

Optimal Sensor and Actuator Placement using Balanced Model Reduction

Krithika Manohar, J. Nathan Kutz, *Member, IEEE*, and Steven L. Brunton, *Member, IEEE*

Abstract—Optimal sensor and actuator placement is a central challenge in high-dimensional estimation and control. Nearly all subsequent control decisions are affected by these sensor/actuator locations, and optimal placement amounts to an intractable brute-force search among the combinatorial possibilities. In this work, we exploit balanced model reduction and greedy optimization to efficiently determine sensor and actuator placements that optimize observability and controllability. In particular, we determine locations that optimize scalar measures of observability and controllability via greedy matrix QR pivoting on the dominant modes of the direct and adjoint balancing transformations. Pivoting runtime scales linearly with the state dimension, making this method tractable for high-dimensional systems. The results are demonstrated on the linearized Ginzburg-Landau system, for which our algorithm approximates well-known optimal placements computed using costly gradient descent methods.

Index Terms—optimal control, balanced truncation, sensor placement, actuator placement, observability, controllability.

I. INTRODUCTION

Optimizing the placement of sensors and actuators is one of the foremost challenges in feedback control [1]. For high-dimensional systems it is impractical to monitor or actuate every state, hence a few sensors and actuators must be carefully positioned for effective estimation and control. Determining optimal placements with respect to a desired objective is an NP-hard selection problem, and in general can only be solved by enumerating all possible placement configurations. This combinatorial growth in complexity is intractable; therefore, the placement of sensors and actuators are typically chosen according to heuristics and intuition. In this paper, we propose a greedy algorithm for sensor and actuator placement based on jointly maximizing observability and controllability in linear time-invariant systems. Our approach (see Fig. 1) exploits low-rank transformations that balance the observability and controllability Gramians to bypass the combinatorial search, enabling favorable scaling for high-dimensional systems.

To understand the challenges of sensor and actuator placement for estimation and control, we will first consider optimal sensor placement, which has mostly been used to reconstruct static signals. The primary challenge of sensor placement is that given n possible locations and a budget of p sensors, there are combinatorially many, $\binom{n}{p}$, configurations to evaluate in a brute-force search. Branch and bound methods can speed up the search, but do not scale to high-dimensional systems. Fortunately, there are heuristics that employ greedy selection of sensors based on maximizing mutual information [2], entropy [3], and other information theoretic criteria [4], [5]. Another popular approach relaxes sensor selection to a weighted convex combination of possible sensors [6], [7], [8], [9], which can be solved using efficient quadratic or semidefinite programming. Both heuristic approaches optimize submodular

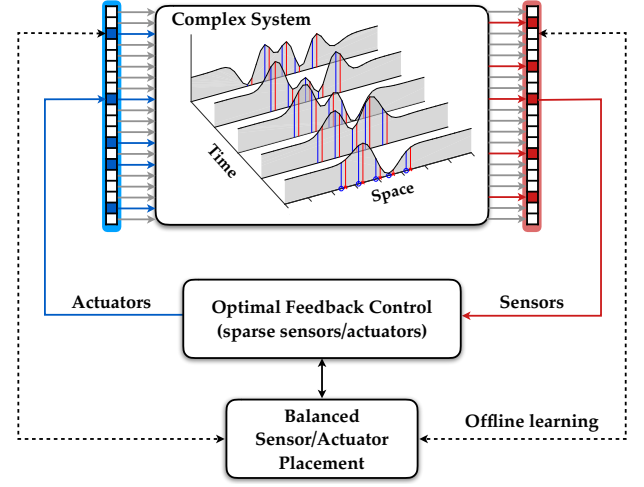


Fig. 1: Schematic of balanced sensor and actuator placement for the optimal control of a high-dimensional system.

objective functions [10], [11], so the distance between heuristic and optimal placement can be bounded. Some objectives, such as those based on the quality of a Kalman filter, are not submodular [12]. Alternatively, sparsity-promoting optimization can be used to determine sensors and actuators [13], [14], [15], although non-differentiability of sparsity promoting terms motivates other optimization techniques [16].

Even such heuristics cannot accommodate the high dimension of many physical models, for example in fluid dynamics. Fortunately, high-dimensional systems often evolve according to relatively few intrinsic degrees of freedom. Thus, it is possible to leverage dimensionality reduction to strategically place sensors. Recent works exploit model reduction, such as proper orthogonal decomposition (POD) [17], to greedily optimize condition number [18], [19], orthogonality [20], or matrix volume [21] of the resulting partially observed subspace. These objectives are closely related to empirical interpolation methods (EIM) [22], [23], [24] for efficient approximation of nonlinear terms in POD-based model reduction.

For systems with actuation, it is necessary to simultaneously consider the placement of sensors and actuators, since the most observable and most controllable subspaces are often different. For optimal control, sensors and actuators should be placed along the most observable and controllable directions, respectively [25], [26], [27], [28], [11], using objective functions based on the associated observability or controllability Gramians. Standard metrics for evaluating a certain sensor/actuator configuration include the H_2 norm, a measure of the average impulse response, and the H_∞ norm to measure the worst case performance. A chief drawback is the need to recompute the controller with each new configuration of sensors and actuators given by either the gradient minimization computation or brute-force searches. Moreover, these methods do not exploit the state-of-the-art in model reduction to optimize sensor and actuator placement.

K. Manohar was with the Department of Applied Mathematics, University of Washington, Seattle, WA, 98195 USA e-mail: kmanohar@uw.edu.

J. N. Kutz and S. L. Brunton are with University of Washington.

Contribution. This work develops a scalable sensor and actuator placement algorithm based on balanced truncation [29], in which modes are hierarchically ordered by their observability and controllability. We extend EIM-based methods to sample the low-rank balancing transformations and find maximally observable and controllable locations in state space. The resulting locations correspond to near-optimal point sensor and actuator configurations. The quality of our optimized configurations are evaluated using H_2 norms of the resulting system, which measure its degree of controllability and observability. The runtime scales linearly with the number of state variables, after a one-time offline computation of the balancing transformation, which is less expensive than iterative alternatives. We demonstrate our method on random state space systems and on the linearized Ginzburg-Landau equation with stochastic disturbances. The resulting sensor and actuator configurations reproduce known optimal locations at a fraction of the computational cost associated with competing convex and gradient optimization methods.

II. PROBLEM SETUP

Consider the following stable linear time-invariant system with a given state-space realization

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^q \quad (1a)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \mathbf{y} \in \mathbb{R}^p, \quad (1b)$$

with q inputs and p outputs. Sensors and actuators will be constrained to be sparse, localized points in state-space. Thus, the sensing and actuation matrices $\mathbf{C} \in \mathbb{R}^{p \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times q}$ must be structured in the following way

$$\mathbf{C} = [\mathbf{e}_{\gamma_1} \quad \mathbf{e}_{\gamma_2} \quad \dots \quad \mathbf{e}_{\gamma_p}]^T \quad (2a)$$

$$\mathbf{B} = [\mathbf{e}_{\beta_1} \quad \mathbf{e}_{\beta_2} \quad \dots \quad \mathbf{e}_{\beta_q}], \quad (2b)$$

where $\mathbf{e}_j$ are the canonical basis vectors for $\mathbb{R}^n$ with a unit entry at index j and zeros elsewhere. Thus the observations in $\mathbf{y}$ consist of p elements selected from $\mathbf{x}$

$$\mathbf{y} = \mathbf{C}\mathbf{x} = [x_{\gamma_1} \quad x_{\gamma_2} \quad \dots \quad x_{\gamma_p}]^T, \quad (3)$$

where $\gamma = \{\gamma_1, \dots, \gamma_p\} \subset \{1, \dots, n\}$ denotes the index set of sensor locations with cardinality $|\gamma| = p$. Similarly, actuator selection indices are given by $\beta = \{\beta_1, \dots, \beta_q\}$. A system with full actuation and sensing corresponds to $\mathbf{B} = \mathbf{C} = \mathbf{I}$. Our sensor & actuator placement problem is posed as follows:

Given a limited budget of $p \ll n$ sensors and $q \ll n$ actuators, what are the best states to observe and actuate?

To answer this question, we must first quantify the degree of observability and controllability for a given set of sensors and actuators, i.e. for a given choice of $\mathbf{C}$ and $\mathbf{B}$. The matrices $\mathbf{C}$ and $\mathbf{B}$ consist of selected rows and columns of the $n \times n$ identity matrix. Optimizing over these directly involves integer programming, and thus a heuristic method must be introduced to make this tractable for high-dimensional systems.

A. Observability and controllability

The degrees of observability and controllability for the state-space system (1) are quantified by the observability Gramian

$$\mathbf{W}_o = \int_0^\infty e^{\mathbf{A}^*t} \mathbf{C}^* \mathbf{C} e^{\mathbf{A}t} dt, \quad (4)$$

and the controllability Gramian

$$\mathbf{W}_c = \int_0^\infty e^{\mathbf{A}t} \mathbf{B} \mathbf{B}^* e^{\mathbf{A}^*t} dt. \quad (5)$$

Explicitly, the maximal energy output from a given initial condition $\mathbf{x}_0$ is quantified by the observability Gramian

$$\begin{aligned} \max \|\mathbf{y}\|^2 &= \int_0^\infty \mathbf{y}(t)^* \mathbf{y}(t) dt \\ &= \int_0^\infty (\mathbf{C} e^{\mathbf{A}t} \mathbf{x}_0)^* \mathbf{C} e^{\mathbf{A}t} \mathbf{x}_0 dt \\ &= \mathbf{x}_0^* \mathbf{W}_o \mathbf{x}_0. \end{aligned} \quad (6)$$

Likewise, the minimal energy required to steer a system to a given state is defined by the inverse controllability Gramian

$$\min \|\mathbf{u}\|_2 = \mathbf{x}_0^* \mathbf{W}_c^{-1} \mathbf{x}_0. \quad (7)$$

The Gramians define ellipsoids that vary with the sensing matrix $\mathbf{C}$ and actuation matrix $\mathbf{B}$ (see Fig. 2, top left). Therefore, they can be used as an evaluation metric to quantify the degree of observability or controllability resulting from a given sensor or actuator placement.

B. H_2 norm metric

The H_2 norm is frequently employed as an evaluation metric for a given set of sensors and actuators. It measures the L_2 norm or root mean square of the impulse response of a system, sometimes called its *output energy*. Given an impulsive input $u_j = \delta(0)$, the output response in component i is $y_{ij}(t) = \mathbf{C}_i e^{\mathbf{A}t} \mathbf{B}_j$. Therefore, the H_2 norm of (1) is

$$\begin{aligned} \|\cdot\|_2^2 &= \int_0^\infty \text{tr}(\mathbf{C} e^{\mathbf{A}t} \mathbf{B} \mathbf{B}^* e^{\mathbf{A}^*t} \mathbf{C}^*) dt \\ &= \text{tr} \left[\mathbf{C} \left(\int_0^\infty e^{\mathbf{A}t} \mathbf{B} \mathbf{B}^* e^{\mathbf{A}^*t} dt \right) \mathbf{C}^* \right] \\ &= \text{tr}(\mathbf{C} \mathbf{W}_c \mathbf{C}^*). \end{aligned} \quad (8)$$

By symmetry of trace, the H_2 norm is equivalently defined by

$$\begin{aligned} \|\cdot\|_2^2 &= \text{tr} \left[\mathbf{B}^* \left(\int_0^\infty e^{\mathbf{A}^*t} \mathbf{C}^* \mathbf{C} e^{\mathbf{A}t} dt \right) \mathbf{B} \right] \\ &= \text{tr}(\mathbf{B}^* \mathbf{W}_o \mathbf{B}). \end{aligned} \quad (9)$$

In the context of sensor and actuator placement, the H_2 norm provides an objective function to be minimized over different choices of sensors and actuators, $\mathbf{C}$ and $\mathbf{B}$. In optimal control, it is desirable that the output energy for given impulse is made as small as possible, since this determines the ability of the controller to quickly stabilize a system upon excitement.

C. A related metric

A closely related metric to the H_2 norm is given by

$$\log \det(\mathbf{C}\mathbf{W}_c\mathbf{C}^*). \quad (10)$$

Summers et al [11] show that maximizing this log determinant is an effective proxy for H_2 norm minimization for choosing optimal actuators. We propose using this as an evaluation metric because it facilitates the efficient, greedy determinant maximization scheme detailed in Section IV.

In [11] the log determinant is shown to be a submodular, monotonically increasing function of the set of actuators, which allows greedy actuator selection:

$$\mathbf{B}_\star = \operatorname{argmax}_{\mathbf{B}} \log \det(\mathbf{C}\mathbf{W}_c(\mathbf{B})\mathbf{C}^*). \quad (11)$$

The drawback of this approach is having to recompute the Gramian in each iteration of the greedy algorithm. Each calculation of the Gramian requires $\mathcal{O}(n^3)$ operations due to an intermediate Cholesky factorization. This cubic scaling may be intractable for high-dimensional systems with large n .

D. Our approach

Our solution is to decouple sensor/actuator optimization from the Gramian calculation, which is performed a single time. We do this by optimizing *sensors* using the full controllability Gramian for $\mathbf{B} = \mathbb{I}$, holding the actuators fixed

$$\mathbf{C}_\star = \operatorname{argmax}_{\mathbf{C}} \log \det(\mathbf{C}\mathbf{W}_c\mathbf{C}^*), \quad (12)$$

omitting the dependence of $\mathbf{W}_c$ on $\mathbf{B}$ so that $\mathbf{W}_c$ need not be recomputed at each step of the optimization. Likewise, we optimize *actuators* using the observability Gramian for $\mathbf{C} = \mathbb{I}$, holding the sensors fixed

$$\mathbf{B}_\star = \operatorname{argmax}_{\mathbf{B}} \log \det(\mathbf{B}^*\mathbf{W}_o\mathbf{B}), \quad (13)$$

in lieu of the trace metric (9). Now, the Gramians no longer depend on the optimization variable and need only be computed once, and both objectives are still fundamentally linked to the H_2 norm of the system. Critically, we will extract the dominant controllable and observable subspaces from a *balanced* coordinate transformation of the Gramians.

III. BALANCED MODEL REDUCTION

Many systems of interest are exceedingly high dimensional, making them difficult to characterize and limiting controller robustness due to significant computational time-delays. However, even if the ambient dimension is large, there may still be a few dominant coherent structures that characterize the system. Thus, significant effort has gone into obtaining efficient reduced-order models that capture the most relevant mechanisms for use in real-time feedback control [1].

The primary goal of model reduction is to find a coordinate transformation $\mathbf{x} \approx \Psi_r \mathbf{a}_r$ giving rise to a related system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$ with similar input-output characteristics, in terms of a state $\mathbf{a}_r \in \mathbb{R}^r$ in a rank- r basis $\Psi_r \in \mathbb{R}^{n \times r}$ with many fewer degrees of freedom, $r \ll n$. Instead of ordering

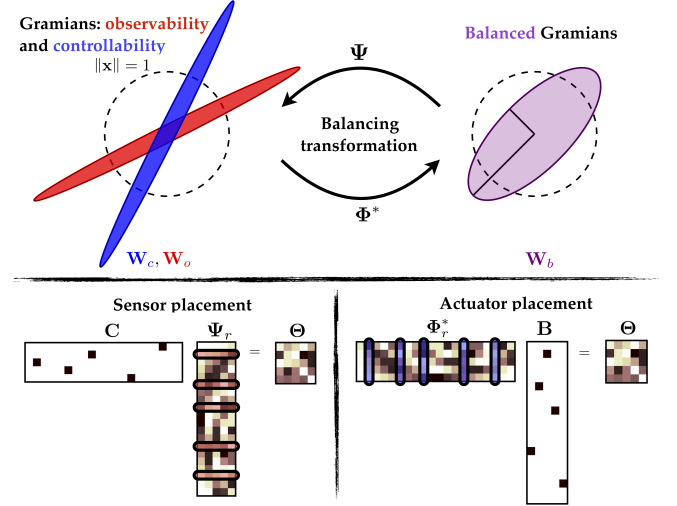


Fig. 2: (top) Illustration of the balancing transformation for Gramians. The reachable set with unit control input is shown in blue, given by $\mathbf{W}_c^{1/2}\mathbf{x}$ for $\|\mathbf{x}\| = 1$. The corresponding observable set is shown in red. Under the balancing transformation Ψ , the Gramians are equal, shown in purple. (bottom) Sensor and actuator placement based on balancing transformation.

modes based on energy, as in POD [17], balanced model reduction identifies a coordinate transformation $\mathbf{x} = \Psi \mathbf{a}$ that hierarchically orders modes to capture the input-output characteristics of the system, as quantified by the controllability and observability Gramians. In the seminal paper by Moore in 1981 [29], he showed that it is possible to find a coordinate system Ψ where the controllability and observability Gramians are equal and diagonal. This results in the balanced model:

$$\begin{aligned} \dot{\mathbf{a}} &= \Phi^* \mathbf{A} \Psi \mathbf{a} + \Phi^* \mathbf{B} \mathbf{u} & \mathbf{a} &\in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^p \\ \mathbf{y} &= \mathbf{C} \Psi \mathbf{a}, & \mathbf{y} &\in \mathbb{R}^q \end{aligned} \quad (14)$$

where Ψ are called *direct* modes and $\Phi^* = \Psi^{-1}$ are called the *adjoint* modes. Note that $\mathbf{u}$ and $\mathbf{y}$ are the same as in (1), even though the system state has been reduced.

The balanced state $\mathbf{a}$ is then *truncated*, keeping only the first $r \ll n$ most jointly controllable and observable states in $\mathbf{a}_r$, so that $\mathbf{x} \approx \Psi_r \mathbf{a}_r$, resulting in the following *balanced truncation* model [29]:

$$\begin{aligned} \dot{\mathbf{a}}_r &= \Phi_r^* \mathbf{A} \Psi_r \mathbf{a}_r + \Phi_r^* \mathbf{B} \mathbf{u} & \mathbf{a}_r &\in \mathbb{R}^r, r \ll n \\ \mathbf{y} &= \mathbf{C} \Psi_r \mathbf{a}_r. \end{aligned} \quad (15)$$

Note that this reduced-order model is formulated in terms of the original system matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and the first r columns of the direct and adjoint modes, Ψ_r and Φ_r^* , respectively.

The controllability and observability Gramians each establish an inner product on state space in terms of how controllable or observable a given state is, respectively. As such, Gramians depend on the particular choice of coordinate system and will transform under a change of coordinates. In the coordinate system $\mathbf{a}$, the controllability Gramian becomes:

$$\tilde{\mathbf{W}}_c = \Phi^* \mathbf{W}_c \Phi. \quad (16)$$

The observability Gramian transforms similarly:

$$\tilde{\mathbf{W}}_o = \Psi^* \mathbf{W}_o \Psi. \quad (17)$$

The coordinate transformation Ψ that makes the controllability and observability Gramians equal and diagonal,

$$\tilde{\mathbf{W}}_c = \tilde{\mathbf{W}}_o = \Sigma, \quad (18)$$

is given by the matrix of eigenvectors of the product of the Gramians $\mathbf{W}_c \mathbf{W}_o$ in the original coordinates:

$$\tilde{\mathbf{W}}_c \tilde{\mathbf{W}}_o = \Phi^* \mathbf{W}_c \mathbf{W}_o \Psi = \Sigma^2 \implies \mathbf{W}_c \mathbf{W}_o \Psi = \Psi \Sigma^2. \quad (19)$$

A. Computational algorithms and history

In practice, computing the Gramians $\mathbf{W}_c$ and $\mathbf{W}_o$ and the eigendecomposition of the product $\mathbf{W}_c \mathbf{W}_o$ in (19) may be prohibitively expensive for high-dimensional systems. Instead, the balancing transformation may be approximated with data from impulse responses of the direct and adjoint systems, utilizing the singular value decomposition for efficient extraction of the relevant subspaces. The method of empirical Gramians is quite efficient and is widely used [29], [30], [31], [32]. Moore's approach computes the entire $n \times n$ balancing transformation, which is not suitable for exceedingly high-dimensional systems. In 2002, Willcox and Peraire [31] generalized the method to high-dimensional systems, introducing a variant based on the rank- r decompositions of $\mathbf{W}_c$ and $\mathbf{W}_o$ obtained from snapshots of direct and adjoint simulations. It is then possible to compute the eigendecomposition of $\mathbf{W}_c \mathbf{W}_o$ using efficient eigenvalue solvers. This approach requires as many adjoint impulse-response simulations as the number of output equations, which may be prohibitively large for full-state measurements. In 2005, Rowley [32] addressed this issue by introducing output projection, which limits the number of adjoint simulations to the number of relevant POD modes in the data. For a complete treatment of balanced model reduction, see Antoulas [33].

IV. SENSOR & ACTUATOR OPTIMIZATION VIA QR PIVOTING

We now motivate and describe an efficient matrix pivoting algorithm to optimize the log determinant over the sensors and actuators, described below and in algorithm 1. Here the representation of the Gramians in balanced truncation coordinates plays a crucial role.

A. Matrix volume objective

Recall the goal of optimizing a set of p point sensors and q actuators out of n possible choices. Here we make the additional assumption that $p \geq r$ and $q \geq r$, where r is the number of balanced modes required to faithfully approximate the full-order model. We begin with the balanced truncation (15) of a state-space system (1), and assume that the observable and controllable subspaces are well characterized by r direct and adjoint modes Ψ_r and Φ_r . The balancing transformation is computed for $\mathbf{B} = \mathbb{I}$ and $\mathbf{C} = \mathbb{I}$, to include the effect of all possible sensors and actuators.

Summers et al [11] show that it suffices to only consider controllable or observable subspaces for selecting sensors and actuators using the log determinant objective. Thus, we can

substitute the rank- r balanced approximation of the Gramian into the log determinant term

$$\begin{aligned} \mathbf{C}_* &= \underset{\mathbf{C}}{\operatorname{argmax}} \log \det(\mathbf{C} \hat{\mathbf{W}}_c \mathbf{C}^*) \\ &= \underset{\mathbf{C}}{\operatorname{argmax}} \log \det(\mathbf{C} \Psi_r \Sigma_r \Psi_r^* \mathbf{C}^*) \\ &= \underset{\mathbf{C}}{\operatorname{argmax}} \det(\mathbf{C} \Psi_r \Sigma_r (\mathbf{C} \Psi_r)^*) \\ &= \underset{\mathbf{C}}{\operatorname{argmax}} (\det \mathbf{C} \Psi_r)^2 \cdot \det \Sigma_r \\ &= \underset{\mathbf{C}}{\operatorname{argmax}} |\det \mathbf{C} \Psi_r|. \end{aligned}$$

By monotonicity of log, the maximizer of the log determinant simplifies to the maximizer of the determinant. The rest follows by applying the product property of determinants for square matrices, then omitting the term that is independent of the sensors, i.e. $\det \Sigma_r$. Likewise, in the actuator case, the objective $\underset{\mathbf{B}}{\operatorname{argmax}} \log \det(\mathbf{B}^* \hat{\mathbf{W}}_o \mathbf{B})$ simplifies to

$$\mathbf{B}_* = \underset{\mathbf{B}}{\operatorname{argmax}} |\det \mathbf{B}^T \Phi|. \quad (20)$$

Consider for now the case of sensor placement. The absolute determinant is a measure of matrix *volume*, and $\mathbf{C}$ is a row selection matrix. The transformed objectives may be viewed as a *submatrix volume maximization* problem, which involves choosing the optimal r -row selection of direct modes Ψ_r with the largest possible determinant. Finding this optimum is an NP-hard, brute-force search over the rows of Ψ_r , which scales combinatorially with the dimension n . However, it can be optimized greedily and efficiently via a one-time matrix QR factorization requiring $\mathcal{O}(nr^2)$ operations, as described next.

B. QR pivoting algorithm

The QR factorization with column pivoting is a greedy submatrix volume optimization scheme that we will use to construct $\mathbf{C}$ and $\mathbf{B}$, given Ψ_r and Φ_r . For any input matrix $\mathbf{V} \in \mathbb{R}^{r \times n}$, the pivoted QR algorithm factors $\mathbf{V}$ into a unitary matrix $\mathbf{Q}$, and upper-triangular matrix $\mathbf{R}$, and column permutation matrix $\mathbf{P}$

$$\mathbf{V}\mathbf{P} = \mathbf{Q}\mathbf{R}. \quad (21)$$

This factorization provides a numerically stable way to compute the determinant as the product of diagonal entries in $\mathbf{R}$

$$|\det \mathbf{V}| = |\det \mathbf{R}| = \prod_{i=1}^r |\mathbf{R}_{ii}|. \quad (22)$$

In each iteration, orthogonal projections are applied to successive columns of $\mathbf{V}$ to introduce subdiagonal zeros in $\mathbf{R}$. For our purposes, $\mathbf{P}$ plays the crucial role: at each step $\mathbf{P}$ stores the column “pivot” index of the column selected at each iteration to guarantee the following *diagonally dominant* structure in $\mathbf{R}$

$$|\mathbf{R}_{ii}|^2 \geq \sum_{j=i}^k |\mathbf{R}_{jk}|^2; \quad 1 \leq i \leq k \leq n. \quad (23)$$

Because the product of these diagonal entries equals the determinant, it can be seen that the pivoting is precisely the greedy procedure needed to optimize the determinant.

Algorithm 1 Businger-Golub QR column pivoting of $\mathbf{V} \in \mathbb{R}^{r \times n}$.

Input: $\mathbf{V}$, p **Output:** γ

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1:  $\gamma \leftarrow []$ 
2: for  $k = 1, \dots, p$  do
3:    $\gamma_k = \operatorname{argmax}_{j \notin \gamma} \|\mathbf{v}_j\|_2$   $\triangleright \mathbf{v}_j$  is the  $j$ th column of  $\mathbf{V}$ 
4:   Compute Householder  $\tilde{\mathbf{Q}}$  such that  $\tilde{\mathbf{Q}} \cdot [v_{kk}, \dots, v_{rk}]^T = [\mathbf{R}_{kk}, 0, \dots, 0]^T$ 
5:    $\mathbf{V} \leftarrow \operatorname{diag}(\mathbf{I}_{k-1}, \tilde{\mathbf{Q}}) \cdot \mathbf{V}$   $\triangleright$  remove from all columns the orthogonal projection onto  $\mathbf{v}_{\gamma_k}$ 
6:    $\gamma \leftarrow [\gamma, \gamma_k]$ 
7: end for
8: return  $\gamma$ 

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To see this, consider the following pivoted QR factorization

$$\Psi_r^* \mathbf{P} = \mathbf{Q} \mathbf{R}. \quad (24)$$

Here, column pivoting of the transpose (row selection of direct modes) corresponds to point sensor selection from the ambient high-dimensional space. Thus $\mathbf{C}$ can be constructed from the first r columns of $\mathbf{P}$ transposed, as in algorithm 1,

$$\mathbf{C} \triangleq \mathbf{P}_{:,j}^T, \text{ where } j : 1 \rightarrow r. \quad (25)$$

Similarly, the actuator placement problem reduces to maximizing the determinant of the term $\mathbf{B}^* \Phi_r$ via submatrix selection, which is computed using one additional QR factorization, this time of the adjoint modes

$$\Phi_r^* \tilde{\mathbf{P}} = \tilde{\mathbf{Q}} \tilde{\mathbf{R}}. \quad (26)$$

Subsequently $\mathbf{B}$ is constructed from the first r columns of $\tilde{\mathbf{P}}$, as in Algorithm 1,

$$\mathbf{B} \triangleq \tilde{\mathbf{P}}_{:,j}, \text{ where } j : 1 \rightarrow r. \quad (27)$$

The QR pivoting procedure is summarized in algorithm 1. Here, we use classical Businger-Golub pivoting [34], which applies Householder projections of the input matrix to introduce subdiagonal zeros in $\mathbf{R}$. In practice, more robust and efficient rank-revealing routines are widely implemented in scientific computing libraries. The operation can be accelerated further using randomization to select pivots [24], [35], [36].

Historical significance. It is well-known that the pivoted QR factorization of an input matrix, $\Psi_r^* \mathbf{P}$, is numerically well conditioned for Gaussian elimination and solving least-squares problems [34]. The idea of using the pivoting matrix $\mathbf{C}$ to select interpolation points for POD modes is first proposed in [24], and its use for sensor placement is detailed in [21]. Due to its favorable numerical properties, this permutation matrix is used to approximate optimal interpolation points for polynomial interpolation [37].

The performance of QR pivot sensors can be analyzed via their ability to recover the high-dimensional state from partial observations through dynamic estimation. Consider the case of sensing in isolation, without actuation. The full state estimate is defined by a projection onto direct modes that depends on the chosen sensors $\mathbf{C}$. The resulting partial observations can be expressed in terms of the balanced coordinates $\mathbf{a}_r$ as

$$\mathbf{y} = \mathbf{C} \mathbf{x} \approx \mathbf{C} \Psi_r \mathbf{a}_r. \quad (28)$$

The best estimate of the balanced state from observations, in the least squares sense, is given by

$$\hat{\mathbf{a}}_r = (\mathbf{C} \Psi_r)^{-1} \mathbf{y}, \quad (29)$$

and the high-dimensional state estimate can be computed as

$$\hat{\mathbf{x}} = \Psi_r (\mathbf{C} \Psi_r)^{-1} \mathbf{y} = \Psi_r (\mathbf{C} \Psi_r)^{-1} \mathbf{C} \mathbf{x}. \quad (30)$$

This can be expressed as a projection $\mathbb{P}_C$ of the true state $\mathbf{x}$ into the observable subspace

$$\mathbb{P}_C \triangleq \Psi_r (\mathbf{C} \Psi_r)^{-1} \mathbf{C}. \quad (31)$$

As we shall see, the upper bound on the approximation error

$$\|\mathbf{x} - \Psi_r (\mathbf{C} \Psi_r)^{-1} \mathbf{C} \mathbf{x}\|_2 \quad (32)$$

is determined by $\|(\mathbf{C} \Psi_r)^{-1}\|_2 = 1/|\mathbf{R}_{rr}|$. Controlling the growth of this term is closely related to maximizing the determinant. In fact, the two objectives are known as *A-optimal* and *D-optimal* criteria in experiment design.

V. ANALYSIS

We now present lower and upper bounds for the approximation error (32) given our choices of $\mathbf{C}$ (25) and $\mathbf{B}$ (27). These bounds rely on the fact that in the case of full observation, $\mathbf{C} = \mathbb{I}$, the projection (31) is simply the approximation in balanced coordinates

$$\mathbf{x}_* \triangleq \Psi_r \Phi_r^* \mathbf{x}, \quad (33)$$

resulting in near-optimal bounds on the approximation error.

Lemma 1 (Antoulas [33]): The ℓ_2 error between the full state and its balanced approximation is bounded by

$$\|\mathbf{x} - \mathbf{x}_*\|_2 \leq 2(\sigma_{r+1} + \dots + \sigma_n), \quad (34)$$

where σ_k are the Hankel singular values, given by the diagonal elements of Σ in (18). An equivalent result holds for the adjoint system, for which the adjoint state projection onto balanced coordinates is given by $\mathbf{z}_* \triangleq \Phi_r \Psi_r^* \mathbf{z}$, and

$$\|\mathbf{z} - \mathbf{z}_*\|_2 \leq 2(\sigma_{r+1} + \dots + \sigma_n). \quad (35)$$

Lemma 2 (Drmac & Gugercin [24]): For any full-rank matrix $\Psi_r \in \mathbb{R}^{n \times r}$ and its submatrix constructed from the first r rows selected by its column permutation matrix, $\mathbf{P}_r = \mathbf{P}_{:,j}, j : 1 \rightarrow r$, from QR factorization (25), the spectral norm of $(\mathbf{P}_r^T \Psi_r)^{-1}$ is bounded by

$$\|(\mathbf{P}_r^T \Psi_r)^{-1}\|_2 = \|(\Psi_r^* \mathbf{P}_r)^{-1}\|_2 \leq \frac{\sqrt{n-r+1}}{\sigma_{\min}(\Psi_r)} \frac{\sqrt{4^r + 6r - 1}}{3}, \quad (36)$$

where the bound grows as $\sqrt{n} \mathcal{O}(2^r)$.

Theorem 3: For any r -truncated row permutation matrix $\mathbf{C}$, the projection error (32) satisfies the following upper bound

$$\|\mathbf{x} - \mathbb{P}_C \mathbf{x}\|_2 \leq \|\Psi_r\|_2 \|(\mathbf{C} \Psi_r)^{-1}\|_2 \|\mathbf{x} - \mathbf{x}_*\|_2. \quad (37)$$

Proof: Define the residual between full state and its projection into balanced coordinates as $\mathbf{v} = \mathbf{x} - \mathbf{x}_*$. Then

$$\begin{aligned} \mathbb{P}_C \mathbf{v} &= \mathbb{P}_C \mathbf{x} - \mathbb{P}_C \mathbf{x}_* \\ &= \mathbb{P}_C \mathbf{x} - \Psi_r (\mathbf{C} \Psi_r)^{-1} \mathbf{C} \Psi_r \Phi_r^* \mathbf{x}_* \\ &= \mathbb{P}_C \mathbf{x} - \Psi_r \Phi_r^* \mathbf{x}_* \\ &= \mathbb{P}_C \mathbf{x} - \mathbf{x}_*, \end{aligned}$$

where we use the fact that the orthogonal projection of $\mathbf{x}_\star$ onto $\mathcal{R}(\Psi_r)$ is $\mathbf{x}_\star$ again.

$$\begin{aligned}\|\mathbf{x} - \mathbb{P}_C \mathbf{x}\|_2 &= \|(\mathbf{v} + \mathbf{x}_\star) - (\mathbb{P}_C \mathbf{v} + \mathbf{x}_\star)\|_2 \\ &= \|(\mathbb{I} - \mathbb{P}_C) \mathbf{v}\|_2 \\ &\leq \|\mathbb{P}_C\|_2 \|\mathbf{x} - \mathbf{x}_\star\|_2 \\ &\leq \|\Psi_r\|_2 \|(\mathbf{C} \Psi_r)^{-1}\|_2 \|\mathbf{C}\|_2 \|\mathbf{x} - \mathbf{x}_\star\|_2 \\ &= \|\Psi_r\|_2 \|(\mathbf{C} \Psi_r)^{-1}\|_2 \|\mathbf{x} - \mathbf{x}_\star\|_2.\end{aligned}$$

Our logic here closely follows that of Chaturantabud and Sorensen [23], in which Ψ_r are restricted to be orthogonal. This property is not true in general for balanced modes, so $\|\Psi_r\|_2$ is not necessarily equal to 1.

Theorem 4: The upper bound for the error between the full state and the projection with QR pivot observations (25) is controlled by the sum of the discarded Hankel singular values and the condition number of direct modes $\kappa(\Psi_r)$, where σ_i represents the i th Hankel singular value of the system

$$\|\mathbf{x} - \mathbb{P}_C \mathbf{x}\|_2 \leq 2\alpha \sum_{i=r+1}^n \sigma_i, \quad (38a)$$

$$\text{where } \alpha = \kappa(\Psi_r) \sqrt{n} \mathcal{O}(2^r). \quad (38b)$$

Proof: Combining Lemmas 3 and 4 with Theorem 1:

$$\begin{aligned}\|\mathbf{x} - \mathbb{P}_C \mathbf{x}\|_2 &\leq \|\Psi_r\|_2 \|(\mathbf{C} \Psi_r)^{-1}\|_2 \left(2 \sum_{i=r+1}^n \sigma_i \right) \\ &\leq \|\Psi_r\|_2 \frac{\sqrt{n-r+1} \sqrt{4^r+6r-1}}{\sigma_{\min}(\Psi_r) 3} \left(2 \sum_{i=r+1}^n \sigma_i \right) \\ &= \kappa(\Psi_r) \sqrt{n} \mathcal{O}(2^r) \sum_{i=r+1}^n 2\sigma_i.\end{aligned}$$

A. Actuator placement

Actuator placement is completely analogous to sensor placement, where we construct $\mathbf{B}$ from QR pivoting of the adjoint modes Φ_r instead of the direct modes. Thus, we seek to place actuators along the most controllable directions in state space. Importantly, the controllability Gramian in the original system is equivalent to the observability Gramian of the adjoint system

$$\dot{\mathbf{z}} = -\mathbf{A}^* \mathbf{z} - \mathbf{C}^* \tilde{\mathbf{y}} \quad (39a)$$

$$\tilde{\mathbf{u}} = \mathbf{B}^* \mathbf{z}. \quad (39b)$$

As before, we desire the best actuation matrix $\mathbf{B}$ in the adjoint system for estimating the adjoint state $\mathbf{z}$. The adjoint system transforms via the same balanced transformation from above

$$\dot{\mathbf{b}}_r = -\Psi_r^* \mathbf{A}^* \Phi_r \mathbf{b}_r \quad (40a)$$

$$\tilde{\mathbf{u}} = \mathbf{B}^* \Phi_r \mathbf{b}_r. \quad (40b)$$

As previously, the adjoint state may be approximated in balanced coordinates using an analogous projection operator

$$\hat{\mathbf{z}} = \Phi_r (\mathbf{B}^* \Phi_r)^{-1} \mathbf{B}^* \mathbf{z} = \mathbb{P}_{B^*} \mathbf{z}. \quad (41)$$

Substituting Φ_r for Ψ_r , $\mathbf{z}$ for $\mathbf{x}$, and $\mathbf{B}^*$ for $\mathbf{C}$ in the proof for Theorem 1, we obtain the following corollary.

Corollary 1: For any r -truncated row permutation matrix $\mathbf{B}^*$, the error from the projection (41) satisfies the following upper bound

$$\|\mathbf{z} - \mathbb{P}_{B^*} \mathbf{z}\|_2 \leq \|\Phi_r\|_2 \|(\mathbf{B}^* \Phi_r)^{-1}\|_2 \|\mathbf{z} - \mathbf{z}_\star\|_2. \quad (42)$$

Corollary 2: The upper bound for the error between the adjoint state and projection with QR pivot actuators (27) is controlled by the sum of the discarded Hankel singular values and the condition number of adjoint modes $\kappa(\Phi_r)$, where σ_i represents the i th Hankel singular value of the system

$$\|\mathbf{z} - \mathbb{P}_{B^*} \mathbf{z}\|_2 \leq 2\alpha \sum_{i=r+1}^n \sigma_i, \quad (43a)$$

$$\text{where } \alpha = \kappa(\Phi_r) \sqrt{n} \mathcal{O}(2^r). \quad (43b)$$

Proof: This result immediately follows from the previous corollary and upon substituting Φ_r for Ψ_r in Lemma 4. ■

The above results are for the case when the number of sensors or actuators equal the rank of the truncated model ($p = q = r$), but they generalize to oversampling because the same upper bounds still hold for $p > r$ and $q > r$.

B. Log determinant objective

We now relate the approximation error bounds using QR pivot sensors and actuators to the log determinant objectives. Specifically lower bounds are established for the maximized objective function value.

Theorem 5: Given direct modes Ψ_r , QR pivot sensors $\mathbf{C}$ guarantee the following lower bound for the log determinant

$$r \log \frac{9\sigma_{\min}(\Psi_r)}{(n-r+1)(4^r+6r-1)} + \sum_{i=1}^r \log \sigma_i \leq \log \det \mathbf{C} \hat{\mathbf{W}}_c \mathbf{C}^T.$$

Proof: Noting the relationship between the singular values of a matrix and its QR factorization, we can express $\det \mathbf{C} \Psi_r$ in terms of the diagonal entries of its $\mathbf{R}$ factor

$$\begin{aligned}|\det \mathbf{C} \Psi_r| &= \sigma_1(\mathbf{C} \Psi_r) \cdot \sigma_2(\mathbf{C} \Psi_r) \cdot \dots \cdot \sigma_r(\mathbf{C} \Psi_r) \\ &= |\mathbf{R}_{11} \cdot \mathbf{R}_{22} \cdot \dots \cdot \mathbf{R}_{rr}| \geq |\mathbf{R}_{rr}|^r, \quad (44)\end{aligned}$$

due to the ordering of singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$. By squaring the inequality and multiplying by $\det \Sigma_r$ we obtain

$$\begin{aligned}\mathbf{R}_{rr}^{2r} \cdot \det \Sigma_r &\leq (\det \mathbf{C} \Psi_r)^2 \cdot \det \Sigma_r \\ &= \det(\mathbf{C} \Psi_r \Sigma_r \Psi_r^* \mathbf{C}^*) = \det \mathbf{C} \hat{\mathbf{W}}_c \mathbf{C}^T,\end{aligned}$$

where taking logarithms yields

$$r \log \mathbf{R}_{rr}^2 + \sum_{i=1}^r \log \sigma_i \leq \log \det \mathbf{C} \hat{\mathbf{W}}_c \mathbf{C}^T.$$

Because $\|(\mathbf{C} \Psi_r)^{-1}\|_2 = 1/|\mathbf{R}_{rr}|$, the upper bound (36) in Lemma 2 is the inverse lower bound for $|\mathbf{R}_{rr}|$, which can now be substituted above to obtain the final result. ■

An analogous lower bound can be obtained for the objective using QR pivot actuators by appropriately substituting $\mathbf{B}^T$, $\tilde{\mathbf{R}}$ and adjoint modes Φ_r in the above proof.

Corollary 3: Given adjoint modes Φ_r , QR pivot actuators $\mathbf{B}$ guarantee the the following lower bound for the log determinant

$$r \log \frac{9\sigma_{\min}(\Phi_r)}{(n-r+1)(4^r+6r-1)} + \sum_{i=1}^r \log \sigma_i \leq \log \det \mathbf{B}^T \hat{\mathbf{W}}_o \mathbf{B}.$$

This final step connects the ℓ_2 approximation error bound to our H_2 norm proxy objective, and establishes QR pivoting as a viable choice for optimal sensor and actuator placement.

VI. RESULTS

This section demonstrates our sensor and actuator placement algorithm on two examples. The first is a random state-space model, for which we consider sensor placement and actuator placement independently, and then joint sensor and actuator placement. The resulting configurations are analyzed in terms of the corresponding log determinant objective. Next, we perform sensor and actuator placement on a linearized Ginzburg-Landau equation with stochastic disturbances, for which the H_2 optimal sensor and actuator placement has been previously determined by Chen and Rowley [25]. We compare the QR pivoting method with their more expensive gradient descent method, and evaluate both sets of placements using their H_2 norm and LQG controller gain.

A. Discrete random state space

For our first example, we investigate sparse sensor and actuator placement on random discrete-time state-space systems, generated using the Matlab command `drss`. First, we empirically compare the results of QR sensor placement against a brute-force search across all possible sensor placements on a system with $n = 25$ states, $p = 7$ point sensors, and full-state actuation $\mathbf{B} = \mathbf{I}$. We evaluate the log determinant objective $\log \det \mathbf{C} \mathbf{W}_c \mathbf{C}^*$ for all possible choices of 7 point sensors, since the system is small enough for the Gramian to be explicitly computed for all $\binom{n}{p} = 480,700$ choices of $\mathbf{C}$. The histogram of these values is plotted in Fig. 3, along with the value resulting from our method (red line). Note that the input to the QR algorithm, the balancing modes, are computed only once from the fully observed and actuated system. It can be seen that the sensors from our method are nearly optimal, with the resulting objective function values exceeding 99.99% of all others. After repeating the experiment for an ensemble of 500 of these randomly generated state-space systems, QR surpasses 99.95% of possible outcomes, on average. Thus, in practice, QR pivot configurations are much closer to optimal than the analysis suggests.

We now investigate performance on a larger random state-space model with $n = 100$ states, and likewise initialize the model with full actuation and sensing, i.e. $p = q = 100$ with $\mathbf{B} = \mathbf{C} = \mathbf{I}$. Figure 4 shows the log determinant objective that is being optimized for various sensor and actuator configurations. The log determinant of the Gramian volume is plotted for the truncated model with QR-optimized sensor and actuator configurations (red circles) and with random configurations (blue violin plots). In panels (a) and (b), the number of

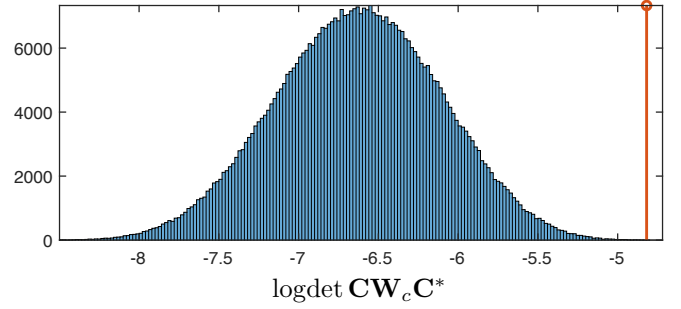


Fig. 3: Histogram of log determinant for a brute force combinatorial search over all possible configurations of seven sensors (blue) versus the QR optimized sensors (red).

balanced modes is fixed at $r = 5$, and the numbers of sensors and actuators are varied, using an extension of QR pivoting for the case when $p > r$ or $q > r$ first described in [21]. With limited sensing and actuation (a), the QR-optimized configurations dramatically outperform random configurations. This is less dramatic in the case of optimized partial sensing with full actuation (b), although the QR optimized configurations still outperform the random distribution. When the number of sensors and actuators are fixed at $p = q = 10$ in (c) we see that as more modes are retained, the chosen sensors and actuators better characterize the input–output dynamics, and their performance gap over random placement increases. This provides empirical confirmation of our approach, which determines placements for the balanced truncated system assuming that they are nearly optimal for the original system.

Because the system is randomly generated and the dynamics do not evolve according to broad, non-localized features in state-space, many sensors and actuators are required to characterize the system. In particular, this is reflected in the slow decay of Hankel singular values. By contrast, the next example is generated by a physical fluid flow model, and has coherent dynamics that allow for a more intuitive, visual interpretation of sensor and actuator placements with enhanced sparsity.

B. Linearized Ginzburg-Landau with stochastic disturbances

As a more sophisticated example, we consider the nonlinear Ginzburg-Landau equation, which models velocity perturbations in a given flow configuration. In the case of small perturbations, the flow is well-described by linearized equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{D}^{1/2}\mathbf{d} \quad (45a)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{N}^{1/2}\mathbf{n}, \quad (45b)$$

where $\mathbf{d}$ is a stochastic process disturbance and $\mathbf{n}$ is noise present everywhere in the domain. The linearized Ginzburg-Landau operator $\mathbf{A}$ is discretized using Hermite pseudospectral differences

$$\mathbf{A} \triangleq -\nu \frac{\partial}{\partial \xi} + \mu(\xi) + \beta \frac{\partial}{\partial \xi^2}. \quad (46)$$

Here ξ is the 1D spatial variable discretized at the roots of weighted Hermite polynomials, ν is the advection speed, β is the diffusion parameter, and $\mu(\xi)$ is the amplification factor. The H_2 optimal sensor and actuator placement for this system

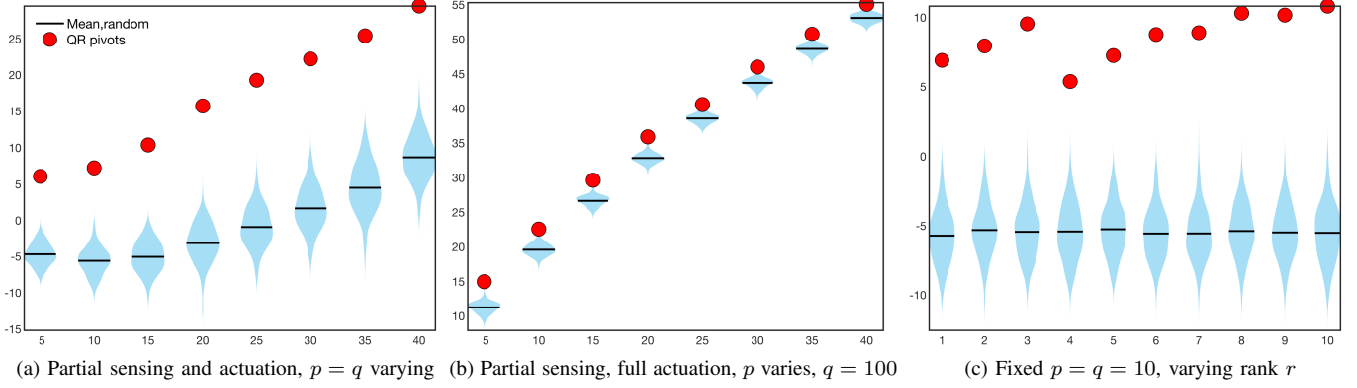


Fig. 4: Sensor and actuator placement in a random state-space system. The log determinant objective is plotted for various sensor and actuator configurations, including QR-optimized configurations (red) and an ensemble of 200 random sensor or actuator placements (blue violin plots). (a) The number of sensors and actuators, p and q , vary with the number of balanced modes fixed at $r = 5$. (b) The number of sensors p varies with full actuation $q = 100$ and $r = 5$ modes. (c) The number of sensors and actuators are fixed, $p = q = 10$, and the number of balanced modes r is varied. The QR-optimized configurations better capture the input-output energy of the original system compared with random placements.

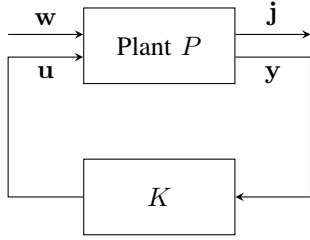


Fig. 5: LQG controller for linearized Ginzburg-Landau

has been determined by Chen and Rowley [25] using gradient minimization, and hence provides a benchmark comparison for our greedy placement.

The observability and controllability Gramians are not defined for this system since it is unstable. It can, however, be stabilized using linear quadratic Gaussian (LQG) control, which seeks to stabilize the system with minimal input using the cost function

$$\mathbf{J} = \mathbf{x}^* \hat{\mathbf{Q}} \mathbf{x} + \mathbf{u}^* \hat{\mathbf{R}} \mathbf{u}, \quad (47)$$

where $\hat{\mathbf{Q}}$ and $\hat{\mathbf{R}}$ refer to user-specified state and input weighting matrices, not to be confused with the matrices in the QR factorization. The structure of the controller is shown in Fig. 5. Here $\mathbf{j}$ and $\mathbf{w}$ are concatenated vectors containing terms in the cost function, and the concatenated sources of process and measurement disturbance, as defined below

$$\mathbf{j} = \begin{bmatrix} \hat{\mathbf{Q}}^{1/2} \mathbf{x} \\ \hat{\mathbf{R}}^{1/2} \mathbf{u} \end{bmatrix}, \mathbf{w} = \begin{bmatrix} \mathbf{d} \\ \mathbf{n} \end{bmatrix}.$$

By minimizing $\mathbf{J}$, the controller minimizes the gain from the disturbances to the weighted cost vectors.

The resulting system is stable, therefore we can compute the balancing transformations using Matlab's `balreal` command. Specifically, the new system matrices $\mathbf{A}_K$, $\mathbf{B}_K$, $\mathbf{C}_K$ are computed using Matlab's `lqg` routine, and its balanced modes are computed using `balreal`. Explicit expressions for the

controller system matrices are given in [25]. In their work, sensing and actuation are defined on smooth spatial Gaussian kernels centered at the sensor/actuator location, and the small-width limit of the kernels corresponds to point sensing and actuation. This permits derivative-based conjugate gradient minimization of the H_2 norm, as well as placement of sensors and actuators at locations that may not be grid points. The major drawback is that each Newton iteration requires $p + q$ solutions of $n \times n$ Lyapunov equations until convergence. Furthermore, the procedure requires optimizing an ensemble of random initial conditions to avoid converging to a local minimum. In [25], the optimal placement is computed using this conjugate gradient optimization for a spatial discretization corresponding to $n = 100$, and this method becomes computationally intractable as the grid resolution increases.

Figure 6 plots sensor and actuator configurations from the QR algorithm and QR-initialized gradient minimization, which may be compared with those from [25]. The resulting placements for the cases $p = q = 1$ to $p = q = 5$ sensors and actuators are plotted vertically, and the horizontal axis is the spatial domain extending from $\xi \in [-12, 12]$, with the wave amplification region shown in gray. For each value of p , we apply QR pivoting to the balanced modes truncated at rank $r = p$. Notably, QR pivoting collocates sensors and actuators, and this agrees with a similar gradient optimization procedure that collocates placements for vibration control [38], [39]. The decoupling of sensor and actuator placement also permits enforcing that sensors are placed away from actuators – an easy modification of the QR pivoting procedure that resembles the optimal placements. The corresponding H_2 norms of the resulting placement are displayed on the y -axis, from which it can be seen that the greedy QR placements closely approximate the optimal placements. In the five sensor and actuator case, the optimal placement [25, Fig. 4] yields a H_2 norm of 27.4, which exactly agrees with our QR initialized optimization and is approximated by the QR pivoted placement (27.8). Our results agree with the optimal placements by Chen

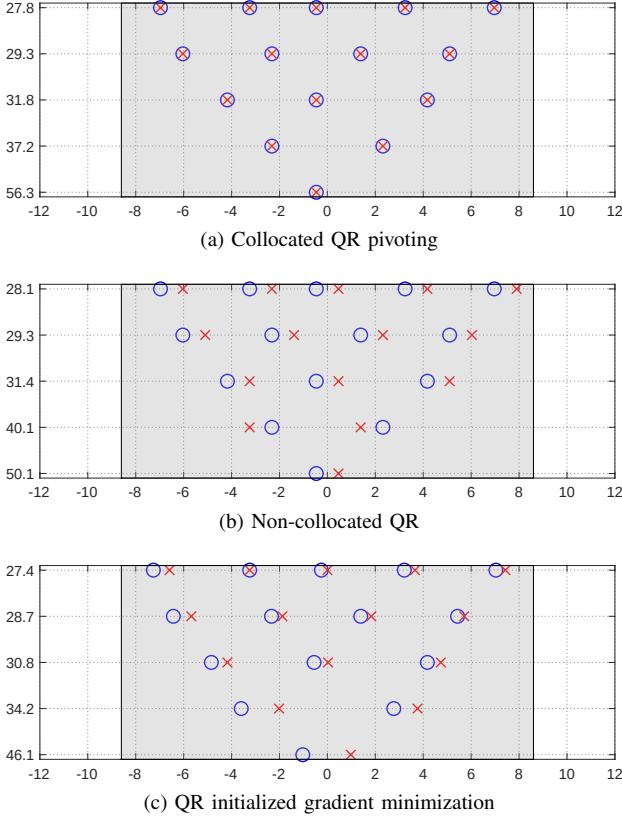


Fig. 6: Sensor ($\times$) and actuator ($\circ$) placement for linearized Ginzburg-Landau. Each row corresponds to the optimized placement for a certain budget of sensors and actuators. Placements based on QR pivoting of balanced truncated modes (a) closely approximate the H_2 norms of the placements determined using gradient minimization (c). It is also possible to modify the QR algorithm to iteratively place sensors and actuators to avoid collocation (b).

and Rowley, both visually and with H_2 norms. Yet another metric for evaluating the placement is the LQG gain of a given signal from a sensor to an actuator, and we compare the greedy QR placement to the QR initialized optimization using this metric over all sensors and actuators in Fig. 7. The results are indistinguishable from each other, and are also indistinguishable from the same LQG gain figure produced by Chen and Rowley [25, Fig. 5]. Due to the dissipative nature of the system, it is observed that low-frequency information from a sensor propagates to upstream actuators, but the majority of high-frequency information only propagates to the nearest corresponding actuator.

QR pivoting runtime scales as $\mathcal{O}(nr^2)$ and the deviation of the resulting placement from optimal (fig. 6) decreases with increasing p, q , and r . Furthermore, the conjugate gradient procedure initialized by the QR pivoted placement converges roughly 50 to 100 iterations to converge. Without QR initialization, the conjugate gradient method requires an ensemble of random initial placements and multiple iterations to converge.

Without balanced model reduction, the Gramians are uninformative for sensor placement. Chen and Rowley show this by placing single sensors & actuators at the extrema of Gramian

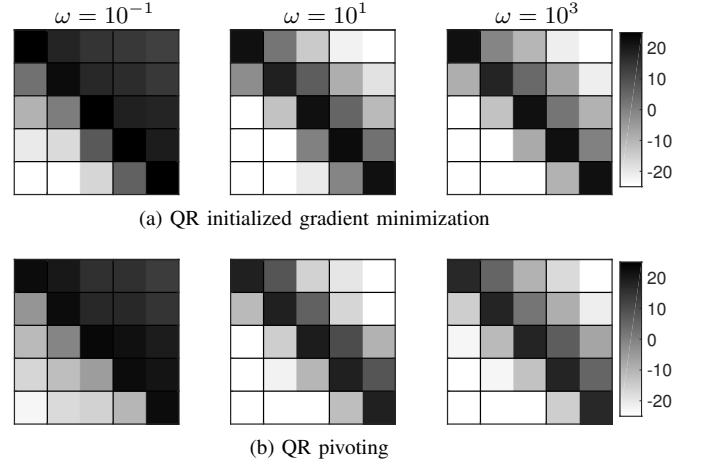


Fig. 7: LQG gain in decibels for a system with five sensors and five actuators. Each block shows the gain from a signal $\exp(i\omega t)$ in sensor k (column) to actuator j (row), ordered upstream to downstream.

eigenvectors, which yields H_2 norms over 200, in contrast to their optimized results, and ours, which have H_2 norm approximately equal to 40. In contrast, balanced truncated modes of the LQG controlled system are indeed useful for optimizing placements, and can be determined from a one-time computation of $\mathcal{O}(n^3)$ to construct the LQG controller, and one more $\mathcal{O}(n^3)$ computation to solve Lyapunov equations and compute the balancing modes.

VII. DISCUSSION AND OUTLOOK

In this work we develop a scalable sensor and actuator placement algorithm whose runtime scales linearly with the number of state variables, after a one-time offline computation of the balanced modes. Our approach relies on balanced model reduction [29], [31], [32], in which modes are hierarchically ordered by their observability and controllability. We extend EIM-based methods to sample the low-rank balancing modes of the system and determine maximally observable and controllable locations (sensor & actuators) in state space. The performance of this algorithm is demonstrated on a random state space system, and to control the linearized Ginzburg-Landau system with stochastic disturbances. Our optimized placements vastly exceed the performance of random placements, as quantified by their corresponding objective function values, and closely approximate H_2 norms of optimal placements determined by expensive gradient minimization methods at a fraction of their runtime.

Point sensors and actuators are critical for feedback control of large high-dimensional complex systems. This work advocates sensor and actuator placement using QR pivots of the direct and adjoint modes of a system's balancing transformation. The resulting placement is empirically shown to preserve the dynamics of the full system. The method has deep connections to system observability, controllability, modal sampling methods and classical experimental design criteria. Furthermore, QR pivoting is more computationally efficient than leading greedy and convex optimization methods, and thus critically enlarge the search space of possible candidate

placements. This is particularly valuable in spatiotemporal models where high-resolution grids result in a large number of possible states, and balanced modes reflect spatial structure in the system that are readily exploited using the QR method.

This work opens a variety of future directions in pivoting sensor and actuator optimization. Our method relies on a known model of the dynamics, but it would also be interesting to generalize the method to data-driven system identification models. In addition, point sensors and actuators are simplifications of constrained or nonlinear sensing and actuation that may occur in practice. Although QR-pivoted sensing can be generalized to incorporate constraints [40], nonlinear sensing constraints remain an open design problem. Moreover, rapid advances in data collection yield extremely large search spaces, in which case the QR pivoting procedure may benefit from accelerations such as randomized or blocked pivoting. Randomized methods may also be used to accelerate the computation of low-rank balancing transformations from data.

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