

5. On a Stochastic Integral Equation.

By Kiyosi Itô.

Mathematical Institute, Nagoya Imperial University.

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In his note "Stochastic Integral"¹⁾ the author has discussed an integral of the type $\int_0^t f(\tau, \omega) d\tau g(\tau, \omega)$, where ω is a variable taking values in a probability field $(\mathcal{Q}, P)$ and $g(t, \omega)$ is a normalized brownian motion on $(\mathcal{Q}, P)$. This note is devoted to the investigation of a stochastic integral equation:

$$(1) \quad x(t, \omega) = c + \int_0^t a(\tau, x(\tau, \omega)) d\tau + \int_0^t b(\tau, x(\tau, \omega)) d\tau g(\tau, \omega),$$

which is closely related to the researches of Markoff process by many authors, especially by S. Bernstein,²⁾ A. Kolmogoroff,³⁾ and W. Feller.⁴⁾

Theorem. Let $a(t, x)$ and $b(t, x)$ be continuous in (t, x) and satisfy

(2) $|a(t, x) - a(t, y)| \leq A|x - y|$, (3) $|b(t, x) - b(t, y)| \leq B|x - y|$, where $0 \leq t \leq 1$ and $-\infty < x, y < \infty$. Then the integral equation (1) has one and only one continuous (in t with P -measure 1) solution.

Proof. Firstly we shall find a solution by the method of successive approximation. We define $x_k(t, \omega)$ for $k = 0, 1, 2, \dots$ as follows,

$$(4) \quad x_0(t, \omega) \equiv c,$$

$$(5) \quad x_k(t, \omega) = c + \int_0^t a(\tau, x_{k-1}(\tau, \omega)) d\tau + \int_0^t b(\tau, x_{k-1}(\tau, \omega)) d\tau g(\tau, \omega);$$

the possibility of these definitions can be verified recursively if we make use of the properties of the stochastic integral shown in S.I..

By (5) we have, for $k = 0, 1, 2, \dots$

$$(6) \quad x_{k+1}(t, \omega) - x_k(t, \omega) = \int_0^t (a(\tau, x_k(\tau, \omega)) - a(\tau, x_{k-1}(\tau, \omega))) d\tau \\ + \int_0^t (b(\tau, x_k(\tau, \omega)) - b(\tau, x_{k-1}(\tau, \omega))) d\tau g(\tau, \omega).$$

Since $a(t, x)$ and $b(t, x)$ are continuous, $|a(t, c)|$ and $|b(t, c)|$ are bounded in $0 \leq t \leq 1$ by a finite upper bound, say M . Then we have

1) These proceedings Vol. XX. No. 8. p. 519. This paper will be cited as S.I. in the following.

2) S. Bernstein: Equations différentielles stochastiques, Actuarités Scientifiques 738.

3) A. Kolmogoroff: Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104, p. 415.

4) W. Feller: Zur Theorie der stochastischen Prozesse. (Existenz und Eindeutigkeitssätze.), Math. Ann. 113, p. 113.

$$\begin{aligned}
 (7) \quad E(x_1(t, \omega) - x_0(t, \omega))^2 &\leq \left\{ \left(E \left(\int_0^t a(\tau, c) d\tau \right)^2 \right)^{\frac{1}{2}} + \left(E \left(\int_0^t b(\tau, c) d\tau g(\tau, \omega) \right)^2 \right)^{\frac{1}{2}} \right\}^2 \\
 &= \left\{ \left| \int_0^t a(\tau, c) d\tau \right| + \left(\int_{\Omega} \int_0^t b(\tau, c)^2 d\tau P(d\omega) \right)^{\frac{1}{2}} \right\}^2 \\
 &\leq (Mt + Mt^{\frac{1}{2}})^2 \leq 4M^2t.
 \end{aligned}$$

In general we can show the following inequalities by mathematical induction for $k = 1, 2, 3, \dots$,

$$(8) \quad E \left(\int_0^t |a(\tau, x_k(\tau, \omega)) - a(\tau, x_{k-1}(\tau, \omega))| d\tau \right)^2 \leq 4A^2(A+B)^{2(k-1)} M^2 \frac{t^{k-1}}{k+1},$$

$$\begin{aligned}
 (9) \quad E \left(\int_0^t (b(\tau, x_k(\tau, \omega)) - b(\tau, x_{k-1}(\tau, \omega))) d\tau g(\tau, \omega) \right)^2 \\
 \leq 4B^2(A+B)^{2(k-1)} M^2 \frac{t^{k-1}}{k+1},
 \end{aligned}$$

$$(10) \quad E(x_k(t, \omega) - x_{k-1}(t, \omega))^2 \leq 4(A+B)^{2(k-1)} M^2 \frac{t^k}{k+1}.$$

By Bienaymé's inequality we can deduce from (8)

$$\begin{aligned}
 (11) \quad P \left\{ \omega; \int_0^1 |a(\tau, x_k(\tau, \omega)) - a(\tau, x_{k-1}(\tau, \omega))| d\tau > \frac{1}{2^{k+1}} \right\} \\
 \leq 4A^2(A+B)^{2(k-1)} M^2 \frac{2^{k+1}}{k+1}.
 \end{aligned}$$

Since the series whose k -th term in the above right side is evidently convergent, we can conclude by Borel-Cantelli's theorem with P -measure 1 that

$$\int_0^1 |a(\tau, x_k(\tau, \omega)) - a(\tau, x_{k-1}(\tau, \omega))| d\tau \leq \frac{1}{2^{k+1}},$$

a fortiori

$$(12) \quad \sup_{0 \leq t \leq 1} \left| \int_0^t (a(\tau, x_k(\tau, \omega)) - (a(\tau, x_{k-1}(\tau, \omega)))) d\tau \right| \leq \frac{1}{2^{k+1}}$$

but for finite exceptional values of k .

By a property of stochastic integral we obtain from (9)

$$\begin{aligned}
 (13) \quad P \left\{ \omega; \sup_{0 \leq t \leq 1} \left| \int_0^t (b(\tau, x_k(\tau, \omega)) - b(\tau, x_{k-1}(\tau, \omega))) d\tau g(\tau, \omega) \right| > \frac{1}{2^{k+1}} \right\} \\
 \leq 4B^2(A+B)^{2(k-1)} M^2 \frac{2^{k+1}}{k+1}
 \end{aligned}$$

and so we can see, by making use of Borel-Cantelli's theorem again, with P -measure 1, that

$$(14) \quad \sup_{0 \leq t \leq 1} \left| \int_0^t (b(\tau, x_k(\tau, \omega)) - b(\tau, x_{k-1}(\tau, \omega))) d\tau g(\tau, \omega) \right| \leq \frac{1}{2^{k+1}}$$

but for finite exceptional values of k .

From (6), (12) and (14) we see, with P -measure 1, that

$$(15) \quad \sup_{0 \leq t \leq 1} |x_k(t, \omega) - x_{k-1}(t, \omega)| \leq \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}$$

but for finite exceptional values of k . Thus the sequence $x_k(t, \omega)$ is uniformly (in $0 \leq t \leq 1$) convergent with P -measure 1. We shall denote the limit by

$x(t, \omega)$, which will be shown to be a solution of (1) in the following.

Integrating both sides of (10) from $t = 0$ to $t = 1$, we obtain

$$(16) \quad \int_0^1 \int_{\Omega} (x_k(t, \omega) - x_{k-1}(t, \omega))^2 P(d\omega) dt \leq 4(A+B)^{2(k-1)} M^2 \frac{1}{|k+1|}.$$

Therefore $x_k(t, \omega)$ is convergent in the norm of $L_2([0, 1] \times \Omega)$ and so the limit $x(t, \omega)$ belongs to $L_2([0, 1] \times \Omega)$. Since $x(t, \omega)$ belongs to $G^{(5)}$ by the definition and it is continuous (in t) as the uniform (in t) limit with P -measure 1, $x(t, \omega)$ belongs to $S^{*,(6)}$. Consequently we have

$$(17) \quad x(t, \omega) \in \bar{S}^{(7)}$$

Therefore we can apply Theorem 2.2 in S.I. to $x(t, \omega)$.

Now we have

$$(18) \quad E \left(\int_0^t a(\tau, x(\tau, \omega)) d\tau - \int_0^t a(\tau, x_k(\tau, \omega)) d\tau \right)^2 \leq t \int_0^t \int_{\Omega} A^2(x(\tau, \omega) - x_k(\tau, \omega))^2 P(d\omega) d\tau$$

$$(19) \quad E \left(\int_0^t b(\tau, x(\tau, \omega)) d\tau g(\tau, \omega) - \int_0^t b(\tau, x_k(\tau, \omega)) d\tau g(\tau, \omega) \right)^2 \leq t \int_0^t \int_{\Omega} B^2(x(\tau, \omega) - x_k(\tau, \omega))^2 P(d\omega) d\tau \leq B^2 \int_0^1 \int_{\Omega} (x(\tau, \omega) - x_k(\tau, \omega))^2 P(d\omega) d\tau,$$

Taking the $L_2([0, 1] \times \Omega)$ -limits of both sides of (5) for $k \rightarrow \infty$, we obtain

$$(20) \quad x(t, \omega) = c + \int_0^t a(\tau, x(\tau, \omega)) d\tau + \int_0^t b(\tau, x(\tau, \omega)) d\tau g(\tau, \omega)$$

with P -measure 1 for any assigned t . But the above both sides are continuous in t with P -measure 1. Therefore (20) holds for any t with P -measure 1. Thus we have obtained a solution $x(t, \omega)$ of (1) which is continuous in t with P -measure 1.

Next we shall the uniqueness of the solution of (1). Let $y(t, \omega)$ and $z(t, \omega)$ be a continuous (in t with P -measure 1) solution of (1):

$$(21) \quad y(t, \omega) = c + \int_0^t a(\tau, y(\tau, \omega)) d\tau + \int_0^t b(\tau, y(\tau, \omega)) d\tau g(\tau, \omega),$$

$$(22) \quad z(t, \omega) = c + \int_0^t a(\tau, z(\tau, \omega)) d\tau + \int_0^t b(\tau, z(\tau, \omega)) d\tau g(\tau, \omega).$$

In the case that $|a(t, x)|$ and $|b(t, x)|$ are bounded by an upper bound G , we have

$$(23) \quad E(y(t, \omega) - z(t, \omega))^2 \leq \left(\left(E \left(\int_0^t G d\tau \right)^2 \right)^{\frac{1}{2}} + \left(E \left(\int_0^t G^2 d\tau \right)^2 \right)^{\frac{1}{2}} \right)^2$$

But we have

$$(24) \quad E(y(t, \omega) - z(t, \omega))^2 \leq (A+B)^2 \int_0^t E(y(\tau, \omega) - z(\tau, \omega))^2 d\tau$$

5) Cf. S.I. 2.

6) Cf. S.I. 2.

7) Cf. S.I. Foot Note 4).

Therefore we have

$$(25) \quad E(y(t, \omega) - z(t, \omega))^2 \leq 4G^2(A + B)^{2k} \frac{t^k}{k}.$$

Thus we obtain, as k tends to ∞ , $E(y(t, \omega) - z(t, \omega))^2 = 0$, and so $y(t, \omega) = z(t, \omega)$ with P -measure 1 for any t . By the continuity (in t) of $y(t, \omega)$ and $z(t, \omega)$, $y(t, \omega) = z(t, \omega)$ holds for any t with P -measure 1

In the general case we obtain, by the assumption,

$$(26) \quad |a(t, y)| \leq |a(t, c)| + |a(t, y) - a(t, c)| \leq M + A|y - c|,$$

$$(27) \quad |b(t, y)| \leq |b(t, c)| + |b(t, y) - b(t, c)| \leq M + B|y - c|.$$

Put

$$(28) \quad \mathcal{Q}_k = \{\omega; \sup_{0 \leq t \leq 1} |y(t, \omega) - c| < K\} \cap \{\omega; \sup_{0 \leq t \leq 1} |z(t, \omega) - c| < K\}$$

$\mathcal{Q}_k$ increases with K and tends to a set $\mathcal{Q}^*$ of P -measure 1 on account of the continuity of $y(t, \omega)$ and $z(t, \omega)$.

Now we have on $\mathcal{Q}_k$

$$(29) \quad |a(t, y(t, \omega))|, |b(t, y(t, \omega))|, |a(t, z(t, \omega))|, |b(t, z(t, \omega))| \\ < M + (A + B)K.$$

Denote the right side by G and define $a_G(t, x)$ and $b_G(t, x)$ as follows.

$$(30) \quad \begin{aligned} a_G(t, x) &= G, & \text{when } a(t, x) \geq G, \\ a_G(t, x) &= a(t, x), & \text{when } |a(t, x)| < G, \\ a_G(t, x) &= -G, & \text{when } a(t, x) \leq -G, \\ b_G(t, x) &= G, & \text{when } b(t, x) \geq G, \\ b_G(t, x) &= b(t, x), & \text{when } |b(t, x)| < G, \\ b_G(t, x) &= -G, & \text{when } b(t, x) \leq -G. \end{aligned}$$

By (29) we have on $\mathcal{Q}_k$

$$(31) \quad a_G(t, y(t, \omega)) = a(t, y(t, \omega)), \quad a_G(t, z(t, \omega)) = a(t, z(t, \omega)), \\ b_G(t, y(t, \omega)) = b(t, y(t, \omega)), \quad b_G(t, z(t, \omega)) = b(t, z(t, \omega)).$$

and so on $\mathcal{Q}_k$ both $y(t, \omega)$ and $z(t, \omega)$ satisfy a stochastic integral equation:

$$(32) \quad x(t, \omega) = c + \int_0^t a_G(\tau, x(\tau, \omega)) d\tau + \int_0^t b_G(\tau, x(\tau, \omega)) d\tau g(\tau, \omega),$$

which has a unique continuous (in t with P -measure 1) solution

by the argument in the above special case. Thus we have $y(t, \omega) = z(t, \omega) = x_k(t, \omega)$ for any t and for any K and so $y(t, \omega) = z(t, \omega)$ for any t on $\mathcal{Q}^*$.

Q.E.D.

Added in proof: The author has published a detailed investigation of the same subject in a more general case. Cf. On stochastic processes (II)—A stochastic differential equation—forth-coming to the Japanese Journal of Mathematics.