x}_0$. It follows that Φ is not almost everywhere injective.

We next consider the case $r < n$. By the Rank Theorem (see [21], Theorem 3.5.1), for any $\mathbf{x}_0 \in U$ there is a decomposition $\mathbb{R}^n = V \oplus \tilde{V}$ where $V, \tilde{V}$ are linear subspaces of $\mathbb{R}^n$ with $\dim(V) = r$ and $\dim(\tilde{V}) = n - r$, such that we can write $\Phi(\mathbf{x})$ in a small neighborhood W of $\mathbf{x}_0$ as

$$\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) + \Phi_2(\mathbf{x}), \qquad \Phi_1(\mathbf{x}) \in V,\ \Phi_2(\mathbf{x}) \in \tilde{V},$$

with the property that the value of $\Phi_2(\mathbf{x})$ is uniquely determined by the value of $\Phi_1(\mathbf{x})$. In other words, if $\Phi_1(\mathbf{x}) = \Phi_1(\mathbf{x}')$ then $\Phi_2(\mathbf{x}) = \Phi_2(\mathbf{x}')$ for any $\mathbf{x}, \mathbf{x}' \in W$. It follows that Φ is almost everywhere injective on W if and only if Φ_1 is almost everywhere injective on W. If the Jacobian of Φ_1 has rank r we have already shown from the first case that Φ_1 cannot be almost everywhere injective, and hence nor can Φ. But if the Jacobian of Φ_1 has rank $< r$ then we can repeat the argument, and eventually yields that Φ cannot be almost everywhere injective. ∎

Through out the rest of this paper, we set

$$\Delta\mathcal{M} := \{\mathbf{x}-\mathbf{y}: \text{ for all } \mathbf{x},\mathbf{y}\in\mathcal{M}\}.$$

Theorem 3.3. *Let $\mathcal{M}$ and V_j be projective varieties in $\mathbb{C}^{p\times q}$, $j=1,\dots,N$. Assume that each V_j is admissible with respect to the maps $\{L_P: P\in\Delta\mathcal{M}, P\neq 0\}$ where $L_P(Q):=\mathrm{Tr}(P^TQ)$. Then for $\mathcal{A}=(A_j)_{j=1}^N$ where $A_j\in V_j$ we have*

(A) *If $N<\dim(\mathcal{M})$ then $\mathcal{A}$ does not have the almost everywhere $\mathcal{M}$-recovery property. On the other hand, if $N>\dim(\mathcal{M})$ then a generic $\mathcal{A}=(A_j)_{j=1}^N$ in $V_1\times V_2\times\cdots\times V_N$ has the almost everywhere $\mathcal{M}$-recovery property.*

(B) *Suppose that $\Delta\mathcal{M}$ is a projective variety. If $N<\dim(\Delta\mathcal{M})$ then $\mathcal{A}$ does not have the $\mathcal{M}$-recovery property. On the other hand, if $N\geq\dim(\Delta\mathcal{M})$ then a generic $\mathcal{A}=(A_j)_{j=1}^N$ in $V_1\times V_2\times\cdots\times V_N$ has the $\mathcal{M}$-recovery property.*

Proof. Let $K=\dim(\mathcal{M})$. First we prove (A). If $N<\dim(M)$ then the map $L_{\mathcal{A}}$ given in (1.1) maps smoothly the higher dimensional manifold $\mathcal{M}$ to the lower dimensional one $\mathbb{C}^N$. For the aim of contradiction, we suppose that $L_{\mathcal{A}}$ is almost everywhere injective. By looking at $\mathcal{M}$ locally we see that there exists a smooth map Φ from a ball B in $\mathbb{C}^K\cong\mathbb{R}^{2K}$, where $K=\dim(\mathcal{M})$, to $\mathbb{C}^N\cong\mathbb{R}^{2N}$ that is almost everywhere injective. But this is impossible by Lemma 3.2.

Now for $N>\dim(\mathcal{M})=K$ let $X\subset\mathbb{C}^{p\times q}\times\mathbb{C}^{p\times q}$ be the quasi-projective variety

$$X:=\Big\{(P,Q): P,Q\in\mathcal{M},\ P\neq Q\Big\}.$$

For each $(P,Q)\in X$ denote $\psi_{(P,Q)}(A)=\mathrm{Tr}(A^T(P-Q))$. As in Theorem 3.1 set

$$Y_{\mathcal{A}}:=\Big\{(P,Q): P,Q\in\mathcal{M},\ P\neq Q, \mathrm{Tr}(A_j(P-Q))=0 \text{ for } 1\leq j\leq N\Big\}.$$

Since each V_j is admissible with respect to the maps $\{\psi_{(P,Q)}: (P,Q)\in X\}$. By Theorem 2.1 we have $\dim(Y_{\mathcal{A}})=\dim(X)-N<2K-K=\dim(\mathcal{M})$. It follows from Theorem 3.1 that $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property.

For (B) it is essentially proved in [28]. We quickly recap it here. For $N<\dim(\Delta\mathcal{M})$ the dimension of the projective variety $U:=\big\{Q\in\Delta\mathcal{M}: \mathrm{Tr}(A_j^TQ)=0, j=1,\dots,N\big\}$ is no less than $\dim(\Delta\mathcal{M})-N>0$. This is because in the complex projective space, through intersection with a hyperplane such as the one given by $\mathrm{Tr}(A_j^TQ)=0$, the dimension of any projective variety can be reduced by at most one. Thus there

exists a $Q = Q_1 - Q_2 \neq 0$ with $Q_1, Q_2 \in \mathcal{M}$ such that $\mathrm{Tr}(A_j^T Q_1) = \mathrm{Tr}(A_j^T Q_2)$ for all j. Hence $\mathcal{A}$ does not have the $\mathcal{M}$-recovery property.

In the case $N \geq \dim(\Delta\mathcal{M})$, we apply Theorem 2.1 with $W = (\Delta\mathcal{M}) \setminus \{0\}$ and $L_j(A, Q) := \mathrm{Tr}(A^T Q)$. Let $V = V_1 \times V_2 \times \cdots \times V_N$. Then there exists a variety $Z \subset V$ with $\dim(Z) < \dim(V)$ such that for all $\mathcal{A} \in V \setminus Z$ there exists no $Q \in W$ with the property $\mathrm{Tr}(A_j^T Q) = 0$ for all j. Thus a generic $\mathcal{A} \in V$ has the $\mathcal{M}$-recovery property. ∎

For the real case the above theorem can be extended. First, for any real variety $V \subseteq \mathbb{R}^d$ it has a natural extension to a variety in $\mathbb{C}^d$. The ideal $I_{\mathbb{R}}(V)$ defining V generates an ideal $I_{\mathbb{C}}(V)$ in $\mathbb{C}^d$, and the variety corresponding to $I_{\mathbb{C}}(V)$ will be our extension, which we denote it by $\bar{V}$. Note that V is clearly the restriction of $\bar{V}$ to $\mathbb{R}^d$, namely $V = \bar{V}_{\mathbb{R}}$ using the terminology in this paper.

Theorem 3.4. *Let $\mathcal{M}$ and V_j be projective varieties in $\mathbb{R}^{p\times q}$, $j = 1, \ldots, N$. Let each $\bar{V}_j$ be admissible with respect to the maps $\{L_P : \ P \in \Delta\bar{\mathcal{M}}, P \neq 0\}$ where $L_P(Q) := \mathrm{Tr}(P^T Q)$. Assume further that $\dim_{\mathbb{R}}(\mathcal{M}) = \dim(\bar{\mathcal{M}})$ and $\dim_{\mathbb{R}}(V_j) = \dim(\bar{V}_j)$ for all j. Then for $\mathcal{A} = (A_j)_{j=1}^N$ where $A_j \in V_j$ we have*

(A) *If $N < \dim_{\mathbb{R}}(\mathcal{M})$ then $\mathcal{A}$ does not have the almost everywhere $\mathcal{M}$-recovery property. On the other hand, if $N > \dim_{\mathbb{R}}(\mathcal{M})$ then a generic $\mathcal{A} = (A_j)_{j=1}^N$ in $V_1 \times V_2 \times \cdots \times V_N$ has the almost everywhere $\mathcal{M}$-recovery property.*

(B) *Assume additionally that $\dim_{\mathbb{R}}(\Delta\mathcal{M}) = \dim(\Delta\bar{\mathcal{M}}) = L$. If $N \geq L$ then a generic $\mathcal{A} = (A_j)_{j=1}^N$ in $V_1 \times V_2 \times \cdots \times V_N$ has the $\mathcal{M}$-recovery property.*

Proof. Denote $K = \dim(\mathcal{M})$ and $V := V_1 \times V_2 \times \cdots \times V_N$. First we prove (A). If $N < \dim(\mathcal{M})$ then the map $L_{\mathcal{A}}$ given in (1.1) maps smoothly the higher dimensional manifold $\mathcal{M}$ to the lower dimensional one $\mathbb{R}^N$. Again if $L_{\mathcal{A}}$ is almost everywhere injective, by looking at $\mathcal{M}$ locally we see that there exists a smooth map Φ from a ball B in $\mathbb{R}^K$ to $\mathbb{R}^N$ that is almost everywhere injective. This is a contradiction.

Now for $N > \dim(\mathcal{M}) = K$ we lift $\mathcal{M}$ and V into the complex projective space to $\bar{\mathcal{M}}$ and $\bar{V}$. Let $X \subset \mathbb{C}^{p\times q} \times \mathbb{C}^{p\times q}$ be the quasi-projective variety

$$X := \Big\{(P, Q) : \ P, Q \in \bar{\mathcal{M}}, \ P \neq Q\Big\}.$$

For each $(P,Q) \in X$ denote $\psi_{(P,Q)}(A) = \mathrm{Tr}(A^T(P-Q))$. As in Theorem 3.1 set

$$\bar{Y}_{\mathcal{A}} := \Big\{(P,Q) : \ P,Q \in \bar{\mathcal{M}},\ P \neq Q, \mathrm{Tr}(A_j^T(P-Q)) = 0 \text{ for } 1 \leq j \leq N\Big\},$$

and let $Y_{\mathcal{A}} = \bar{Y}_{\mathcal{A}} \cap \mathbb{R}^{p\times q} \times \mathbb{R}^{p\times q}$ be its restriction to the reals. Note that each $\bar{V}_j$ is admissible with respect to the maps $\{\psi_{(P,Q)} : (P,Q) \in X\}$. By Theorem 2.1 there exists a subvariety $\bar{Z} \subset \bar{V}$ with $\dim(\bar{Z}) < \dim(\bar{V})$ such that for any $\mathcal{A} \in \bar{V} \setminus \bar{Z}$ we have $\dim(\bar{Y}_{\mathcal{A}}) = \dim(X) - N < 2K - K = \dim(\bar{\mathcal{M}})$. By assumption we have $\dim_{\mathbb{R}}(V) = \dim(\bar{V})$ so the restriction $Z = \bar{Z}_{\mathbb{R}}$ of $\bar{Z}$ to the reals must have $\dim_{\mathbb{R}}(Z) < \dim(V)$. Furthermore, $\dim_{\mathbb{R}}(Y_{\mathcal{A}}) \leq \dim(\bar{Y}_{\mathcal{A}}) < K$. It follows from Theorem 3.1 that any $\mathcal{A} \in V \setminus Z$ has the almost everywhere $\mathcal{M}$-recovery property. This proves (A).

For (B) we follow the same strategy, which has been used for phase retrieval in [28]. We apply Theorem 2.1 with $\bar{W} = (\Delta\bar{\mathcal{M}}) \setminus \{0\}$ and $L_j(A,Q) := \mathrm{Tr}(A^TQ)$. Let $V, \bar{V}$ be as in part (A). Since $N \geq \dim(\Delta\bar{\mathcal{M}})$ it follows from Theorem 2.1 that there exists a variety $\bar{Z} \subset \bar{V}$ with $\dim(\bar{Z}) < \dim(\bar{V})$ such that for any $\mathcal{A} \in \bar{V} \setminus \bar{Z}$ there exists no $Q \in \bar{W}$ with the property $\mathrm{Tr}(A_j^TQ) = 0$ for all j. Now

$$\dim_{\mathbb{R}}(Z) \leq \dim(\bar{Z}) < \dim(\bar{V}) = \dim_{\mathbb{R}}(V).$$

Thus any $\mathcal{A} \in V \setminus Z$ has the $\mathcal{M}$-recovery property, which means a generic $\mathcal{A} \in V$ has the $\mathcal{M}$-recovery property. ∎

4. Cases of Almost Everywhere Matrix Recovery

In this section we provide several applications of almost everywhere matrix recovery on algebraic variety of interest.

4.1. **Algebraic varieties satisfying the admissibility condition.**

According to Theorem 3.3, the admissibility condition plays a key role in studying the almost everywhere matrix recovery. Then, we list many algebraic varieties as follows which satisfy this condition:

Proposition 4.1. *Let V be one of the following projective varieties in $\mathbb{C}^{p\times q}$. Then V is admissible with respect to any set of nontrivial linear functions on $\mathbb{C}^{p\times q}$:*

(A) $V = \mathcal{M}_{p\times q,s}(\mathbb{C})$*, the set of all $p\times q$ complex matrices of rank s or less, where* $1 \leq s \leq \min(p,q)$.

(B) *$q \geq p$ and V is the set of all scalar multiples of matrices P satisfying $PP^T = I$.*

(C) *$p = q$ and V is the set of all scalar multiples of projection matrices P, i.e. $P^2 = P$.*

Proof. (A) To this end, we just need show that for a generic $P_0 \in V$ and any nontrivial linear function f on $\mathbb{C}^{p\times q}$ we have $f(P) \not\equiv 0$ on any neighborhood of P_0. Note that there exists a nonzero $Q_0 \in \mathbb{C}^{p\times q}$ such that $f(P) = \mathrm{Tr}(PQ_0)$ for all P. If $\mathrm{Tr}(P_0Q_0) \neq 0$ then we are done. For the case $\mathrm{Tr}(P_0Q_0) = 0$, there always exist two matrices S_1, S_2 so that $\mathrm{Tr}(S_1P_0S_2Q_0) \neq 0$. Take $P_t = (I+tS_1)P_0(I+tS_2) \in V$. Then

$$\mathrm{Tr}(P_tQ_0) = t^2\mathrm{Tr}(S_1P_0S_2Q_0) + t\mathrm{Tr}(S_1P_0Q_0) + t\mathrm{Tr}(P_0S_2Q_0).$$

Clearly $\mathrm{Tr}(P_tQ_0) \not\equiv 0$ for sufficiently small $t \neq 0$. We thus arrive at the conclusion.
(B) It is sufficient to show that a generic point $P_0 \in V$ and any nonzero $Q_0 \in \mathbb{C}^{p\times q}$ we must have $\mathrm{Tr}(PQ_0) \not\equiv 0$ in any small neighborhood of P_0 in V.
If $P_0Q_0 \neq 0$, then set $P_t := e^{tS}P_0$, where S is skew-symmetric $q \times q$ matrices. A simple observation is that $P_t \in V$. Then all we need to show is that for some S and arbitrarily small $t \neq 0$, we have $\mathrm{Tr}(P_tQ_0) \not\equiv 0$. Then

$$\mathrm{Tr}(P_tQ_0) = \mathrm{Tr}(e^{tS}P_0Q_0) = \mathrm{Tr}(P_0Q_0) + \sum_{n=1}^{+\infty} \frac{1}{n!}t^n\mathrm{Tr}(S^nP_0Q_0).$$

If $\mathrm{Tr}(P_0Q_0) \neq 0$ then we are done. We next assume that $\mathrm{Tr}(P_0Q_0) = 0$. To this end, we show there is a S_0 such that $\mathrm{Tr}(S_0P_0Q_0) \neq 0$.
We first consider the case where P_0Q_0 is not a symmetric matrix. Then there exists $1 \leq i < j \leq q$ such that $(P_0Q_0)_{ij} \neq (P_0Q_0)_{ji}$ and we can define S by setting all the entries to be zero except $(S_0)_{ij} = 1$ and $(S_0)_{ji} = -1$. Then we have $\mathrm{Tr}(S_0P_0Q_0) = (P_0Q_0)_{ji} - (P_0Q_0)_{ij} \neq 0$.
If P_0Q_0 is a symmetric matrix, we claim that there exists a skew-symmetric matrix S_1 such that $e^{t_1S_1}P_0Q_0$ is not symmetric for $t_1 \in (0, 1]$. Then we can take $P_{t_0,t_1} = e^{t_0S_0+t_1S_1}P_0Q_0$ and the above statement will hold. To verify the claim, notice that

$$e^{t_1S_1}P_0Q_0 = P_0Q_0 + \sum_{n=1}^{+\infty} \frac{1}{n!}t_1^nS_1P_0Q_0,$$

it is sufficient to show that there is a skew-symmetric matrix S_1 such that $S_1P_0Q_0$ is not symmetric. Since $P_0Q_0 \neq 0$, there exists $1 \leq i, j \leq q$ such that $(P_0Q_0)_{ij} \neq 0$, then choose $1 \leq k \leq q$ such that $k \neq i$, $k \neq j$, and define S_1 by setting all the entries to be zeros except $(S_1)_{ik} = 1$ and $(S_1)_{ki} = -1$. Then we have

$$(S_1P_0Q_0)_{kj} = -(P_0Q_0)_{ij} \neq 0 = (S_1P_0Q_0)_{jk}.$$

It remains to discuss the case where $P_0Q_0 = 0$. In that case, since $P_0, Q_0 \neq 0$, we claim that there exists a skew-symmetric matrix S_2 such that $P_0e^{t_2S_2}Q_0 \neq 0$ for any $t_2 \in (0, 1]$. Then we can set $P_{t_0,t_1,t_2} = e^{t_0S_0+t_1S_1}P_0e^{t_2S_2}Q_0$ and the above statement will hold. To verify the claim, notice that

$$P_0e^{t_2S_2}Q_0 = P_0Q_0 + \sum_{n=1}^{+\infty} \frac{1}{n!}t^nP_0S_2Q_0,$$

and it is sufficient to show that there exists a skew symmetric matrix S_2 such that $P_0S_2Q_0 \neq 0$. Assume the above claim does not hold. Since $P_0, Q_0 \neq 0$, we can choose k, l such that the k-th row of P_0, denoted by $(P_0)^k$, and the l-th column of Q_0, denoted by $(Q_0)_l$, are nonzero. Then for any $1 \leq i < j \leq q$, if we define S_2 by setting all the entries to be zero except $(S_2)_{ij} = 1$ and $(S_2)_{ji} = -1$, and we will have $(P_0S_2Q_0)_{kl} = (P_0)_{ki}(Q_0)_{jl} - (P_0)_{kj}(Q_0)_{il} = 0$. And we have

$$((P_0)_{ki}, (P_0)_{kj}) = \lambda_{ij,kl}((Q_0)_{il}, (Q_0)_{jl})$$

for any $1 \leq i < j \leq q$ such that $|(Q_0)_{jl}|^2 + |(Q_0)_{il}|^2 \neq 0$. Obviously $(P_0)_k$ and $(Q_0)_l$ satisfy $(P_0)^k = \lambda_{kl}(Q_0)_l^T$ where $\lambda_{kl} \neq 0$. Then we have $(P_0Q_0)_{kl} = \lambda_{kl}||(P_0)^k||^2 \neq 0$, which contradicts to the assumption that $P_0Q_0 = 0$. This completes the proof.

(C) Let $d = p = q$. It is sufficient to show that at a generic point $P_0 \in V$ and any nonzero $Q_0 \in \mathbb{C}^{d\times d}$ we must have $\mathrm{Tr}(PQ_0) \not\equiv 0$ in any small neighborhood of P_0 in V.

Since $P_0^2 = P_0$, there exists a nonsingular matrix R such that $P_0 = RJ_sR^{-1}$ where

$$J_s = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{d\times d}.$$

The property of trace implies that $\mathrm{Tr}(P_0Q_0) = \mathrm{Tr}(J_s(R^{-1}Q_0R))$. Hence, without loss of generality, we just need consider the case where $P_0 = J_s$. Set $P_t = (I + tS)J_s(I + tS)^{-1}$. Then all we need to show is that for some S and arbitrary small $t \neq 0$ we have

$\mathrm{Tr}(P_tQ_0) \neq 0$. Since $(I+tS)^{-1} = \sum_{n=0}^{\infty}(-1)^n t^n S^n$, we have

$$\mathrm{Tr}(P_tQ_0) = \mathrm{Tr}(P_0Q_0) + \sum_{n=1}^{\infty}(-1)^{n-1}t^n\mathrm{Tr}((SJ_s - J_sS)S^{n-1}Q_0).$$

If there exists a $S \in \mathbb{C}^{d\times d}$ such that $\mathrm{Tr}((SJ_s - J_sS)S^{n-1}Q_0) \neq 0$ for some $n \geq 1$ then we are done.

For $n = 1$,

$$\mathrm{Tr}((SJ_s - J_sS)S^{n-1}Q_0) = \mathrm{Tr}((SJ_s - J_sS)Q_0) = \mathrm{Tr}(S(J_sQ_0 - Q_0J_s)).$$

We first consider the case where $J_sQ_0 - Q_0J_s \not\equiv 0$. Then we can take $S = (J_sQ_0 - Q_0J_s)^*$ and obtain $\mathrm{Tr}(S(J_sQ_0 - Q_0J_s)) \neq 0$. We are done.

We next only consider the case where $J_sQ_0 - Q_0J_s = 0$. Then Q_0 must have the form

$$Q_0 = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

where $Q_1 \in \mathbb{C}^{s\times s}$ and $Q_2 \in \mathbb{C}^{(d-s)\times(d-s)}$. Consider now $n = 2$ and we have

$$\begin{aligned}(SJ_s - J_sS)S^{n-1}Q_0 &= \begin{pmatrix} 0 & -S_{12} \\ S_{21} & 0 \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \\ &= \begin{pmatrix} -S_{12}S_{21}Q_1 & -S_{12}S_{22}Q_2 \\ S_{21}S_{11}Q_1 & S_{21}S_{12}Q_2 \end{pmatrix},\end{aligned}$$

which yields $\mathrm{Tr}((SJ_s - J_sS)SQ_0) = \mathrm{Tr}(-S_{12}S_{21}Q_1 + S_{21}S_{12}Q_2)$.

Assume that Q_1, Q_2 are not both scalar multiples of identity matrices. Without loss of generality, suppose $Q_1 \neq \lambda I_s$ for any $\lambda \in \mathbb{C}$, where $I_s \in \mathbb{C}^{s\times s}$ is the identity matrix. Then there exist $\mathbf{u}, \mathbf{v} \in \mathbb{C}^s$ such that $\mathbf{u}^*\mathbf{v} = 0$ but $\mathbf{u}^*Q_1\mathbf{v} \neq 0$. Take $S_{12} = \mathbf{u}\mathbf{x}^*$ and $S_{21} = \mathbf{x}\mathbf{v}^*$ where $\mathbf{x} \in \mathbb{C}^{d-s}$ and $\mathbf{x} \neq 0$. Then

$$\mathrm{Tr}((SJ_s - J_sS)SQ_0) = -\mathbf{u}^*Q_1\mathbf{v}\|\mathbf{x}\|^2 \neq 0.$$

We next consider the case where both Q_1, Q_2 are scalar multiples of identity matrices, that is, $Q_1 = \lambda_1 I_s$ and $Q_2 = \lambda_2 I_{d-s}$ where $\lambda_1, \lambda_2 \in \mathbb{C}$. We claim that $\lambda_1 \neq \lambda_2$, otherwise we have $Q_0 = \lambda I_d$ where $\lambda \in \mathbb{C}$ and $\lambda \neq 0$. Then $\mathrm{Tr}(P_0Q_0) = \lambda s \neq 0$, which contradicts to the assumption above. Thus we can take $S_{12} = \mathbf{x}\mathbf{y}^*$ and $S_{21} = \mathbf{y}\mathbf{x}^*$ where $\mathbf{x} \in \mathbb{C}^s, \mathbf{x} \neq 0$ and $\mathbf{y} \in \mathbb{C}^{d-s}, \mathbf{y} \neq 0$. Then

$$\mathrm{Tr}((SJ_s - J_sS)SQ_0) = (\lambda_2 - \lambda_1)\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \neq 0.$$

This completes the proof. ∎

Remark 4.2. *It should be noted that there are indeed some projective varieties V that are not admissible with respect to certain class of linear functions on $\mathbb{C}^{p\times q}$. For example, if V is the set of all symmetric $p \times p$ matrices, then $f(P) = \mathrm{Tr}(P^T Q) \equiv 0$ on V for any skew-symmetric Q. Hence, the V is* not *admissible with respect to $\{\mathrm{Tr}(\cdot Q) : Q \in \mathbb{C}^{p\times p}, Q^T = -Q\}$. Nevertheless the admissibility condition for many application such as phase retrieval does hold, but often needs to be individually checked.*

The case most people study is the set of all matrices of rank r or less. Theorem 1.1 addresses the random measurements case. By combining the results above, we can prove much more general results.

Theorem 4.3. *Let $\mathcal{M}$ and $\Delta\mathcal{M}$ be a projective varieties in $\mathbb{C}^{p\times q}$. Let $\mathcal{A} = (A_j) \in (\mathbb{C}^{p\times q})^N$ be randomly chosen under an absolutely continuous probability distribution in $(\mathbb{C}^{p\times q})^N$. Then with probability one $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property if $N > \dim(\mathcal{M})$, and it has the $\mathcal{M}$-recovery property if $N \geq \dim(\Delta\mathcal{M})$.*

Proof. We apply Theorem 3.3 with $V_j = \mathbb{C}^{p\times q}$. Proposition 4.1 implies that the admissibility condition in the theorem is met by any $\mathcal{M}$. Thus a generic $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property if $N > \dim(\mathcal{M})$, and it has the $\mathcal{M}$-recovery property if $N \geq \dim(\Delta\mathcal{M})$. This implies the theorem. ∎

For a real projective variety $\mathcal{M}$, recall that it has a lift $\bar{\mathcal{M}}$ into the complex space as defined in the previous section. We have the following theorem.

Theorem 4.4. *Let $\mathcal{M}$ and $\Delta\mathcal{M}$ be a projective varieties in $\mathbb{R}^{p\times q}$. Let $\mathcal{A} = (A_j) \in (\mathbb{R}^{p\times q})^N$ be randomly chosen under an absolutely continuous probability distribution in $(\mathbb{R}^{p\times q})^N$. Then $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property if $N > \dim_{\mathbb{R}}(\mathcal{M})$ and $\dim(\bar{\mathcal{M}}) = \dim_{\mathbb{R}}(\mathcal{M})$. It has the $\mathcal{M}$-recovery property if $N \geq \dim_{\mathbb{R}}(\Delta\mathcal{M})$ and $\dim(\Delta\bar{\mathcal{M}}) = \dim_{\mathbb{R}}(\Delta\mathcal{M})$.*

Proof. We apply Theorem 3.4 with $V_j = \mathbb{R}^{p\times q}$. According to Proposition 4.1, the admissibility condition in the theorem are met by $\mathcal{M}$. Thus a generic $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property if $N > \dim_{\mathbb{R}}(\mathcal{M})$, and it has the $\mathcal{M}$-recovery property if $N \geq \dim_{\mathbb{R}}(\Delta\mathcal{M})$. This implies the theorem. ∎

Corollary 4.5. *Let $\mathcal{M}$ be a projective varieties in $\mathbb{C}^{p\times q}$. Let $\mathcal{A} = (A_j) \in (\mathbb{C}^{p\times q})^N$ where $A_j \in V_j$ is a generic element and each V_j is one of the projective varieties in (A)-(C) of Proposition 4.1. Then $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property if $N > \dim(\mathcal{M})$, and it has the $\mathcal{M}$-recovery property if $N \geq \dim(\Delta\mathcal{M})$.*

Proof. All the arguments for Theorem 4.3 are valid, and the result readily follows. ∎

Corollary 4.6. *Let $\mathcal{M}$ be a projective varieties in $\mathbb{R}^{p\times q}$. Let $\mathcal{A} = (A_j) \in (\mathbb{R}^{p\times q})^N$ where $A_j \in V_j$ is a generic element and $\bar{V}_j$ for each j is one of the projective varieties in (A)-(C) of Proposition 4.1. Then $\mathcal{A}$ has the almost everywhere $\mathcal{M}$-recovery property if $N > \dim_{\mathbb{R}}(\mathcal{M})$ and $\dim(\bar{\mathcal{M}}) = \dim_{\mathbb{R}}(\mathcal{M})$. It has the $\mathcal{M}$-recovery property if $N \geq \dim_{\mathbb{R}}(\Delta\mathcal{M})$ and $\dim(\Delta\bar{\mathcal{M}}) = \dim_{\mathbb{R}}(\Delta\mathcal{M})$.*

Proof. Again, all the arguments for Theorem 4.4 are valid, and the result readily follows. ∎

Remark 4.7. *Note that Theorem 1.1 and Theorem 1.2 follow from Theorem 4.3 and Corollary 4.6, respectively. Because the admissibility condition holds for many classes of varieties, the almost everywhere rank r matrix recovery property will hold in far greater generalities. It should also be pointed out that if we replace the "generic" stipulation in our results by random choices over some absolute continuous probability distribution over the varieties then the almost everywhere rank r matrix recovery property holds with probability 1.*

4.2. **Rank one measurements.** The recovery of Hermitian low rank matrices from rank one measurements attract much attention recently [25, 6, 29]. In this topic, one is interested in recovering a Hermitian low rank matrix $P \in \mathbb{C}^{p\times p}$ from $\{\mathbf{x}_j^* P \mathbf{x}_j\}_{j=1}^N$ where $\mathbf{x}_j \in \mathbb{C}^p$ (For the real case, P is assumed to be symmetric). One already develops many algorithms to compute it. Here, we focus on the theoretical sides. Particularly, we are interested in the following question: *how many measurements are needed to recover Hermitian low rank matrices from the rank one measurements.*

Although Hermitian matrices are complex, they do not form a complex variety. Thus the theorems we have here on complex recovery cannot be applied directly to

the recovery of Hermitian matrices. However, they can be formulated as the affine image of a real projective variety, and from which our theorems can be applied.

In the following lemma, we present the real dimension of the set of symmetric/Hermitian matrices of rank at most r which are from [17] and [19], respectively.

Lemma 4.8. *([17, 19])*

(A) *Let $\mathcal{M} \subset \mathbb{R}^{p\times p}$ be the set of all real symmetric matrices of rank at most r. Then $\mathcal{M}$ is a real projective variety of dimension $pr - r(r-1)/2$.*

(B) *Let $\mathcal{M} \subset \mathbb{C}^{p\times p}$ be the set of all Hermetian matrices of rank at most r. Then $\mathcal{M}$ is a real projective variety of dimension $2pr - r^2$.*

Now we are ready to present the following theorem:

Theorem 4.9. (A) *Let $\mathcal{M} \subset \mathbb{R}^{p\times p}$ be the set of all real symmetric matrices of rank at most r where $r \leq p/2$. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ be randomly chosen vectors in $\mathbb{R}^p$ according to some absolutely continuous probability distribution. Then any $P \in \mathcal{M}$ can be recovered with probability one from $\{\mathbf{x}_j^T P \mathbf{x}_j\}_{j=1}^N$ if $N \geq 2pr - 2r^2 + r$, and almost all $P \in \mathcal{M}$ can be recovered with probability one from $\{\mathbf{x}_j^T P \mathbf{x}_j\}_{j=1}^N$ if $N \geq pr - r(r-1)/2 + 1$.*

(B) *Let $\mathcal{M} \subset \mathbb{C}^{p\times p}$ be the set of all Hermitian matrices of rank at most r where $r \leq p/2$. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ be randomly chosen vectors in $\mathbb{C}^p$ according to some absolutely continuous probability distribution. Then from $\{\mathbf{x}_j^* P \mathbf{x}_j\}_{j=1}^N$, any $P \in \mathcal{M}$ can be recovered with probability one if $N \geq 4pr - 4r^2$, and almost all $P \in \mathcal{M}$ can be recovered with probability one if $N \geq 2pr - r^2 + 1$.*

Proof. (A) Let $V_j \subset \mathbb{C}^{p\times p}$ be the set of symmetric matrices of rank at most 1. Note that $\mathbf{x}_j^T P \mathbf{x}_j = \mathrm{Tr}(A_j P)$ where $A_j = \mathbf{x}_j \mathbf{x}_j^T \in V_j$. The admissibility condition of V_j is already verified in the previous paper (see the proof of Theorem 4.1 in [28]). By Lemma 4.8, $\mathcal{M}$ is a real projective variety with $\dim_{\mathbb{R}}(\mathcal{M}) = pr - r(r-1)/2$. Observe that $\Delta\mathcal{M}$ is all the real symmetric varieties with rank at most $2r$, and then $\dim_{\mathbb{R}}(\Delta\mathcal{M}) = 2pr - r(2r-1)$. Then by Theorem 4.4, we only need to show $\dim(\bar{\mathcal{M}}) = \dim_{\mathbb{R}}(\mathcal{M})$ and $\dim(\Delta\bar{\mathcal{M}}) = \dim_{\mathbb{R}}(\Delta\mathcal{M})$, where $\bar{\mathcal{M}}$ is the lift of $\mathcal{M}$ into complex space.

Let $\bar{\mathcal{M}} \subset \mathbb{C}^{p\times p}$ be the set of all complex symmetric matrices of rank at most r. $\bar{\mathcal{M}}$ is complex projective variety and $\bar{\mathcal{M}} \cap \mathbb{R}^{p\times p} = \mathcal{M}$.

Then using the same approach as in the proof of Lemma 4.8, replacing $\mathbb{R}$ with $\mathbb{C}$, we have $\dim(\bar{\mathcal{M}}) = 2pr - r(2r-1)$. Thus $\dim(\bar{\mathcal{M}}) = \dim_{\mathbb{R}}(\mathcal{M})$. Similarly, we can show $\dim(\Delta\bar{\mathcal{M}}) = \dim_{\mathbb{R}}(\Delta\mathcal{M})$.

For (B), consider the map $\varphi : \mathbb{C}^{p\times p} \to \mathbb{C}^{p\times p}$ defined by

$$\varphi(A) = \frac{1}{2}(A + A^T) + \frac{i}{2}(A - A^T).$$

Then φ is a isomorphism on $\mathbb{C}^{p\times p}$. Let

$$\bar{\mathcal{N}} = \{A \in \mathbb{C}^{p\times p} : \operatorname{rank}(\varphi(A)) \le r\}$$

and

$$\mathcal{N} = \{A \in \mathbb{R}^{p\times p} : \operatorname{rank}(\varphi(A)) \le r\}.$$

Then $\bar{\mathcal{N}} \cap \mathbb{R}^{p\times p} = \mathcal{N}$. Besides, we have $\bar{\mathcal{N}} = \varphi^{-1}(\{B \in \mathbb{C}^{p\times p} \text{ with rank at most } r\})$ and $\mathcal{N} = \varphi^{-1}(\mathcal{M})$. We only need to show that any $B \in \mathcal{N}$ can be recovered from $\{\mathbf{x}_j^*\varphi^{-1}(B)\mathbf{x}_j\}_{j=1}^N$ if $N \ge 4pr - 4r^2$, and almost all $B \in \mathcal{N}$ can be recovered with probability one if $N \ge 2pr - r^2 + 1$. Let $V_j \subset \mathbb{C}^{p\times p}$ be the set of Hermitian matrices with rank at most 1. A simple observation is that $\mathbf{x}_j^*\varphi^{-1}(B)\mathbf{x}_j = \operatorname{Tr}(A_j\varphi^{-1}(B))$ where $A_j = \mathbf{x}_j\mathbf{x}_j^* \in V_j$. Recall that the Hermitian matrix set V_j satisfies the admissibility condition (see Theorem 4.1 in [28]). The admissibility condition naturally holds since φ is a linear transformation on $\mathbb{C}^{p\times p}$. According to lemma 4.8, $\dim_{\mathbb{R}}(\mathcal{N}) = \dim_{\mathbb{R}}(\mathcal{M}) = 2pr - r^2$. Since φ is a linear transform, we have $\dim(\bar{\mathcal{N}}) = 2pr - r^2$. Hence, $\dim_{\mathbb{R}}(\mathcal{N}) = \dim(\bar{\mathcal{N}})$. The conclusion follows from Theorem 3.4. ∎

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Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

E-mail address: yrong@ust.hk

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

E-mail address: yangwang@ust.hk

LSEC, Inst. Comp. Math., Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing, 100091, China

E-mail address: xuzq@lsec.cc.ac.cn